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NATIONAL INSTITUTE OF SCIENCES OF INDIA

NOTICE

The National Institute of Sciences of India, since its inception in 1935, has been publishing regularly the *Proceedings* every year of which twenty volumes were published till January, 1955. It contained miscellaneous articles on Physical and Biological Sciences, as well as important official proceedings of the Institute.

The Publication Advisory Board of the Institute had been considering for some time past whether it would be possible to have the *Proceedings* published in two Parts, that is, one Part on *Physical Sciences* and the other Part on *Biological Sciences* and to have the official proceedings published in the Year Book, on the lines of the publications of the Royal Society. An effort was made last year in this direction and the articles on Physical Sciences and Biological Sciences were published in alternate issues of the *Proceedings* (Vol. XX, 1954). On a report from the Publication Advisory Board that with the articles coming in for publication by the Institute it will be possible to have the *Proceedings* published in two parts, the Council decided that with effect from Volume XXI (1955) the *Proceedings* should be published in two Parts, that is, Part 'A' for Physical Sciences and Part 'B' for Biological Sciences and they be issued monthly in alternate months.

In view of this new form of publication, the titles of the two Parts of the *Proceedings* which will be issued from 1955 will be:

Proceedings, National Institute of Sciences of India, Part A, Physical Sciences, and

Proceedings, National Institute of Sciences of India, Part B, Biological Sciences.

The Year Book will be an annual publication which will contain the official proceedings of the Institute as were hitherto published in the *Proceedings*.

As regards the division of the *Proceedings* into Parts 'A' and 'B', the Volume and Year numbers will be the same for the two Parts, with their own Title Page, page numbering and Index, both the Parts starting with No. 1, as Physical Science, Part A, No. 1 and Biological Science, Part B, No. 1 and aiming to have yearly six issues for each Part.

The following abbreviations are suggested for use in References:

For *Proceedings*, National Institute of Sciences of India, Part 'A', Physical Sciences—*PNISIPS*.

For *Proceedings*, National Institute of Sciences of India, Part 'B', Biological Sciences—*PNISIBS*.

J. M. SEN,

Editor of Publications,

National Institute of Sciences of India.

Calcutta,

26th January, 1955.

SELF-SUPERPOSABILITY IN AXIALLY SYMMETRICAL FLOWS

by PREM PRAKASH, Lucknow University

(Communicated by R. S. Varma, F.N.I.)

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INTRODUCTORY

Several papers have been written to define and subsequently develop the theory of superposition of fluid motions (Ram Ballabh, 1940, 69, 85; Strang, J. A., 1942; Ram Ballabh, 1952). The same has been done here for the case of axially symmetrical flows. The condition of superposability has been derived and some examples of self-superposable flows have been given.

Let us take the axis of symmetry as the x -axis and denote by $\hat{\omega}$ the co-ordinate perpendicular to x , also let u, v be the components of the velocity and X, Ω those of the external force along x and $\hat{\omega}$ respectively. Then the equations of motion along x and $\hat{\omega}$ directions are

$$\frac{\partial u}{\partial t} - v\zeta = -\frac{\partial}{\partial x} \left(\frac{1}{2}q^2 + \frac{p}{\rho} \right) + X + \nu \nabla^2 u \quad \dots \quad (1)$$

and

$$\frac{\partial v}{\partial t} + u\zeta = -\frac{\partial}{\partial \hat{\omega}} \left(\frac{1}{2}q^2 + \frac{p}{\rho} \right) + \Omega + \nu \left(\nabla^2 v - \frac{v}{\hat{\omega}^2} \right) \quad \dots \quad (2)$$

respectively, where p is the pressure, ρ the density, ν the kinematic coefficient of viscosity, ζ the vorticity

$$\left(= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial \hat{\omega}} \right), \quad q^2 = u^2 + v^2$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \hat{\omega}^2} + \frac{1}{\hat{\omega}} \frac{\partial}{\partial \hat{\omega}} \quad \dots \quad (3)$$

is the equivalent Laplacian operator in the present system of co-ordinates.

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial \hat{\omega}} + \frac{v}{\hat{\omega}} = 0. \quad \dots \quad (4)$$

Putting (u_1, v_1, p_1) , (u_2, v_2, p_2) and $(u_1+u_2, v_1+v_2, p_1+p_2+\Pi)$ in (1), simplifying and arranging the terms, we get under the set of external forces (X_1, Ω_1) , (X_2, Ω_2) and $(X_1+X_2, \Omega_1+\Omega_2)$ respectively

$$v_2\zeta_1 + v_1\zeta_2 = \frac{\partial}{\partial x} \left(\frac{\Pi}{\rho} + u_1u_2 + v_1v_2 \right).$$

Similarly (2) gives

$$u_2\zeta_1 + u_1\zeta_2 = -\frac{\partial}{\partial \hat{\omega}} \left(\frac{\Pi}{\rho} + u_1u_2 + v_1v_2 \right).$$

Elimination of Π furnishes

$$\frac{\partial}{\partial x} (u_2 \zeta_1 + u_1 \zeta_2) + \frac{\partial}{\partial \bar{\omega}} (v_2 \zeta_1 + v_1 \zeta_2) = 0$$

which with (4) becomes

$$-\frac{v_1 \zeta_2 + v_2 \zeta_1}{\bar{\omega}} + u_1 \frac{\partial \zeta_2}{\partial x} + v_1 \frac{\partial \zeta_2}{\partial \bar{\omega}} + u_2 \frac{\partial \zeta_1}{\partial x} + v_2 \frac{\partial \zeta_1}{\partial \bar{\omega}} = 0. \quad \dots (5)$$

If the motion is self-superposable (5) reduces to

$$-\frac{1}{\bar{\omega}} \frac{\partial \psi}{\partial \bar{\omega}} \cdot \frac{\partial \zeta}{\partial x} + \frac{1}{\bar{\omega}} \frac{\partial \psi}{\partial x} \frac{\partial \zeta}{\partial \bar{\omega}} - \frac{\zeta}{\bar{\omega}^2} \frac{\partial \psi}{\partial x} = 0$$

where Stokes's current function ψ has been introduced. This is the same as

$$-\frac{\partial \psi}{\partial \bar{\omega}} \frac{\partial}{\partial x} \left(\frac{\zeta}{\bar{\omega}} \right) + \frac{\partial \psi}{\partial x} \frac{\partial}{\partial \bar{\omega}} \left(\frac{\zeta}{\bar{\omega}} \right) = 0$$

or

$$\frac{\zeta}{\bar{\omega}} = f(\psi), \text{ i.e. } \zeta = \bar{\omega} f(\psi) \quad \dots \quad \dots \quad \dots (6)$$

f being any function. This according to Stokes is the condition for steady motion of a non-viscous flow with axial symmetry (Lamb, 1945, 245).

If the forces are conservative the condition of integrability of the equations of motion is

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} (u \zeta) + \frac{\partial}{\partial \bar{\omega}} (v \zeta) = \nu \left(\nabla^2 \zeta - \frac{\zeta}{\bar{\omega}^2} \right).$$

In the case of self-superposable flows this reduces to

$$\frac{\partial \zeta}{\partial t} = \nu \left(\nabla^2 \zeta - \frac{\zeta}{\bar{\omega}^2} \right). \quad \dots \quad \dots \quad \dots (7)$$

To find a self-superposable flow we have to find a ψ satisfying (6), (7) and the relation

$$\zeta = \frac{1}{\bar{\omega}} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial \bar{\omega}^2} - \frac{1}{\bar{\omega}} \frac{\partial \psi}{\partial \bar{\omega}} \right). \quad \dots \quad \dots \quad \dots (8)$$

SOLUTIONS

Case (i): From (6) and (8) ψ cannot be a function of x only. To effect simplification we assume ψ to be a function of $\bar{\omega}$ alone. In this case (6) and (8) are satisfied and (7) becomes

$$\frac{\partial \zeta}{\partial t} = \nu \left\{ \frac{\partial^2 \zeta}{\partial \bar{\omega}^2} + \frac{1}{\bar{\omega}} \frac{\partial \zeta}{\partial \bar{\omega}} - \frac{\zeta}{\bar{\omega}^2} \right\}. \quad \dots \quad \dots \quad \dots (9)$$

Making the substitution $\zeta = \bar{\omega} \phi(\bar{\omega})$, eq. (9) simplifies into

$$\bar{\omega} \frac{\partial \phi}{\partial t} = \nu \left\{ 3 \frac{\partial \phi}{\partial \bar{\omega}} + \bar{\omega} \frac{\partial^2 \phi}{\partial \bar{\omega}^2} \right\}. \quad \dots \quad \dots \quad \dots (10)$$

This is clearly satisfied by $\phi = \text{constant} = A$. This gives

$$\zeta = A\hat{\omega} \text{ and } u = -\left(\frac{A\hat{\omega}^2}{2} + B\right) \quad \dots \quad (11)$$

B being another constant. Since $u = 0$ at $\hat{\omega} = \sqrt{\frac{-2B}{A}}$ (11) represents a steady flow of a viscous liquid in a circular pipe of cross-section $\sqrt{\frac{-2B}{A}}$. The velocity on the axis is $-B$. The solution is already known (Lamb, 1945, 585).

In the case of steady flow the solution of (10) is $\phi = \frac{-C}{2\hat{\omega}^2} + D$; C and D being arbitrary constants. This gives

$$\zeta = \frac{-C}{2\hat{\omega}} + D\hat{\omega}; \text{ and } u = \frac{C}{2} \log \hat{\omega} - \frac{D}{2} \hat{\omega}^2 - E.$$

If $u = 0$ at $\hat{\omega} = a$ and also at $\hat{\omega} = b$

$$-\frac{C}{2} \log a + \frac{D}{2} a^2 + E = 0 = -\frac{C}{2} \log b + \frac{D}{2} b^2 + E.$$

These give

$$D = \frac{C \log \frac{b}{a}}{b^2 - a^2} \text{ and } E = \frac{C}{2} \left(\log a - \frac{\log \frac{b}{a}}{b^2 - a^2} \cdot a^2 \right).$$

Hence

$$u = \frac{C}{2} \left(\log \hat{\omega} - \frac{\log \frac{b}{a}}{b^2 - a^2} \cdot \hat{\omega}^2 - \log a + \frac{\log \frac{b}{a}}{b^2 - a^2} \cdot a^2 \right). \quad \dots \quad (12)$$

This therefore gives the flow of a viscous liquid in a circular pipe of annular cross-section. This solution too is already known (Lamb, 1945, 586).

Both flows are self-superposable.

Case (ii): We now proceed to find some unsteady solutions of (10) believed to be new. An approach to the solution of (6), (8) and (9) is furnished by the assumption $f(\psi) = A\psi$, A being a function of the time, and $\psi = \psi(\hat{\omega})$ as before.

Then (8) with (6) gives

$$A\hat{\omega}\psi = \frac{d}{d\hat{\omega}} \left(\frac{1}{\hat{\omega}} \frac{d\psi}{d\hat{\omega}} \right). \quad \dots \quad (13)$$

Writing $\psi = \psi(\hat{\omega}^2)$ has the advantage of reducing (13) to $\psi'' = \frac{A}{4}\psi$; dashes denoting differentiation with respect to $\hat{\omega}^2$. The solutions are

$$\left. \begin{aligned} \text{(i)} \quad \psi &= B \sin \left(\frac{\sqrt{-A}}{2} \hat{\omega}^2 \right), \\ \text{(ii)} \quad \psi &= B \cos \left(\frac{\sqrt{-A}}{2} \hat{\omega}^2 \right), \\ \text{(iii)} \quad \psi &= Be^{\pm \frac{\sqrt{A}}{2} \hat{\omega}^2}, \end{aligned} \right\} \quad \dots \quad (14)$$

according as A is negative or positive. The vorticity is given by

$$\left. \begin{aligned} \text{(i)} \quad \zeta &= AB \hat{\omega} \sin \left(\frac{\sqrt{-A}}{2} \hat{\omega}^2 \right), \\ \text{(ii)} \quad \zeta &= AB \hat{\omega} \cos \left(\frac{\sqrt{-A}}{2} \hat{\omega}^2 \right), \\ \text{(iii)} \quad \zeta &= AB \hat{\omega} e^{\pm \frac{\sqrt{A}}{2} \hat{\omega}^2}. \end{aligned} \right\} \dots \dots \dots (15)$$

It can be seen that (i) and (ii) of (15) do not satisfy (9) and hence have to be neglected. In (iii) the sign of the exponential power should be taken negative to ensure vanishing of the motion at infinity. In that case (9) becomes

$$\hat{\omega} \left\{ \frac{d}{dt} (AB) + \frac{AB}{4\sqrt{A}} \hat{\omega}^2 \frac{dA}{dt} \right\} e^{-\frac{\sqrt{A}}{2} \hat{\omega}^2} = -\nu AB (4\sqrt{A} \hat{\omega} + A \hat{\omega}^3) e^{-\frac{\sqrt{A}}{2} \hat{\omega}^2}.$$

Hence,

$$\nu A^2 B = \frac{-AB}{4\sqrt{A}} \frac{dA}{dt} \text{ and } 4\nu AB\sqrt{A} = -\frac{d}{dt} (AB).$$

These give,

$$\sqrt{A} = \frac{1}{2\nu t + \frac{C}{2}} \text{ and } B = \text{constant.} \text{ Hence (14) gives } \psi = B e^{-\hat{\omega}^2/(4\nu t + C)} \text{ and the velo-}$$

city is given by

$$u = \frac{2B}{4\nu t + C} e^{-\hat{\omega}^2/(4\nu t + C)} \dots \dots \dots (16)$$

the vorticity being

$$\zeta = \frac{B\hat{\omega}}{\left(2\nu t + \frac{C}{2}\right)^2} e^{-\hat{\omega}^2/(4\nu t + C)} \dots \dots \dots (17)$$

The motion decays with time, vanishes at infinity and admits of no finite boundaries. The stream lines lie on coaxial cylinders. The vorticity distribution when the flow takes place in circles about an axis, and the radial flow of heat in two dimensions, are noted for rapid decay with time (Lamb, 1945, 592). The vorticity given by (17) is capable of an interesting comparison with the flows referred to above and decays even more rapidly with time due to viscosity.

Case (iii): The motion in circles and the radial flow of heat are characterised by

$$\zeta = \frac{k}{4\nu t + C_1} e^{-\hat{\omega}^2/(4\nu t + C_1)} \dots \dots \dots (18)$$

At a fixed point $\hat{\omega}_1$, this increases from $\frac{k}{C_1} e^{-\hat{\omega}_1^2/C_1}$ at $t = 0$ to $\frac{k}{\hat{\omega}_1^2} e^{-1}$ at $t = \frac{\hat{\omega}_1^2 - C_1}{4\nu}$

and then asymptotically tends to zero. At a fixed time t_1 , this is equal to $\frac{k}{4\nu t_1 + C_1}$ at $\hat{\omega} = 0$, does not rise but asymptotically falls to zero for $\hat{\omega} = \infty$.

As regards the behaviour of ζ in (17), for a fixed $\hat{\omega}$ it rises from $\frac{B\hat{\omega}_1}{C^2/4}e^{-\hat{\omega}_1^2/C}$ at $t = 0$ to $\frac{16Be^{-2}}{\hat{\omega}_1^3}$ at $t = \frac{\hat{\omega}_1^2 - 2C}{8\nu}$ and then asymptotically falls to zero. At a fixed time t_1 , it rises from zero at $\hat{\omega} = 0$ to $\frac{Be^{-\frac{1}{2}}}{\left(2\nu t_1 + \frac{C}{2}\right)^{\frac{3}{2}}}$ at $\hat{\omega} = \sqrt{2\nu t_1 + \frac{C}{2}}$ and then asymptotically falls to zero. Thus the difference in behaviour with (18) occurs when variation with $\hat{\omega}$ is taken into account.

To find an unsteady flow of a viscous liquid in a circular pipe we take $\phi = e^{-avt}F(\hat{\omega})$. Then (10) becomes

$$ve^{-avt} \left\{ \hat{\omega} \frac{d^2 F}{d\hat{\omega}^2} + 3 \frac{dF}{d\hat{\omega}} + a\hat{\omega}F \right\} = 0. \quad \dots \quad (19)$$

$$\text{Let } F(\hat{\omega}) = \hat{\omega}^C \sum_0^{\infty} a_n \hat{\omega}^n.$$

Then,

$$\frac{dF}{d\hat{\omega}} = \hat{\omega}^C \sum_1^{\infty} n a_n \hat{\omega}^{n-1} + C \hat{\omega}^{C-1} \sum_0^{\infty} a_n \hat{\omega}^n; \text{ and}$$

$$\frac{d^2 F}{d\hat{\omega}^2} = \hat{\omega}^C \sum_2^{\infty} n(n-1) a_n \hat{\omega}^{n-2} + 2C \hat{\omega}^{C-1} \sum_1^{\infty} n a_n \hat{\omega}^{n-1} + C(C-1) \hat{\omega}^{C-2} \sum_0^{\infty} a_n \hat{\omega}^n.$$

Substituting these in (19) and equating to zero the coefficient of the lowest power of $\hat{\omega}$ we get the indicial equation to be

$$a_0 \{3C + C(C-1)\} = 0, \text{ i.e. } C = 0 \text{ or } -2. \quad \dots \quad (20)$$

Next, we get similarly

$$aa_0 + 6a_2 + 3Ca_2 + 2a_2 + 4Ca_2 + a_2C(C-1) = 0$$

i.e.,

$$a_2 = \frac{-aa_0}{8 + 6C + C^2}$$

and in general

$$a_{n+1} = \frac{-aa_{n-1}}{(n+1)(3+n+2C) + 2C + C^2} \quad \dots \quad (21)$$

When $C = -2$, a_2 becomes infinite and so become $a_4, a_6 \dots$. Hence this value of C does not give a solution of (19). Corresponding to $C = 0$ the solution is

$$\begin{aligned} F(\hat{\omega}) = & a_0 \left(1 - \frac{a\hat{\omega}^2}{2 \cdot 4} + \frac{a^2\hat{\omega}^4}{2 \cdot 4^2 \cdot 6} - \frac{a^3\hat{\omega}^6}{2 \cdot 4^2 \cdot 6^2 \cdot 8} + \dots + \frac{(-a)^n \hat{\omega}^{2n}}{2 \cdot 4^2 \cdot 6^2 \dots (2n)^2 (2n+2)} + \dots \right) \\ & + a_1 \left(\hat{\omega} - \frac{a\hat{\omega}^3}{3 \cdot 5} + \frac{a^2\hat{\omega}^5}{3 \cdot 5^2 \cdot 7} - \frac{a^3\hat{\omega}^7}{3 \cdot 5^2 \cdot 7^2 \cdot 9} + \dots + \frac{(-a)^n \hat{\omega}^{2n+1}}{3 \cdot 5^2 \cdot 7^2 \dots (2n+1)^2 (2n+3)} + \dots \right) \quad \dots \quad (22) \end{aligned}$$

It can easily be seen that the power series (22) is absolutely convergent for any finite value of $\hat{\omega}$ and hence is uniformly convergent in the same interval. Hence it defines a continuous function and can be integrated term by term, the integrated series possessing the same properties. This gives

$$\zeta = e^{-a\omega t} \left\{ a_0 \left(\hat{\omega} - \frac{a\hat{\omega}^3}{2 \cdot 4} + \frac{a^2\hat{\omega}^5}{2 \cdot 4^2 \cdot 6} - \frac{a^3\hat{\omega}^7}{2 \cdot 4^2 \cdot 6^2 \cdot 8} + \dots \right. \right. \\ \left. \left. + \frac{(-a)^n \hat{\omega}^{2n+1}}{2 \cdot 4^2 \cdot 6^2 \dots (2n)^2(2n+2)} + \dots \right) \right. \\ \left. + a_1 \left(\hat{\omega}^2 - \frac{a\hat{\omega}^4}{3 \cdot 5} + \frac{a^2\hat{\omega}^6}{3 \cdot 5^2 \cdot 7} - \frac{a^3\hat{\omega}^8}{3 \cdot 5^2 \cdot 7^2 \cdot 9} + \dots \right. \right. \\ \left. \left. + \frac{(-a)^n \hat{\omega}^{2n+2}}{3 \cdot 5^2 \cdot 7^2 \dots (2n+1)^2(2n+3)} + \dots \right) \right\}$$

and the velocity is

$$u = -e^{-a\omega t} \left\{ \frac{a_0}{2} \left(\hat{\omega}^2 - \frac{a\hat{\omega}^4}{4^2} + \frac{a^2\hat{\omega}^6}{4^2 \cdot 6^2} - \frac{a^3\hat{\omega}^8}{4^2 \cdot 6^2 \cdot 8^2} + \dots + \frac{(-a)^n \hat{\omega}^{2n+2}}{4^2 \cdot 6^2 \dots (2n+2)^2} + \dots \right) \right. \\ \left. + \frac{a_1}{3} \left(\hat{\omega}^3 - \frac{a\hat{\omega}^5}{5^2} + \frac{a^2\hat{\omega}^7}{5^2 \cdot 7^2} - \frac{a^3\hat{\omega}^9}{5^2 \cdot 7^2 \cdot 9^2} + \dots + \frac{(-a)^n \hat{\omega}^{2n+3}}{5^2 \cdot 7^2 \dots (2n+3)^2} + \dots \right) \right. \\ \left. + c \right\} \quad \dots \quad (23)$$

where c is an arbitrary constant or a function of time. Since (23) represents a continuous velocity, by proper adjustment of c , u can be made to vanish for $\hat{\omega} = \alpha$. Hence we get the proposed motion decaying due to viscosity exponentially with time.

Since the expression for velocity contains four arbitrary constants a , a_0 , a_1 and c , the flow may be subjected to more conditions. To illustrate a particular case we take $c = 0$ in (23). Then in the case of a flow in a circular pipe of annular cross-section we have from (23)

$$\frac{a_0}{2} \left(\alpha^2 - \frac{a\alpha^4}{4^2} + \frac{a^2\alpha^6}{4^2 \cdot 6^2} - \dots + \frac{(-a)^n \alpha^{2n+2}}{4^2 \cdot 6^2 \dots (2n+2)^2} + \dots \right) \\ + \frac{a_1}{3} \left(\alpha^3 - \frac{a\alpha^5}{5^2} + \frac{a^2\alpha^7}{5^2 \cdot 7^2} - \dots + \frac{(-a)^n \alpha^{2n+3}}{5^2 \cdot 7^2 \dots (2n+3)^2} + \dots \right) = 0 \quad \dots \quad (24)$$

and

$$\frac{a_0}{2} \left(\beta^2 - \frac{a\beta^4}{4^2} + \frac{a^2\beta^6}{4^2 \cdot 6^2} - \dots + \frac{(-a)^n \beta^{2n+2}}{4^2 \cdot 6^2 \dots (2n+2)^2} + \dots \right) \\ + \frac{a_1}{3} \left(\beta^3 - \frac{a\beta^5}{5^2} + \frac{a^2\beta^7}{5^2 \cdot 7^2} - \dots + \frac{(-a)^n \beta^{2n+3}}{5^2 \cdot 7^2 \dots (2n+3)^2} + \dots \right) = 0 \quad \dots \quad (25)$$

where $\hat{\omega} = \alpha$ and $\hat{\omega} = \beta$ are the two co-axial surfaces of the pipe. Next if the rate of total flux through the pipe be $Ae^{-a\omega t}$ we have the equation

$$A = -2\pi \int_{\alpha}^{\beta} \left\{ \frac{a_0}{2} \left(\hat{\omega}^2 - \frac{a\hat{\omega}^4}{4^2} + \frac{a^2\hat{\omega}^6}{4^2 \cdot 6^2} - \dots + \frac{(-a)^n \hat{\omega}^{2n+2}}{4^2 \cdot 6^2 \dots (2n+2)^2} + \dots \right) \right. \\ \left. + \frac{a_1}{3} \left(\hat{\omega}^3 - \frac{a\hat{\omega}^5}{5^2} + \frac{a^2\hat{\omega}^7}{5^2 \cdot 7^2} - \dots + \frac{(-a)^n \hat{\omega}^{2n+3}}{5^2 \cdot 7^2 \dots (2n+3)^2} + \dots \right) \right\} \hat{\omega} d\hat{\omega}$$

$$A = -2\pi \times$$

$$\left\{ \frac{a_0}{2} \left(\frac{\beta^4 - \alpha^4}{4} - a \frac{\beta^6 - \alpha^6}{4^2 \cdot 6} + a^2 \frac{\beta^8 - \alpha^8}{4^2 \cdot 6^2 \cdot 8} - \dots + (-a)^n \frac{\beta^{2n+4} - \alpha^{2n+4}}{4^2 \cdot 6^2 \dots (2n+2)^2 (2n+4)} + \dots \right) \right. \\ \left. + \frac{a_1}{3} \left(\frac{\beta^5 - \alpha^5}{5} - a \frac{\beta^7 - \alpha^7}{5^2 \cdot 7} + a^2 \frac{\beta^9 - \alpha^9}{5^2 \cdot 7^2 \cdot 9} - \dots + (-a)^n \frac{\beta^{2n+5} - \alpha^{2n+5}}{5^2 \cdot 7^2 \dots (2n+3)^2 (2n+5)} + \dots \right) \right\} \quad \dots (26)$$

Thus the three arbitrary constants a , a_0 , a_1 are determined in terms of α and β from the equations (24), (25) and (26) and when substituted in (23) with $c = 0$, we obtain the unsteady flow of a viscous liquid in a circular pipe of annular cross-section with a given flux. Another solution of (10) consistent with a fixed circular boundary is given by

$$\phi = A\bar{\omega}^2 + 8avt + c; \quad a \text{ and } c \text{ being constants,}$$

which gives

$$\xi = \bar{\omega} (a\bar{\omega}^2 + 8avt + c); \text{ and}$$

$$u = - \left(\frac{a\bar{\omega}^4}{4} + 4avt\bar{\omega}^2 + \frac{c\bar{\omega}^2}{2} \right) + D(t) \quad \dots \quad (27)$$

$D(t)$ being an arbitrary function of t . For $\bar{\omega} = \alpha$ the velocity

$$u_\alpha = - \left(\frac{a\alpha^4}{4} + 4avt\alpha^2 + \frac{c\alpha^2}{2} \right) + D(t).$$

Superimposing the velocity $-u_\alpha$ over u , which is permissible by (5), we get another unsteady flow inside a fixed circular pipe to be

$$u = \frac{a}{4} (\alpha^4 - \bar{\omega}^4) + 4av(\alpha^2 - \bar{\omega}^2)t + \frac{c}{2} (\alpha^2 - \bar{\omega}^2) \quad \dots \quad (28)$$

This flow is essentially an accelerated one and holds for finite values of t only.

When ψ is a function of x and $\bar{\omega}$ both, a steady self-superposable non-viscous flow is given by Hill's Spherical Vortex (Lamb, 1945, 245)

$$\psi = \frac{1}{2} A \bar{\omega}^2 (a^2 - x^2 - \bar{\omega}^2) \quad \dots \quad (29)$$

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SUMMARY

The paper deals with superposability in axially symmetrical flows. Two flows in circular pipes and one extending up to infinity have been obtained, all of them being self-superposable. Besides some known flows have also occurred as self-superposable flows.

REFERENCES

- Lamb, H. (1945). *Hydrodynamics*, Dover.
 Ram Ballabh (1940). Superposable Fluid Motions. *Proc. Benares Math. Soc.*, 2, 69-79.
 ——— (1940). Self-superposable Motions of the Type $\xi = \lambda u$, etc. *Ibid.*, 85-89.
 ——— (1952). On Two-dimensional Superposable Flows. *Jour. Ind. Math. Soc. (N.S.)*, 16, 191-197.
 Strang, J. A. (1942). Self-superposable Motion of Viscous Incompressible Fluid Referred to Rotating Axes. *Proc. Benares Math. Soc.*, 4, 9-18.

ON 'SIMILARITY' SOLUTIONS OF PRANDTL'S BOUNDARY LAYER EQUATIONS

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INTRODUCTION

In two recent papers, Goldstein (1939) and Mangler (1943) have examined the question of the existence of 'similarity' solutions of two-dimensional boundary layer equations of an incompressible viscous fluid. By assuming the longitudinal velocity profile $u(x, y)$ to be affine, i.e., assuming that the values of the longitudinal component $u(x, y)$ of the velocity for two different values of x are the same when suitable scale factors depending on x are introduced in u and y , they have found that the partial differential equations of the boundary layer reduce to an ordinary differential equation. They have naturally taken the known velocity $U(x)$ of the potential flow outside the boundary layer as the scale factor for the longitudinal component $u(x, y)$ of the velocity within the boundary layer and have deduced from this assumption that, in order that 'similarity' solutions of the boundary layer equations may exist, $U(x)$ must be proportional to x^m or to e^{px} where m and p are constants.

It has been shown in this paper that under more general conditions 'similarity' solutions of the boundary layer equations can be proved to exist. This has been done by discarding the scale factor $U(x)$ for the longitudinal component $u(x, y)$ of the velocity in the boundary layer and introducing, in its place, an unknown scale factor $h(x)$ of the dimensions of a velocity. This scale factor $h(x)$ and also the scale factor $g(x)$ for y are then determined by examining under what conditions the boundary layer equations reduce to an ordinary differential equation. The velocity $U(x)$ of the potential flow outside the boundary layer is then easily determined and it is found that the value of $U(x)$ is more general than that found by Goldstein or Mangler. The 'similarity' solutions obtained in this paper have therefore a wider range of applications.

DIMENSIONLESS STREAM FUNCTION

In the steady two-dimensional flow in the boundary layer of an incompressible viscous fluid along a curved boundary, the equation of motion is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2}, \quad \dots \dots \dots (1)$$

where x is measured along the boundary of the curve, y perpendicular to it, and $U(x)$ is the velocity in the irrotational flow just outside the boundary layer. The equation of continuity in the boundary layer is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad \dots \dots \dots (2)$$

If there be a free stream line in the liquid along which the boundary layer approximations are valid, the equation (1) is the equation of motion and the equation (2)

is the equation of continuity in a thin layer about that stream line, x being now measured along the stream line. In particular, these equations hold in the flow in a two-dimensional jet and in two-dimensional wakes behind a cylinder.

Introducing the stream function ψ , we can write

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \dots \dots \dots (3)$$

Taking l to be a suitable reference length, V a suitable reference velocity, $R = \frac{Vl}{\nu}$ the Reynolds number and $g(x)$ a dimensionless scale factor, we introduce dimensionless variables

$$\xi = \frac{x}{l}, \quad \eta = \frac{y\sqrt{R}}{lg(x)} \dots \dots \dots (4)$$

We further introduce a dimensionless stream function $f(\xi, \eta)$ defined by

$$f(\xi, \eta) = \frac{\psi(x, y)\sqrt{R}}{lh(x)} \dots \dots \dots (5)$$

where $h(x)$ has the dimensions of a velocity. The components of velocity are then

$$u = \frac{\partial \psi}{\partial y} = \frac{h}{g} f',$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{l}{\sqrt{R}} \left[f \frac{dh}{dx} + h \left(\frac{1}{l} \frac{\partial f}{\partial \xi} - \frac{\eta f'}{g} \frac{dg}{dx} \right) \right].$$

We then have

$$\frac{\partial u}{\partial x} = f' \frac{d}{dx} \left(\frac{h}{g} \right) + \frac{h}{gl} \frac{\partial f'}{\partial \xi} - \frac{\eta h f''}{g^2} \frac{dg}{dx},$$

$$\frac{\partial u}{\partial y} = \frac{h\sqrt{R}}{lg^2} f'',$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{R}{l^2 g^3} h f''',$$

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{hV}{lg^3} f''',$$

where the dash denotes differentiation of f with respect to η .

Substituting in (1), we get after a little simplification

$$\frac{lg^2}{V} \frac{d}{dx} \left(\frac{h}{g} \right) f'^2 + \frac{gh}{V} \left(f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right) - \frac{lg}{V} f f'' \frac{dh}{dx} = \frac{lg^3}{hV} U \frac{dU}{dx} + f'''. \dots \dots (6)$$

Putting

$$\left. \begin{aligned} \alpha &= \frac{lg}{V} \frac{dh}{dx}, \\ \beta &= \frac{lg^3}{hV} U \frac{dU}{dx}, \\ \gamma &= \frac{lg^2}{V} \frac{d}{dx} \left(\frac{h}{g} \right), \end{aligned} \right\} \dots \dots \dots (7)$$

we can write the equation (6) as

$$f''' + \alpha f f'' + \beta - \gamma f'^2 = \frac{gh}{V} \left(f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right). \quad \dots \quad (8)$$

In order that the differential equation (8) may reduce to an ordinary differential equation in f and η only, it is necessary that f and f' should be independent of ξ , and that the coefficients α , β , γ appearing in (8) should be independent of x . Hence, when 'similarity' solutions of the boundary layer equations exist, the dimensionless stream function f satisfies the ordinary differential equation

$$f''' + \alpha f f'' + \beta - \gamma f'^2 = 0. \quad \dots \quad (9)$$

This equation is more general than the corresponding equation of the dimensionless stream function obtained by Goldstein and Mangler. When α , β , γ are constants, the equations (7) determine $g(x)$, $h(x)$ and $U(x)$.

DETERMINATION OF SCALE FACTORS AND THE VELOCITY OF IRROTATIONAL FLOW

From (7), we get at once

$$(2\alpha - \gamma) = \frac{l}{V} \frac{d}{dx} (gh),$$

which gives, on integration

$$gh = \frac{V}{l} \left[(2\alpha - \gamma)x + A \right], \quad \dots \quad (10)$$

where A is a constant.

Eliminating g between (10) and the first equation of (7), we get

$$\frac{1}{h} \frac{dh}{dx} = \frac{\alpha}{(2\alpha - \gamma)x + A}. \quad \dots \quad (11)$$

Assuming that $(2\alpha - \gamma) \neq 0$, we get from (11) after integration

$$h = k \left[(2\alpha - \gamma)x + A \right]^{\frac{\alpha}{2\alpha - \gamma}}, \quad \dots \quad (12)$$

where k is a constant.

From (10) and (12), we then get

$$g = \frac{V}{lk} \left[(2\alpha - \gamma)x + A \right]^{\frac{(\alpha - \gamma)}{(2\alpha - \gamma)}}. \quad \dots \quad (13)$$

The second equation of (7) then gives

$$U \frac{dU}{dx} = \frac{\beta h V}{l g^3} = \frac{\beta k^4 l^2}{V^2} \left[(2\alpha - \gamma)x + A \right]^{\frac{(3\gamma - 2\alpha)}{(2\alpha - \gamma)}}.$$

If $\gamma \neq 0$, we get on integration

$$U^2 = \frac{\beta k^4 l^2}{\gamma V^2} \left[(2\alpha - \gamma)x + A \right]^{\frac{2\gamma}{(2\alpha - \gamma)}} + B, \quad \dots \quad (14)$$

where B is a constant.

The equations (12) and (13) determine the scale factors $h(x)$ and $g(x)$ for $u(x, y)$ and y and the equation (14) determines the flow outside the boundary layer.

PARTICULAR CASES

Case I

When $B = 0$, $\beta \neq 0$, $\gamma \neq 0$ and $2\alpha - \gamma \neq 0$, we have a 'similarity' solution for which

$$\begin{aligned} U &= \frac{k^2 l}{V} \sqrt{\frac{\bar{\beta}}{\gamma}} \left[(2\alpha - \gamma)x + A \right]^{\frac{\gamma}{2\alpha - \gamma}}, \\ g &= \frac{V}{lk} \left[(2\alpha - \gamma)x + A \right]^{\frac{(\alpha - \gamma)}{(2\alpha - \gamma)}}, \\ h &= k \left[(2\alpha - \gamma)x + A \right]^{\frac{\alpha}{(2\alpha - \gamma)}}, \end{aligned}$$

so that

$$h = \sqrt{\frac{\bar{\beta}}{\gamma}} (gU).$$

If there is a stagnation point on the boundary, we measure x along the boundary from this point. Then $U = 0$, when $x = 0$. This condition requires that

$$A = 0 \text{ and } \frac{\gamma}{2\alpha - \gamma} > 0.$$

Hence we have for this 'similarity' solution

$$\left. \begin{aligned} U &= \frac{k^2 l}{V} \sqrt{\frac{\bar{\beta}}{\gamma}} \left[(2\alpha - \gamma)x \right]^{\frac{\gamma}{(2\alpha - \gamma)}}, \\ g &= \frac{V}{lk} \left[(2\alpha - \gamma)x \right]^{\frac{(\alpha - \gamma)}{(2\alpha - \gamma)}}, \\ h &= \sqrt{\frac{\bar{\beta}}{\gamma}} (gU). \end{aligned} \right\} \quad \dots \quad (15)$$

In this case we can take, without loss of generality, $\beta = \gamma$. This particular case of the general 'similarity' solution obtained here has been discussed by Mangler.

There cannot be a stagnation point on the boundary, when

$$\frac{\gamma}{2\alpha - \gamma} < 0.$$

Taking $A = 0$ in this case, so that U , g and h are given by (15), we see that U becomes infinite when $x = 0$. Therefore the point on the boundary from which x is measured corresponds to a singularity of the irrotational flow outside the boundary layer.

If $\alpha \neq 0$, we can take $\alpha = 1$, without any loss of generality. Putting

$$m = \frac{\gamma}{2 - \gamma},$$

so that

$$, \frac{2m}{m+1}$$

we find that in both the above cases (when $x = 0$ is a stagnation point or a singularity of the flow outside the boundary layer) U is proportional to x^m , where m is any number, positive or negative, except 0 and -1 . Here also we can take $\beta = \gamma$, without any loss of generality. This case has been considered by Falkner and Skan (1930) and the corresponding equation for the dimensionless stream function f has been considered by Hartree (1937). If, on the other hand, $\alpha = 0$, U is proportional to $\frac{1}{x}$, which means that there is a source or a sink at $x = 0$. g is then proportional to x and h is a constant. This 'similarity' solution corresponds to the two-dimensional flow in the boundary layer along a wall of a converging or diverging channel with straight boundaries. An exact solution of the equation satisfied by the dimensionless stream function has been given by Pohlhausen (1921).

Case II

When $2\alpha - \gamma = 0$, $\alpha \neq 0$, the equation (11) becomes

$$\frac{1}{h} \frac{dh}{dx} = \frac{\alpha}{A},$$

which gives, on integration

$$h = C \exp\left(\frac{\alpha}{A} x\right),$$

where C is a constant. The equation (10) then gives

$$g = \frac{VA}{lC} \exp\left(-\frac{\alpha}{A} x\right). \quad \dots \quad (16)$$

Substituting in (7) and integrating, we get

$$U^2 = \frac{\beta}{2\alpha} \frac{l^2 C^4}{V^2 A^2} \exp\left(\frac{4\alpha}{A} x\right) + D. \quad \dots \quad (17)$$

This case has been considered by Goldstein (1939). Assuming $D = 0$ and making use of the conditions to be satisfied on the boundaries of the boundary layer, Goldstein has shown that there cannot be any 'similarity' solution of the boundary layer equations in this case, unless $\frac{\alpha}{A}$ is positive.

Case III

When $B \neq 0$, $2\alpha - \gamma \neq 0$, $\beta = 0$ we get from (12), (13) and (14)

$$U = \text{constant},$$

$$g = \frac{V}{lk} \left[(2\alpha - \gamma)x + A \right]^{\frac{(\alpha - \gamma)}{(2\alpha - \gamma)}},$$

$$h = k \left[(2\alpha - \gamma)x + A \right]^{\frac{\alpha}{(2\alpha - \gamma)}}.$$

If, in addition, $\gamma = 0$ and $A = 0$ we have

$$\left. \begin{aligned} U &= \text{constant}, \\ g &= \frac{V}{\sqrt{k}} (2\alpha x)^{\frac{1}{2}}, \\ h &= k(2\alpha x)^{\frac{1}{2}}. \end{aligned} \right\} \dots \dots \dots (18)$$

This case corresponds to the 'similarity' solution in the boundary layer along a flat plate placed in a uniform stream parallel to the plane of the plate and perpendicular to its edge. This case has been discussed by Blasius (1908).

Case IV

When $B = 0$, $\beta = 0$, $\gamma \neq 0$ and $2\alpha - \gamma \neq 0$, we have a 'similarity' solution for which

$$\begin{aligned} U &= 0, \\ g &= \frac{V}{\sqrt{k}} \left[(2\alpha - \gamma)x + A \right]^{\frac{(\alpha - \gamma)}{(2\alpha - \gamma)}}, \\ h &= k \left[(2\alpha - \gamma)x + A \right]^{\frac{\alpha}{(2\alpha - \gamma)}}. \end{aligned}$$

This solution cannot be obtained from the 'similarity' solution discussed by Goldstein and Mangler. The reason for this is that both Goldstein and Mangler have taken $U(x)$ as the scale factor for the longitudinal component $u(x, y)$ of the velocity in the boundary layer. The success achieved here is due to the fact that a scale factor $h(x)$ different from $U(x)$ has been assumed for $u(x, y)$ in this paper. It is found that this scale factor can be so determined as to yield a 'similarity' solution of the boundary layer equations.

Taking $A = 0$, $\alpha = 1$, $\gamma = -1$ we have

$$\left. \begin{aligned} U &= 0, \\ g &= \frac{V}{\sqrt{k}} (3x)^{\frac{1}{2}}, \\ h &= k(3x)^{\frac{1}{2}}. \end{aligned} \right\} \dots \dots \dots (19)$$

This case corresponds to the flow in a two-dimensional jet issuing from a thin slit in an incompressible fluid at rest. The corresponding equation satisfied by the dimensionless stream function has been considered by Schlichting (1933) and Bickley (1937).

In conclusion, I wish to express my thanks to Dr. S. Ghosh for helpful suggestions and guidance.

REFERENCES

- Blasius, H. (1908). Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Zeit. f. Math. u. Phys., Berlin*, 56, 1.
 Bickley, W. (1937). The plane jet. *Phil. Mag., London*, (7) 23, 727.
 Falkner, V. M. and Skan, S. W. (1930). Some approximate solutions of the boundary layer equations. *Rep. Memor. Aero. Res. Comm., London*, No. 1314.

- Goldstein, S. (1939). A note on the boundary layer equations. *Proc. Camb. Phil. Soc.*, **35**, 338.
- Hartree, D. R. (1937). On an equation occurring in Falkner and Skan's approximate treatment of the equations of the boundary layer. *Ibid.*, **33**, 223.
- Mangler, W. (1943). Die ähnlichen Lösungen der Prandtl'schen Grenzschichtgleichungen. *Zeit. f. angew. Math. u. Mech., Berlin*, **23**, 243.
- Pohlhausen, K. (1921). Zur näherungsweise Integration der Differentialgleichung der laminaren Reibungsschicht. *Zeit. f. angew. Math. u. Mech., Berlin*, **1**, 235.
- Schlichting, H. (1933). Laminare Strahlausbreitung. *Ibid.*, **13**, 260.

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AN X-RAY STUDY ON THE MICELLAR AND INTERMICELLAR ZONES OF SOME CELLULOSE FIBRES

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On account of manifold uses of fibres in everyday life, the importance of the investigations of fibres, in general, is felt universally. It is accepted that fibres are long chain molecules. The natural fibres are usually polysaccharides or proteins and cellulose is the main constituent of polysaccharides. Published information concerning the physical and chemical properties of cellulose fibres are voluminous and in that connection the work of Haworth, Astbury, Sission, Clerk, Hermans, Meyer, Mark, etc., may be specially mentioned. The X-ray study on cellulose reveals that the unit cell is monoclinic with $a = 8.35 \text{ \AA}$, $b = 10.3 \text{ \AA}$, $c = 7.9 \text{ \AA}$ and $\beta = 84^\circ$ and the space group is P_2 . Usually the internal structure is built up of small crystallites separated by amorphous or intercrystalline area and the entire crystal area is built up by repetition of such unit cells. In the present investigations our study is confined to four varieties of cellulose fibres, namely, (1) Sida, (2) Mesta, (3) Chukai and (4) Alissima.

Fibres have been studied very extensively by X-ray diffraction. In India investigations on jute fibres and some other vegetable fibres have been carried out in different laboratories and on that line valuable informations have been presented in the papers of Banerjee (K.) and Roy (1941), Sarkar, Rudra and Saha (1944), Banerjee (N. G.), Basak and Sen (1945), Bose and Ahmed (1946) and Banerjee (B. K.) (1946). All these papers deal with the general X-ray fibre diagrams of different varieties of jute. Average sizes of the crystalline region and changes in the fibre diagrams due to stretching, and chemical treatment are also discussed in these papers.

The present X-ray investigations on different varieties are primarily intended to compare in detail the order of orientations of cellulose crystallites with respect to the fibre axis for different varieties of jute as well as to ascertain the size and nature of the intermicellar spaces.

A special camera of 3 cms. radius was constructed for this purpose and X-rays from a Philips sealed tube (Cu-rays) was used and run at voltage of 45 kV with a tube current of 15 mA. The rays were usually incident on a bundle of fibres about 1 mm. in diameter normal to the length of the fibre. Usually exposures of $6\frac{1}{2}$ hrs. were given.

The glancing angles were calculated from the following relation, $\cos \theta \cos \mu = \cos 2\theta_0$, where θ and μ , the angular co-ordinates, were obtained in the usual way. The glancing angles are compared with the Bragg angles of various planes and comparison of these yield the identification of the spots.

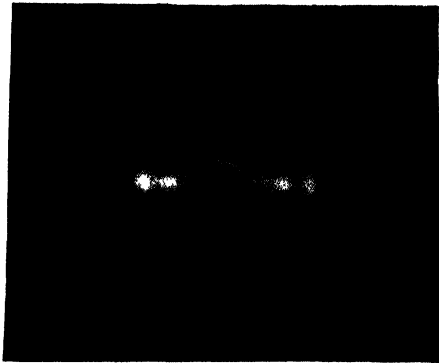
Details of the indices, intensity and character of each variety of the fibres are given in Tables I to IV. The X-ray pictures are shown in plate I (figs 1 to 4).

TABLE I
Sida Fibres

Indices	Intensity	Character
101	s.	Sharp and well resolved.
002	v.s.	Sharp and distinct.
004	w.	Extended along the radial direction.
310	m.	Extended in radial direction well resolved.
120 } 221 }	m.s.	Slightly broad.
130 } 032 }	m.w.	Diffuse.
131	m.w.	Extended into arc and diffuse.
042	w.	Extended into arc.

TABLE II
Chukai Fibres

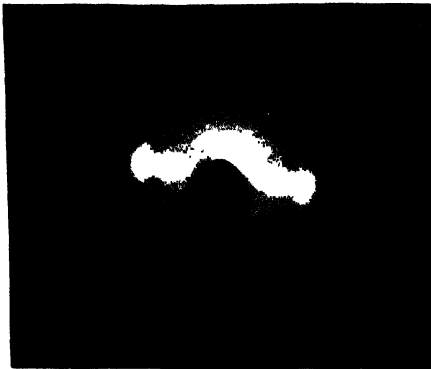
Indices	Intensity	Character
101	s.	Sharp and well resolved.
002	v.s.	Sharp and bent into arc.
004	w.	Extended along the radial direction.
310	m.w.	Extended in radial direction and well resolved.
221 } 120 }	m.s.	Slightly broad.
130	w.	Diffuse.
032	m.w.	Diffuse and extended into arc.
040	w.	Diffuse.
042	w.	Extended into arc.



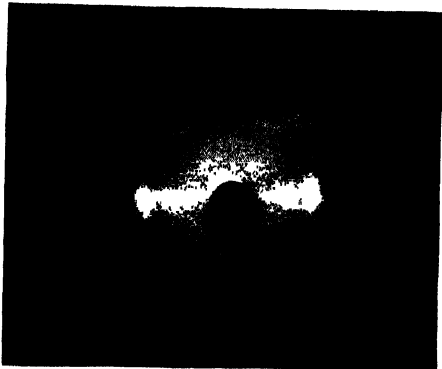
1



3



2



4

X-ray diffraction patterns of

- | | |
|------------|--------------|
| 1. CHIKAL. | 3. ALISSIMA. |
| 2. SIDA. | 4. MESTA. |

TABLE III

Mesta Fibres

Indices	Intensity	Character
(101)	s.	Sharp and well resolved.
002	v.s.	Sharp and bent along the arc of the circle on which the spot lie.
310	v.w.	Extended into arc.
120 } 221 }	m.	Well resolved.
032 } 130 }	m.w.	Bent along the circumference of a circle on which spots lie.
040	v.w.	Diffuse.

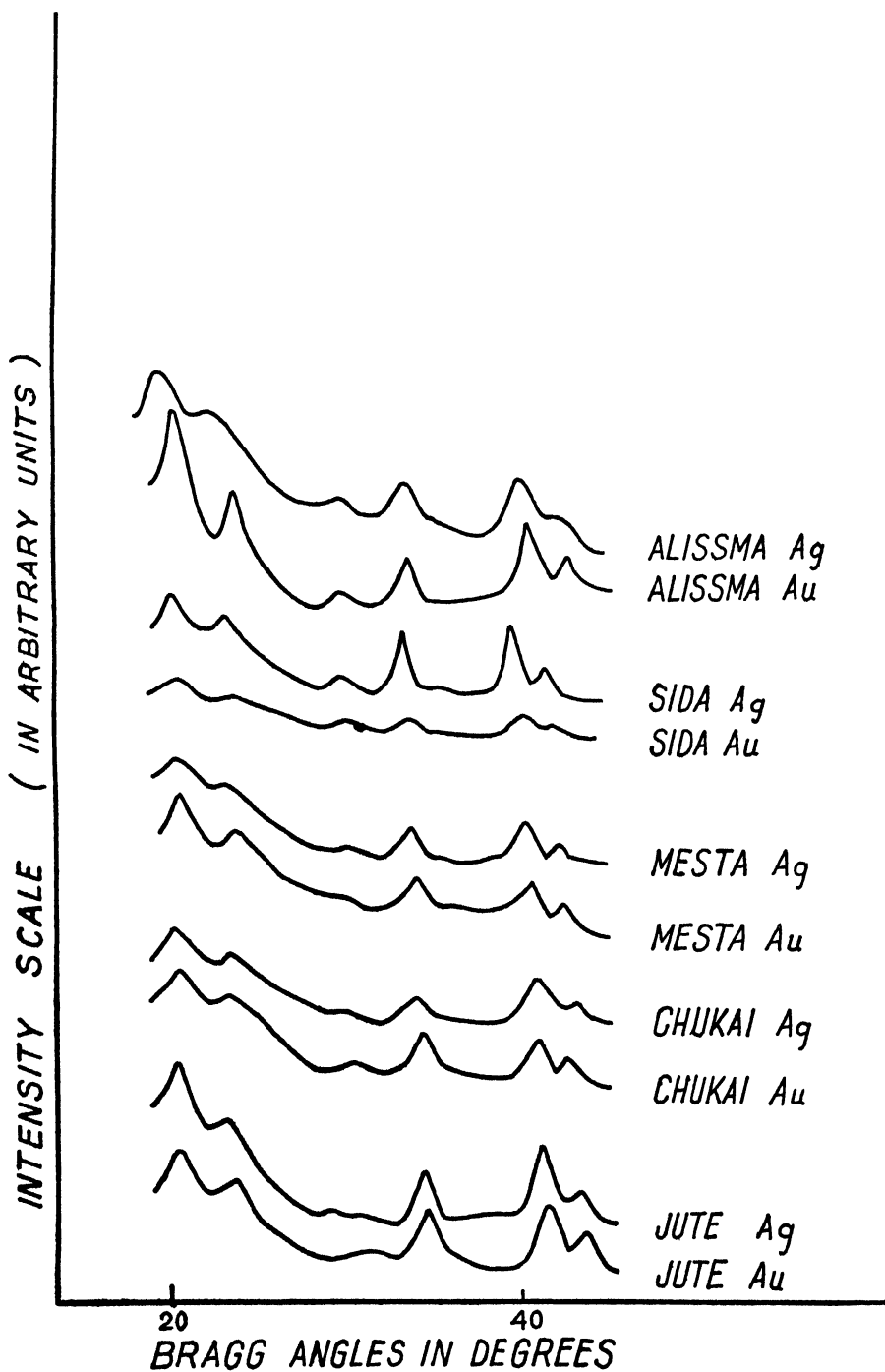
TABLE IV

Alissima Fibres

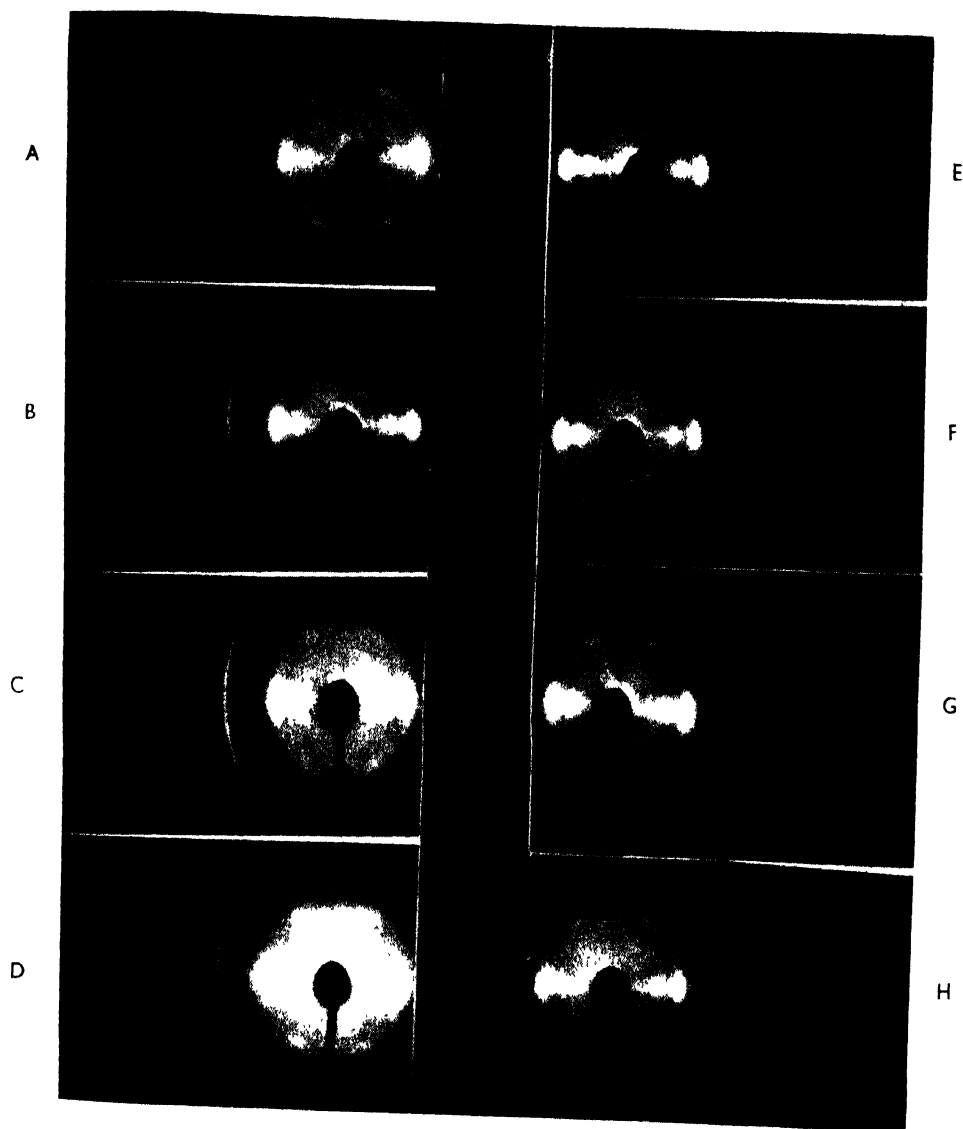
Indices	Intensity	Character
101	s.	Sharp and well resolved.
002	v.s.	Sharp and bent along the arc of the circle on which the spots lie.
004	v.v.w.	Extended into arc. Diffuse.
310	w.	Extended into arc.
221 } 021 }	m.s.	Well resolved.
130	m.w.	Diffuse.
032 } 131 }	m.w.	Extended into arc.
040	w.	Diffuse.
042	w.	Extended into arc.

It is observed that the orientation of 'b' axis of Chukai, Alissima and Mesta is much more parallel to the fibre axis than that of the Sida fibre. The spots on the equator are clearly broader than those in other layer lines and therefore the dimension of the crystallites in the perpendicular direction is much smaller than in the direction of fibre axis.

Apart from the preferential oriented crystalline domains of the cellulose crystallites, there are regions known as narrow intermicellar spaces. With a view to obtain some informations for this space, following Frey-Wyssling (1937) we introduced such foreign substances as would reveal the dimensions of the intermicellar



Note.—Read ALISSIMA for ALISSMA.



X ray diffraction patterns of

- | | |
|---------------------------------------|---|
| <i>A.</i> CHUKAI (with silver lines). | <i>E.</i> ALISSIMA (with silver lines). |
| <i>B.</i> CHUKAI (with gold lines). | <i>F.</i> ALISSIMA (with gold lines). |
| <i>C.</i> SIDA (with silver lines). | <i>G.</i> MESTA (with silver lines). |
| <i>D.</i> SIDA (with gold lines). | <i>H.</i> MESTA (with gold lines). |

spaces. For this purpose gold and silver, in the form of colloidal gold and colloidal silver are dispersed following the procedure adopted by Ambronn. To disperse colloidal silver in the lattice fresh fibres are usually soaked in freshly prepared 7.5% silver nitrate solution and then finally treated in hydrazine hydrate. The treated fibre is then dried and mounted in the usual way in a fibre camera and X-ray pictures are taken. In addition to the fibre diagrams the lines due to crystalline silver also come in the X-ray pictures. The same procedure is repeated to disperse gold particles in which case the fibre is first treated in 7.5% auric chloride solution and then in hydrazine hydrate. Finally X-ray pictures are taken. The gold and silver lines of each varieties of fibres are then photometered in Moll's type of microphotometer and the breadths at half maximum are measured from the photometric curves and finally the particle size is calculated from Scherrer's equations. The particle sizes of each variety are given in Table V and the photometric curves given in the Figure. The X-ray pictures are shown in Plate II.

TABLE V

Fibres of different varieties	Particle size calculated from gold lines	Particle size calculated from silver lines
(1) Chukai	77 Å	77 Å
(2) Alissima	97 Å	96 Å
(3) Mesta	94 Å	96 Å
(4) Sida	103 Å	103 Å
(5) Jute (low variety) ..	80 Å	84 Å

It is evident from the preceding table that the size of two different dispersoids in the case of same fibre comes to a similar value and the measurement of particle sizes of gold and silver lines incrustated within the fibre framework varies from 77 A.U. to 103 A.U. width for different fibres obtained from calculations. Similar observations were also made by Frey-Wyssling in the case of ramie fibres where he found the intermicellar spaces as 84 Å-85 Å width by the same method as above but with lower concentration of gold and silver solutions. In his case the lines due to gold and silver were weaker apparently due to the total intermicellar space in ramie being less than the fibres studied by us. The results of measurements given in the present communication can, we think, represent the nearest approach to the correct value of the dimensions of the intermicellar spaces in the fibre framework. Our works support the view expressed by Frey-Wyssling on fibres (Frey-Wyssling, Sub-microscopic morphology of Protoplasms and its derivatives).

Thanks are due to Prof. K. Banerjee, D.Sc., F.N.I., and Dr. A. Bose, D.Sc., for their keen interest in the work.

ABSTRACT

In this communication, X-ray pictures of cellulose fibres of different varieties, namely, Alissima, Sida, Mesta and Chukai are studied in detail. The orientation of cellulose crystallites with respect to the fibre axis has been discussed and further, in order to get an insight of the structure of intermicellar zones, in each of these varieties gold and silver are incrustated in the form of colloidal gold and colloidal silver wherein it is observed that the particle size of the colloidal dispersoid values from 77 Å to 103 Å (depending on the type of fibres) which gives an insight into the intermicellar regions of the above-mentioned fibres.

REFERENCES

- Banerjee, B. K. (1947). X-ray Study of Vegetable Fibres. *Ind. Jour. of Phys.*, **21**, 259.
- Banerjee, N. G., Basak, B. S. and Sen, R. K. (1945). X-ray Diffraction Study of Dyed mercerised and intensively delignified Jute Fibres. *Ibid.*, **19**, 217.
- Banerjee, K. and Roy, A. (1941). The Structure of Jute Fibre by X-ray Diffraction method. *Proc. Nat. Inst. Sci.*, **7**, 376.
- Bose, C. R. and Ahmed, A. (1946). A study of some vegetable fibres by X-ray diffraction method. *Ind. Jour. of Phys.*, **20**, 105.
- Frey-Wyssling, A. (1937). Röntgenometric measurement of submicroscopic spaces in the framework of substances. *Protoplasma*, **27**, 372.
- (1937). Ultramicroscopic investigation of submicroscopic spaces in framework of substances. *Ibid.*, **27**, 563.
- (1938). Submicroscopic morphology of Protoplasm and its derivatives. Elsevier Publishing Company, Inc., New York. (*English translation.*)
- and Speich, H. (1942). The permeability of cellulose fibres. *Helv. Chim. Acta*, **25**, 1474.
- Sircar, S. C., Saha, N. N. and Rudra, R. M. (1945). X-ray Analysis of Jute Fibre of different qualities under various conditions. *Proc. Nat. Inst. Sci.*, **10**, 325.

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STATISTICAL THERMODYNAMICS OF MIXTURE OF BOSE-EINSTEIN AND FERMI-DIRAC 'LIQUIDS'

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1. INTRODUCTION

Recent experiments of Abraham, Osborne and Weinstock (1949a) and of Daunt and Heer (1950) indicate that He^3 does not show superfluid properties even down to 0.25°K . unlike He^4 which becomes superfluid at about 2.19°K . This seems to confirm London's (1938) hypothesis that the peculiar phase transition in Helium (i.e. He^4) can be interpreted as the 'condensation' phenomenon exhibited by an ideal Bose gas so that no transition would be expected in the case of He^3 , which obeys Fermi statistics. Hence, it is now generally held that the difference in the Behaviour of He^3 and He^4 arises from the different statistics that they obey.

Heer and Daunt (1951) have proposed 'smoothed' potential models for considering mixtures of Bose and Fermi liquids. They have assumed that the 'Fermi liquid' can be regarded as an ideal Fermi gas in a constant potential well $-\chi_F^0$, and the 'Bose liquid' as an ideal Bose gas with a constant potential well $-\chi_B^0$. In the case of the mixture they have assumed that the liquids are incompressible and form statistically independent systems. Thus they have written for the total volume V , and the total potential χ , of the mixture the expressions

$$V = N_F V_F^0 + N_B V_B^0.$$

$$\chi = N_F \chi_F^0 + N_B \chi_B^0,$$

respectively, where V_F^0 and V_B^0 are the respective atomic volumes. Further, they have assumed the vapours to be perfect gases. Heer and Daunt (1951) and more recently Daunt, Tseng and Heer (1952) have applied this model to calculate the partial vapour pressures and the distribution coefficients of their 'model solutions'. They find that the theoretical predictions are in good agreement with the observed properties of He^3 - He^4 mixtures.

The present paper is concerned with a detailed discussion of the thermodynamic properties of mixtures of Bose-Einstein and Fermi-Dirac liquids when the distribution laws are of the more general type:

$$N_B(\epsilon) d\epsilon = \frac{N_B V_B^0 C_B \epsilon^{s_B-1} d\epsilon}{\frac{1}{A_B^0} \exp \left[\frac{\epsilon}{kT} - \frac{\chi_B^0}{kT} \right] - 1}, \quad \dots \quad (1)$$

$$N_F(\epsilon) d\epsilon = \frac{N_F V_F^0 C_F \epsilon^{s_F-1} d\epsilon}{\frac{1}{A_F^0} \exp \left[\frac{\epsilon}{kT} - \frac{\chi_F^0}{kT} \right] + 1}, \quad \dots \quad (2)$$

for a Bose and a Fermi 'liquid' respectively. In the sequel we shall refer to them as Liq. *B* and Liq. *F*. Clearly, our results would reduce to those of Daunt and Heer when

$$s_B = s_F = \frac{3}{2},$$

and

$$C = \frac{2\pi g(2m)^{\frac{3}{2}}}{h^3},$$

where g is the weight factor, being 1 for He⁴ and 2 for He³. As usual, the conditions for the onset of extreme degeneracy are

$$A_B^0 \exp\left(\frac{\chi_B^0}{kT}\right) = A_B^{0*} = 1,$$

$$A_F^0 \exp\left(\frac{\chi_F^0}{kT}\right) = A_F^{0*} \gg 1.$$

Kothari and Singh (1941, 1942) have studied the properties of pure Bose and Fermi gases obeying general distribution laws of the type given above. Wherever possible, we shall use their notation.

In the sequel we shall consider the following cases of mixtures of Liq. *B* and Liq. *F*:

- (i) Liq. *F* Boltzmannian ($A_F^* \rightarrow 0$),
- (ii) Liq. *F* Non-degenerate ($A_F^* \ll 1$), and
- (iii) Liq. *F* Degenerate ($A_F^* \gg 1$).

In each of these cases we distinguish between two ranges of temperature, viz., (a) below the λ -temperature and (b) above the λ -temperature, of the mixture. It is of interest to do so because below T_λ Liq. *B* is degenerate ($A_B^* = 1$) and above T_λ Liq. *B* is non-degenerate ($A_B^* \ll 1$).

2. THE PURE LIQUIDS

The generalized distribution laws lead to the following expressions for the Helmholtz free energy:—

$$F_B^0 = -N_B \chi_B^0 - N_B kT \left[1 - \log A_{0B}^{0*} + b_2 A_{0B}^{0*} + \frac{b_3}{2} (A_{0B}^{0*})^2 + \dots \right], \dots \quad (3)$$

$$F_F^0 = -N_F \chi_F^0 - N_F kT \left[1 - \log A_{0F}^{0*} - b_2 A_{0F}^{0*} - \frac{b_3}{2} (A_{0F}^{0*})^2 - \dots \right], \dots \quad (4)$$

of non-degenerate and

$$F_B^0 = -N_B \chi_B^0 - \frac{N_B kT}{A_{0B}^{0*}} \zeta(s_B + 1), \quad \dots \quad \dots \quad \dots \quad (5)$$

$$F_F^0 = -N_F \chi_F^0 + \frac{s_F}{s_F + 1} N_F \xi_0^0 \left[1 - \frac{\pi^2}{6} (s_F + 1) \left(\frac{kT}{\xi_0^0} \right)^2 \right], \quad \dots \quad (6)$$

of degenerate Bose and Fermi liquids, respectively. Here ξ_0^0 is the Fermi degeneracy energy for pure Liq. F^* and

$$A_0^{0*} = \sum_n \left\{ \frac{(A_0^{0*})^n}{n^s} \right\} = \frac{N}{C\Gamma(s) (kT)^s N V^0},$$

$$b_2 = \frac{1}{2^{s+1}}, b_3 = \frac{2}{3^{s+1}} - \frac{1}{4^s}.$$

Following London (1938) we have for the degeneracy temperature of Liq. B the expression

$$T_0 = \frac{1}{k} \left[\frac{1}{C_B V_B^0 \Gamma(s_B) \zeta(s_B)} \right]^{\frac{1}{s_B}}. \quad \dots \quad (7)$$

For completely non-interacting systems (Heer and Daunt's case) $s_B = \frac{3}{2}$ and

$$C_B = \frac{2\pi(2m)^{3/2}}{\hbar^3}$$

and eq. (7) reduces to the well-known expression due to London. Eq. (7) shows that the introduction of the potential well does not, in the first approximation†, modify the degeneracy temperature. It may be noted that in this paper the degeneracy temperature will always refer to the condensation temperature of the Bose assembly, unless otherwise specified.

3. THE λ -TEMPERATURE

We shall now consider the case of a mixture of Liq. F and Liq. B . Since they are assumed to form statistically independent assemblies the modification of the degeneracy temperature is purely due to the change in the volume of the Bose assembly. We shall therefore get the degeneracy temperature, T_λ , of the Bose assembly (Liq. B) in the mixture by replacing the volume $N_B V_B^0$ of pure Liq. B in eq. (7) by the volume $(N_F V_F^0 + N_B V_B^0)$ of the mixture, thus giving

$$T_\lambda = \frac{1}{k} \left[\frac{N_B}{N_F V_F^0 + N_B V_B^0} \cdot \frac{1}{C_B \Gamma(s_B) \zeta(s_B)} \right]^{\frac{1}{s_B}}, \quad \dots \quad (8)$$

This, on combination with eq. (7), leads to

$$\frac{T_\lambda}{T_0} = \left[\frac{N_B V_B^0}{N_F V_F^0 + N_B V_B^0} \right]^{\frac{1}{s_B}} = \left[\frac{1 - X_F}{1 + X_F \left(\frac{V_F^0}{V_B^0} - 1 \right)} \right]^{\frac{1}{s_B}}, \quad \dots$$

* Throughout this paper the superscript zero (0) is used to indicate quantities referring to the pure liquids.

† Our model must be regarded as a first approximation in so far as the variation in χ^0 , with volume, has been neglected. In practice it is well-known that the λ -temperature of He⁴ depends upon the external pressure. Since contributions to the pressure will come from (i) a kinetic term and (ii) a statical term due to the variation of χ^0 , we expect that in the second approximation, i.e., when $\frac{\partial \chi^0}{\partial V^0}$ is taken into account, T_0 would be modified by the introduction of χ^0 .

whence

$$\frac{\partial T_\lambda}{\partial X_F} = -\frac{T_0}{s_B} \cdot \frac{(1-X_F)^{\frac{1}{s_B}-1}}{\left[1+X_F\left(\frac{V_F^0}{V_B^0}-1\right)\right]^{\frac{1}{s_B}+1}} \cdot \frac{V_F^0}{V_B^0}, \quad \dots \quad (10)$$

where $X_F = \frac{N_F}{N_F+N_B}$ and $X_B = \frac{N_B}{N_F+N_B}$ are, respectively, the concentrations of Liq. F and Liq. B in the mixture.

4. PROPERTIES OF MIXTURES IN CASE (i)

As pointed out in Sec. 1 we shall get two sets of expressions according as the temperature is above or below the λ -point of the mixture.

(a) Below the λ -point.

For a mixture of volume $(N_F V_F^0 + N_B V_B^0)$ eqs. (4) and (5) lead to the following expression for the free energy:—

$$F = -N_F \chi_F^0 - N_B \chi_B^0 + N_F kT \log A_{0F}^* - C_B \Gamma(s_B) (kT)^{s_B+1} \zeta(s_B+1) \times \\ \times (N_F V_F^0 + N_B V_B^0), \quad \dots \quad (11)$$

whereas, the free energy of mixing, $\Delta F = F - F_F^0 - F_B^0$ will be given by

$$\Delta F = N_F kT \log \frac{A_{0F}^*}{A_{0F}^0} - C_B \Gamma(s_B) (kT)^{s_B+1} \zeta(s_B+1) N_F V_F^0, \quad \dots \quad (12)$$

where all terms of the order of A_{0F}^* have been neglected corresponding to the Boltzmannian case.

It is usual to assume, for a liquid, that $G \approx F$, G being the Gibbs free energy. In our model the 'liquids' are actually gases with certain potential wells. It is, therefore, of interest to note that if this assumption is not introduced the pressure values so obtained are higher by about 20 per cent for low temperatures and by about 10 per cent for high temperatures (Bhagat and Katti 1954a). However, following Daunt and Heer we assume that

$$G = F + P(N_F V_F^0 + N_B V_B^0), \\ \approx F,$$

for sufficiently small pressures. Then, using the usual thermodynamic relations we obtain for the thermodynamic potentials the expressions

$$\mu_F^0 = -\chi_F^0 - kT - kT \log \left\{ C_F \Gamma(s_F) V_F^0 (kT)^{s_F} \right\}, \quad \dots \quad (13)$$

$$\mu_B^0 = -\chi_B^0 - C_B V_B^0 (kT)^{s_B+1} \Gamma(s_B) \zeta(s_B+1), \quad \dots \quad (14)$$

for the pure liquids and

$$\mu_F = -\chi_F^0 - X'_F kT + kT \log \frac{X'_F}{V_F^0} - kT \log C_F \Gamma(s_F) (kT)^{s_F} - C_B V_F^0 (kT)^{s_B+1} \Gamma(s_B) \zeta(s_B+1). \quad (15)$$

$$\mu_B = -\chi_B^0 - C_B V_B^0 (kT)^{s_B+1} \Gamma(s_B) \zeta(s_B+1) \quad (16)$$

for the liquids in the mixture. Since the vapours are assumed to be perfect gases we use the relations:

$$\mu_F - \mu_F^0 = kT \log \frac{p_F}{p_F^0},$$

$$\mu_B - \mu_B^0 = kT \log \frac{p_B}{p_B^0},$$

and obtain for the partial vapour pressures

$$\frac{p_F}{p_F^0} = X'_F \exp \left[-\frac{V_F^0 \zeta(s_B+1)}{V_B^0 \zeta(s_B)} X'_B \left(\frac{T}{T_0} \right)^{s_B} + X'_B \right], \quad (17)$$

$$\frac{p_B}{p_B^0} = \exp \left[-\frac{V_B^0}{V_F^0} X'_F \right], \quad (18)$$

where

$$X'_F = \frac{N_F V_F^0}{N_F V_F^0 + N_B V_B^0} \quad \text{and} \quad X'_B = \frac{N_B V_B^0}{N_F V_F^0 + N_B V_B^0}.$$

Further, the total energy U , entropy S , and specific heat C are obtained from the expression for the free energy by using the well-known thermodynamic relations

$$U = F - T \left(\frac{\partial F}{\partial T} \right), \quad S = - \left(\frac{\partial F}{\partial T} \right), \quad \text{and} \quad C = \left(\frac{\partial U}{\partial T} \right),$$

whence,

$$U = X_F s_F T - \frac{1}{R} [N_F \chi_F^0 + N_B \chi_B^0] + C_B \Gamma(s_B+1) (kT)^{s_B+1} \frac{(X_F V_F^0 + X_B V_B^0)}{k}, \quad (19)$$

$$S = X_F (s_F+1) - X_F \log A_{0F}^* + \frac{C_B}{s_B} \Gamma(s_B+2) \zeta(s_B+1) (kT)^{s_B} (X_F V_F^0 + X_B V_B^0), \quad (20)$$

$$C = X_F s_F - C_B \Gamma(s_B+2) (kT)^{s_B} \zeta(s_B+1) (X_F V_F^0 + X_B V_B^0), \quad (21)$$

in units of $R (= k(N_F + N_B))$.

(b) *Above the λ -point.*

Combining eqs. (3) and (7) we obtain the free energy of non-degenerate Liq. B in the form

$$F_B^0 = -N_B \chi_B^0 - N_B kT \left[1 - \log \zeta(s_B) + \log \left(\frac{T}{T_0} \right)^{s_B} + b_2 \zeta(s_B) \left(\frac{T_0}{T} \right)^{s_B} + \frac{b_3}{2} [\zeta(s_B)]^2 \left(\frac{T_0}{T} \right)^{2s_B} + \dots \right], \quad (22)$$

whence the thermodynamic potential of pure Liq. B becomes

$$\mu_B^0 = -\chi_B^0 - kT \left[1 - \log \zeta(s_B) + \log \left(\frac{T}{T_0} \right)^{s_B} + b_2 \zeta(s_B) \left(\frac{T_0}{T} \right)^{s_B} + \frac{b_3}{2} [\zeta(s_B)]^2 \left(\frac{T_0}{T} \right)^{2s_B} + \dots \right], \quad \dots \quad (23)$$

On combining the expressions for the free energies of Liq. B and Liq. F we obtain for the free energy of the mixture and the free energy of mixing (for $T > T_0$) the expressions:

$$F = -N_F \chi_F^0 - N_B \chi_B^0 - N_F kT \log A_{0F}^* - N_B kT \left[1 - \log \zeta(s_B) + \log \left(\frac{T}{T_\lambda} \right)^{s_B} + b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} + \frac{b_3}{2} [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right], \quad \dots \quad (24)$$

$$\Delta F = -kT \left[-N_F \log \frac{A_{0F}^*}{A_{0F}^*} + N_B \log \left(\frac{T_0}{T_\lambda} \right)^{s_B} + b_2 N_B \frac{\zeta(s_B)}{T^{s_B}} (T^{s_B} - T_0^{s_B}) + \frac{b_3}{2} \left[\frac{\zeta(s_B)}{T^{s_B}} \right]^2 (T^{2s_B} - T_0^{2s_B}) + \dots \right]. \quad \dots \quad (25)$$

Thence the potentials in the mixed state take the form

$$\mu_F = -\chi_F^0 - X'_F kT + kT \log A_{0F}^* - X'_B \frac{V_F^0}{V_B^0} kT \left[1 - b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} - b_3 [\zeta(s_B)]^2 \times \left(\frac{T_\lambda}{T} \right)^{2s_B} - \dots \right], \quad \dots \quad (26)$$

$$\mu_B = -\chi_B^0 - X'_F \frac{V_F^0}{V_B^0} kT - kT \left[X'_B - \log \zeta(s_B) + \log \left(\frac{T}{T_\lambda} \right)^{s_B} + (1 + X'_F) b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} + (1 + 2X'_F) \frac{b_3}{2} [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right], \quad \dots \quad (27)$$

which when combined with the expressions for the potentials of the pure liquids, eqs. (13) and (23), give for the partial vapour pressures

$$\frac{p_B}{p_B^0} = X'_B \exp \left[-X'_B - X'_F \frac{V_B^0}{V_F^0} + \log \left(\frac{T_0}{T} \right)^{s_B} + \frac{\zeta(s_B + 1)}{\zeta(s_B)} \left(\frac{T}{T_0} \right)^{s_B} + \log \zeta(s_B) - (1 + X'_F) b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} - (1 + 2X'_F) \frac{b_3}{2} [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} - \dots \right], \quad \dots \quad (28)$$

for $T_\lambda < T < T_0$,

$$\frac{p_B}{p_B^0} = X'_B \exp \left[\left(1 - \frac{V_B^0}{V_F^0} \right) X'_F - b_2 \zeta(s_B) \left(1 + X'_F - \frac{1}{X'_B} \right) \left(\frac{T_\lambda}{T} \right)^{s_B} - \frac{b_3}{2} [\zeta(s_B)]^2 \left(1 + 2X'_F - \frac{1}{(X'_B)^2} \right) \left(\frac{T_\lambda}{T} \right)^{2s_B} - \dots \right], \quad \dots \quad (28')$$

for $T > T_0$, and

$$\frac{p_F}{p_F^0} = X'_F \exp \left[\left(1 - \frac{V_F^0}{V_B^0} \right) X'_B + X'_B \frac{V_F^0}{V_B^0} \left\{ b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} + b_3 [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right\} \right]. \quad (29)$$

It is important to note here that the distinction between the two temperature ranges $T_\lambda < T < T_0$ and $T > T_0$ above, as well as in the following sections, has to be introduced only because the partial pressures have been expressed in terms of the saturated pressures. Otherwise, the pressure of Liq. B in the mixture is given by the same expression both above and below the degeneracy temperature, T_0 , of pure Liq. B .

The expression for the free energy of the mixture when substituted in the thermodynamic relations quoted above gives for the energy U , entropy S , and specific heat C , in units of R , the expressions

$$U = -\frac{1}{R} [N_F \chi_F^0 + N_B \chi_B^0] + (s_F X_F + s_B X_B) T - s_B X_B T \left[b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} + b_3 [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right], \quad (30)$$

$$S = 1 - X_F \log A_{0F}^* - X_B \log \zeta(s_B) + X_B \log \left(\frac{T}{T_\lambda} \right)^{s_B} + (s_F X_F + s_B X_B) - \left[b_2 X_B (s_B - 1) \left(\frac{T_\lambda}{T} \right)^{s_B} \zeta(s_B) + \frac{b_3}{2} X_B (2s_B - 1) \left(\frac{T_\lambda}{T} \right)^{2s_B} [\zeta(s_B)]^2 + \dots \right], \quad (31)$$

$$C = (s_F X_F + s_B X_B) + X_B \left[b_2 s_B (s_B - 1) \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} + b_3 s_B (2s_B - 1) [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right]. \quad (32)$$

From eqs. (32) and (21) we obtain for the jump in the specific heat at the λ -temperature of the mixture the expression

$$\Delta C = X_B s_B \left[1 + b_2 \zeta(s_B) (s_B - 1) + b_3 [\zeta(s_B)]^2 (2s_B - 1) + \dots \right] - C_B \Gamma(s_B + 2) (kT_\lambda)^{s_B} (X_F V_F^0 + X_B V_B^0) \zeta(s_B + 1). \quad (33)$$

5. LIQ. F NON-DEGENERATE

In our considerations so far Liq. F has been regarded as essentially Boltzmannian in character. We shall now study the case when Liq. F is considered to be a non-degenerate 'Fermi liquid'. For this purpose we shall have to include the higher order terms in the series expression for the free energy of Liq. F , i.e., eq. (4). The results of the preceding section then take the generalized form given below:

(a) *Below the λ -point*: the free energy expression becomes

$$F = -N_F \chi_F^0 - N_B \chi_B^0 - C_B \Gamma(s_B) \zeta(s_B + 1) (N_F V_F^0 + N_B V_B^0) (kT)^{s_B + 1} - N_F kT \left[1 - \log A_{0F}^* - b_2 A_{0F}^* - \frac{b_3}{2} (A_{0F}^*)^2 - \dots \right], \quad (34)$$

which gives, on combination with eqs. (8) and (9), for the free energy of mixing, ΔF ,

$$\Delta F = -N_F kT \left[\log \frac{A_{0F}^*}{A_{0F}^{0*}} + b_2 (A_{0F}^{0*} - A_{0F}^*) + \frac{b_3}{2} \{ (A_{0F}^{0*})^2 - (A_{0F}^*)^2 \} + \dots \right] - C_B \Gamma(s_B) \zeta(s_B + 1) N_F V_F^0 (kT)^{s_B + 1}. \quad (35)$$

In this case the thermodynamic potentials of Liq. F take the form

$$\mu_F^0 = -\chi_F^0 - kT \left[\log A_{0F}^{0*} + A_{0F}^{0*} + \frac{b_3}{2} (A_{0F}^{0*})^2 + \dots \right], \quad (36)$$

when pure, and

$$\mu_F = -\chi_F^0 - C_B V_F^0 (kT)^{s_B + 1} \zeta(s_B + 1) + kT \left[\log A_{0F}^* + 2b_2 A_{0F}^* + \frac{3b_3}{2} (A_{0F}^*)^2 + \dots \right] - X'_F kT [1 + b_2 A_{0F}^* + b_3 (A_{0F}^*)^2 + \dots], \quad (37)$$

when in solution. Also, the potential of Liq. B , when in solution, takes the form

$$\mu_B = -\chi_B^0 - C_B (kT)^{s_B + 1} V_B^0 \zeta(s_B + 1) - kT X'_F \left[\frac{V_B^0}{V_F^0} + b_2 A_{0F}^* + \frac{b_3}{2} (A_{0F}^*)^2 + \dots \right]. \quad (38)$$

These then give for the partial vapour pressures the expressions

$$\frac{p_B}{p_B^0} = \exp \left[-\frac{V_B^0}{V_F^0} X'_F - X'_F (b_2 A_{0F}^* + b_3 (A_{0F}^*)^2 + \dots) \right], \quad (39)$$

$$\frac{p_F}{p_F^0} = X'_F \exp \left[X'_B \frac{V_F^0}{V_B^0} \frac{\zeta(s_B + 1)}{\zeta(s_B)} \left(\frac{T}{T'_B} \right)^{s_B} + X'_B \left\{ 1 + b_2 A_{0F}^* + b_3 (A_{0F}^*)^2 + \dots \right\} + \left\{ b_2 (A_{0F}^* - A_{0F}^{0*}) + \frac{b_3}{2} ([A_{0F}^*]^2 - [A_{0F}^{0*}]^2) + \dots \right\} \right]. \quad (40)$$

The expressions for the energy, entropy and specific heat (all in units of R) will, from eq. (34), come out to be

$$U = -\frac{1}{R} (N_F \chi_F^0 + N_B \chi_B^0) + C_B \Gamma(s_B + 1) \zeta(s_B + 1) (kT)^{s_B + 1} \frac{X_F V_F^0 + X_B V_B^0}{k} + X_F s_F [T + b_2 A_{0F}^* + b_3 (A_{0F}^*)^2 + \dots], \quad (41)$$

$$S = \frac{C_B}{s_B} \Gamma(s_B + 2) \zeta(s_B + 1) (kT)^{s_B} (X_F V_F^0 + X_B V_B^0) + X_F \left[1 + s_F - \log A_{0F}^* + b_2 (s_F - 1) A_{0F}^* + \frac{b_3}{2} (2s_F - 1) (A_{0F}^*)^2 + \dots \right], \quad (42)$$

$$C = C_B \Gamma(s_B + 2) \zeta(s_B + 1) (kT)^{s_B} (X_F V_F^0 + X_B V_B^0) + \frac{X_F s_F^2}{T} \left[\frac{T}{s_F} - b_2 A_{0F}^* - 2b_3 (A_{0F}^*)^2 - \dots \right]. \quad (43)$$

(b) *Above the λ -temperature*: each expression will involve two series since both Liq. F and Liq. B (in solution) are non-degenerate in character. Thus the free energy of the mixture and the free energy of mixing ($T > T_0$)* will be given by

$$F = -N_F \chi_F^0 - N_B \chi_B^0 - N_F kT \left[1 - \log A_{0F}^* - b_2 A_{0F}^* - \frac{b_3}{2} (A_{0F}^*)^2 - \dots \right] - \\ - N_B kT \left[1 - \log \zeta(s_B) + \log \left(\frac{T}{T_\lambda} \right)^{s_B} + b_2 \zeta(s_B) \left(\frac{T}{T_\lambda} \right)^{s_B} + \right. \\ \left. + \frac{b_3}{2} [\zeta(s_B)]^2 \left(\frac{T}{T_\lambda} \right)^{2s_B} + \dots \right], \quad \dots \quad (44)$$

$$\Delta F = -N_B kT \left[\log \left(\frac{T_0}{T_\lambda} \right)^{s_B} + \frac{b_2 \zeta(s_B)}{T^{s_B}} (T_\lambda^{s_B} - T_0^{s_B}) + \right. \\ \left. + \frac{b_3}{2} \left\{ \frac{\zeta(s_B)}{T^{s_B}} \right\}^2 (T_\lambda^{2s_B} - T_0^{2s_B}) + \dots \right] - \\ - N_F kT \left[\log \frac{A_{0F}^{0*}}{A_{0F}^*} - b_2 (A_{0F}^* - A_{0F}^{0*}) - \frac{b_3}{2} \{ (A_{0F}^*)^2 - (A_{0F}^{0*})^2 \} - \dots \right]. \quad \dots \quad (45)$$

Eq. (44) gives for the thermodynamic potentials the expressions

$$\mu_F = -\chi_F^0 + kT \left[\log A_{0F}^* - X'_F + b_2 (2 - X'_F) A_{0F}^* + b_3 \left(\frac{3}{2} - X'_F \right) (A_{0F}^*)^2 + \dots \right] - \\ - X_B \frac{V_F^0}{V_B^0} kT \left[1 - b_2 \zeta(s_B) \left(\frac{T}{T_\lambda} \right)^{s_B} - b_3 [\zeta(s_B)]^2 \left(\frac{T}{T_\lambda} \right)^{2s_B} - \dots \right], \quad \dots \quad (46)$$

$$\mu_B = -\chi_B^0 - X'_F \left[\frac{V_B^0}{V_F^0} kT + b_2 A_{0F}^* + b_3 (A_{0F}^*)^2 + \dots \right] - \\ - kT \left[X'_B + \log \left(\frac{T}{T_\lambda} \right)^{s_B} + (1 + X'_F) b_2 \zeta(s_B) \left(\frac{T}{T_\lambda} \right)^{s_B} + \right. \\ \left. + (1 + 2X'_F) \frac{b_3}{2} [\zeta(s_B)]^2 \left(\frac{T}{T_\lambda} \right)^{2s_B} + \dots \right], \quad \dots \quad (47)$$

which, applying the usual method, give for the partial pressures

$$\frac{p_B}{p_B^0} = X'_B \exp \left[-X'_B - X'_F \frac{V_B^0}{V_F^0} - X'_F \{ b_2 A_{0F}^* + b_3 (A_{0F}^*)^2 + \dots \} + \frac{\zeta(s_B + 1)}{\zeta(s_B)} \left(\frac{T}{T_\lambda} \right)^{s_B} - \right. \\ \left. - \left\{ \log \left(\frac{T}{T_0} \right)^{s_B} + b_2 \zeta(s_B) (1 + X'_F) \left(\frac{T}{T_\lambda} \right)^{s_B} + \right. \right. \\ \left. \left. + \frac{b_3}{2} [\zeta(s_B)]^2 (1 + 2X'_F) \left(\frac{T}{T_\lambda} \right)^{2s_B} + \dots \right\} \right], \quad \dots \quad (48)$$

* It may be noted that here, as elsewhere, the range $T > T_0$ has to be specified because in the range $T_\lambda < T < T_0$ pure Liq. B is degenerate and therefore the expression for ΔF will be different according as $T > T_0$ or $T_\lambda < T < T_0$.

for $T_\lambda < T < T_0$,

$$\frac{p_B}{p_B^0} = X'_B \exp \left[X'_F \left\{ 1 - \frac{V_B^0}{V_F^0} - b_2 A_{0F}^* - b_3 (A_{0F}^*)^2 - \dots \right\} - \right. \\ \left. - \left\{ b_2 \zeta(s_B) \left(1 + X'_F - \frac{1}{X_B} \right) \left(\frac{T_\lambda}{T} \right)^{s_B} + \frac{b_3}{2} [\zeta(s_B)]^2 \left(1 + 2X'_F - \frac{1}{(X'_B)^2} \right) \times \right. \right. \\ \left. \left. \times \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right\} \right], \quad \dots \quad (48')$$

for $T > T_0$ and,

$$\frac{p_F}{p_F^0} = X'_F \exp \left[X'_B \left\{ 1 - \frac{V_F^0}{V_B^0} + b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} + \frac{b_3}{2} [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right\} + \right. \\ \left. + X'_B \left\{ b_2 A_{0F}^* + b_3 (A_{0F}^*)^2 + \dots \right\} + b_2 \left\{ (A_{0F}^* - A_{0F}^{0*}) + \right. \right. \\ \left. \left. + \frac{b_3}{2b_2} [(A_{0F}^*)^2 - (A_{0F}^{0*})^2] + \dots \right\} \right]. \quad \dots \quad (49)$$

Using eq. (44) we obtain for the other thermodynamic properties of the mixture the relations

$$U = -\frac{1}{R} (N_F \chi_F^0 + N_B \chi_B^0) + X_F s_F \left[T + b_2 A_{0F}^* + b_3 (A_{0F}^*)^2 + \dots \right] - \\ - s_B X_B T \left[-1 + b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} + b_3 [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right], \quad \dots \quad (50)$$

$$S = X_B \left[1 + s_B - \log \zeta(s_B) - b_2 (s_B - 1) \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} - \right. \\ \left. - \frac{b_3}{2} (2s_B - 1) [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} - \dots \right] + \\ + X_F \left[1 + s_F - \log A_{0F}^* + b_2 (s_F - 1) A_{0F}^* + \frac{b_3}{2} (2s_F - 1) (A_{0F}^*)^2 + \dots \right], \quad \dots \quad (51)$$

$$C = \frac{X_F s_F^2}{T} \left[\frac{T}{s_F} - b_2 A_{0F}^* + \frac{b_3}{2} (A_{0F}^*)^2 + \dots \right] + s_B X_B \left[1 + b_2 (s_B - 1) \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} + \right. \\ \left. + \frac{b_3}{2} (2s_B - 1) [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right]. \quad \dots \quad (52)$$

From eqs. (52) and (43) we get for the jump in the specific heat at the λ -temperature of the mixture

$$\Delta C = X_B s_B \left[1 + b_2 \zeta(s_B) (s_B - 1) + b_3 [\zeta(s_B)]^2 (2s_B - 1) + \dots \right] - \\ - C_B \Gamma(s_B + 2) (kT_\lambda)_{s_B} (X_F V_F^0 + X_B V_B^0) \zeta(s_B + 1). \quad \dots \quad (53)$$

6. LIQ. F DEGENERATE IN THE PURE STATE AND NON-DEGENERATE WHEN IN THE MIXTURE

Evidently, this case is of interest only for very dilute mixtures at temperatures below the Fermi degeneracy temperature, 4.85°K. , for the pure Fermi gas (with liquid density). Also it is important only in so far as it is customary to express the partial pressures of the liquids in the mixture in terms of their saturated pressures. As regards the partial pressures themselves they would still be the same as in the previous case. As in the previous case we shall obtain two sets of expressions according as the temperature is above or below the λ -temperature of the mixture. Now, we have for the thermodynamic potential of pure Liq. F

$$\mu_F^0 = -\chi_F^0 + \frac{s_F}{s_F+1} \xi_0^0 \left[1 - (s_F+1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0^0} \right)^2 \right]. \quad \dots \quad (54)$$

The other thermodynamic potentials are unaltered and therefore only the expressions for $\frac{p_F}{p_F^0}$ would be modified. Combining eq. (54) with eqs. (37) and (46) we obtain

(a) for temperatures below the λ -point:

$$\begin{aligned} \frac{p_F}{p_F^0} &= \frac{X'_F}{\Gamma(s_F+1)} \left(\frac{\xi_0^0}{kT} \right)^{s_F} \times \\ &\times \exp \left[-X'_F - \frac{\zeta(s_B+1)}{\zeta(s_B)} \frac{V_F^0}{V_B^0} \left(\frac{T}{T_\lambda} \right)^{s_B} - \frac{s_F}{s_F+1} \frac{\xi_0^0}{kT} \left\{ 1 - (s_F+1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0^0} \right)^2 \right\} + \right. \\ &\left. + \left\{ b_2(1+X'_F) A_{0F}^* + b_3 \left(\frac{1}{2} + X'_F \right) (A_{0F}^*)^2 + \dots \right\} \right], \quad \dots \quad (55) \end{aligned}$$

and (b) for temperatures above the λ -point:

$$\begin{aligned} \frac{p_F}{p_F^0} &= \frac{X'_F}{\Gamma(s_F+1)} \left(\frac{\xi_0^0}{kT} \right)^{s_F} \times \\ &\times \exp \left[-X'_F + X'_B \frac{V_F^0}{V_B^0} \left\{ -1 + b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} + b_3 [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right\} + \right. \\ &\left. + \left\{ b_2(1+X'_B) A_{0F}^* + \frac{b_3}{2} (1+2X'_B) (A_{0F}^*)^2 + \dots \right\} - \right. \\ &\left. - \frac{s_F}{s_F+1} \frac{\xi_0^0}{kT} \left\{ 1 - (s_F+1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0^0} \right)^2 \right\} \right]. \quad \dots \quad (56) \end{aligned}$$

Further, the expressions for the other thermodynamic quantities are left unaltered since the expression for the free energy of the mixture is the same as in Sec. 5.

In the extreme (Boltzmannian) case $A_F^* \rightarrow 0$ and eqs. (55) and (56) reduce to the form

$$\frac{p_F}{p_F^0} = \frac{X'_F}{\Gamma(s_F+1)} \left(\frac{\xi_0^0}{kT} \right)^{s_F} \exp \left[-X'_F - \frac{s_F}{s_F+1} \frac{\xi_0^0}{kT} \left\{ 1 - (s_F+1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0^0} \right)^2 \right\} - \frac{\zeta(s_B+1)}{\zeta(s_B)} \frac{V_F^0}{V_B^0} X'_B \left(\frac{T_\lambda}{T} \right)^{s_B} \right], \quad \dots (55')$$

and

$$\begin{aligned} \frac{p_F}{p_F^0} &= \frac{X'_F}{\Gamma(s_F+1)} \left(\frac{\xi_0^0}{kT} \right)^{s_F} \times \\ &\times \exp \left[-X'_F + X'_B \frac{V_F^0}{V_B^0} \left\{ -1 + b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} + b_3 [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right\} - \right. \\ &\quad \left. - \frac{s_F}{(s_F+1)} \frac{\xi_0^0}{kT} \left\{ 1 - (s_F+1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0^0} \right)^2 \right\} \right], \quad \dots \dots \dots (56') \end{aligned}$$

respectively.

7. LIQ. F DEGENERATE

Now we deal with the other extreme case ($A_F^* \gg 1$). It is expected that this case will be of importance for mixture with very high Liq. F concentration, that is, mixtures in which the density of Liq. F is comparable with the density of pure Liq. F . We shall obtain two sets of expressions according as $T > T_\lambda$ or $T_\lambda < T < 4.85^\circ\text{K}$. For $T > 4.85^\circ\text{K}$, the expressions derived in Sec. 5 would apply. Hence, we find that

(a) *Below the λ -point*: the expressions for the free energy of the mixture and the free energy of mixing get modified to

$$\begin{aligned} F &= -N_F \chi_F^0 + \frac{s_F}{s_F+1} N_F \xi_0 \left[1 - (s_F+1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0} \right)^2 \right] - \\ &\quad - N_B \chi_B^0 - C_B \Gamma(s_B) \zeta(s_B+1) (N_F V_F^0 + N_B V_B^0) (kT)^{s_B+1}, \quad \dots (57) \end{aligned}$$

and

$$\begin{aligned} \Delta F &= \frac{s_F}{s_F+1} N_F (\xi_0 - \xi_0^0) + s_F N_F (kT)^2 \left[\frac{1}{\xi_0^0} - \frac{1}{\xi_0} \right] \frac{\pi^2}{6} \\ &\quad - C_B \Gamma(s_B) (kT)^{s_B+1} \zeta(s_B) N_F V_F^0, \quad \dots (58) \end{aligned}$$

respectively. Consequently, the thermodynamic potentials of Liq. F and Liq. B , when in the mixture, take the form

$$\mu_F = \frac{s_F + X'_B}{s_F + 1} \xi_0 \left[1 - \frac{\pi^2}{6} (s_F+1) \left(\frac{kT}{\xi_0} \right)^2 \right] - \chi_F^0 - \frac{\zeta(s_B+1)}{\zeta(s_B)} X'_B \frac{V_F^0}{V_B^0} kT \left(\frac{T}{T_\lambda} \right)^{s_B}, \quad (59)$$

$$\mu_B = -\chi_B^0 - C_B V_B^0 (kT)^{s_B+1} \Gamma(s_B) \zeta(s_B+1)$$

$$- \frac{X'_F}{s_F+1} \frac{V_B^0}{V_F^0} \frac{\xi_0}{kT} \left[1 + \frac{\pi^2}{6} (s_F+1) \left(\frac{kT}{\xi_0} \right)^2 \right]. \quad \dots (60)$$

These relations, when combined with the expressions for the potentials of pure Liq. F and Liq. B , give for the partial vapour pressures

$$\frac{p_B}{p_B^0} = \exp \left[-\frac{X'_F}{s_F+1} \frac{V_B^0}{V_F^0} \frac{\xi_0}{kT} \left\{ 1 + (s_F+1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0} \right)^2 \right\} \right], \quad \dots \quad (61)$$

$$\log \frac{p_F}{p_F^0} = \frac{X'_B + s_F}{1+s_F} \cdot \frac{\xi_0}{kT} \left[1 - (s_F+1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0} \right)^2 \right] - \frac{s_F}{s_F+1} \frac{\xi_0}{kT} \left[1 - (s_F+1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0} \right)^2 \right] - \frac{\zeta(s_B+1)}{\zeta(s_B)} X'_B \frac{V_F^0}{V_B^0} \left(\frac{T}{T_\lambda} \right)^{s_B}. \quad \dots \quad (62)$$

Again, the modified expression for the free energy, Eq. (57), gives for the other thermodynamic properties (in units of R) the expressions

$$U = -\frac{1}{R} (N_F \chi_F^0 + N_B \chi_B^0) + \frac{X_F s_F}{1+s_F} \frac{\xi_0}{kT} \left[1 + (1+s_F) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0} \right)^2 \right] + C_B \Gamma(s_B+1) \zeta(s_B+1) \frac{(X_F V_F^0 + X_B V_B^0)}{k} (kT)^{s_B+1}, \quad \dots \quad (63)$$

$$S = \frac{\pi^2}{3} s_F X_F \frac{kT}{\xi_0} + \frac{C_B}{s_B} \Gamma(s_B+2) \zeta(s_B+1) (kT)^{s_B} (X_F V_F^0 + X_B V_B^0), \quad \dots \quad (64)$$

$$C = \frac{\pi^2}{3} s_F X_F \frac{kT}{\xi_0} + C_B \Gamma(s_B+2) \zeta(s_B+1) (kT)^{s_B} (X_F V_F^0 + X_B V_B^0), \quad \dots \quad (65)$$

and (b) *Above the λ -point*: Liq. B , when in the mixture, must be considered as non-degenerate so that the free energy of the mixture will be given by

$$F = -N_F \chi_F^0 - N_B \chi_B^0 + \frac{s_F}{1+s_F} \xi_0 \left[1 - (1+s_F) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0} \right)^2 \right] - N_B kT \left[1 - \log \zeta(s_B) + \log \left(\frac{T}{T_\lambda} \right)^{s_B} + b_2 \zeta(s_B) \left(\frac{T}{T_\lambda} \right)^{s_B} + \frac{b_3}{2} \{ \zeta(s_B) \}^2 \left(\frac{T}{T_\lambda} \right)^{2s_B} + \dots \right]. \quad \dots \quad (66)$$

In the temperature range $T > T_0$ the free energy of mixing will take the form

$$\Delta F = N_F s_F \frac{\pi^2}{6} (kT)^2 \left[\frac{1}{\xi_0^0} - \frac{1}{\xi_0} \right] + \frac{s_F}{1+s_F} N_F (\xi_0 - \xi_0^0) - N_B kT \left[\log \left(\frac{T_0}{T_\lambda} \right)^{s_B} + b_2 \frac{\zeta(s_B)}{T^{s_B}} (T_\lambda^{s_B} - T_0^{s_B}) + \frac{b_3}{2} \left\{ \frac{\zeta(s_B)}{T^{s_B}} \right\}^2 (T_\lambda^{2s_B} - T_0^{2s_B}) + \dots \right]. \quad \dots \quad (67)$$

Eq. (66) gives for the potentials in the mixed state

$$\mu_F = -\chi_F^0 - \frac{X'_B + s_F}{1 + s_F} \cdot \xi_0 \left[1 - (s_F + 1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0} \right)^2 \right] - \\ - X'_B \frac{V_F^0}{V_B^0} kT \left[1 - b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} - b_3 \{ \zeta(s_B) \}^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right], \quad \dots \quad (68)$$

$$\mu_B = -\chi_B^0 - \frac{X'_F}{1 + s_F} \frac{V_B^0}{V_F^0} \cdot \xi_0 \left[1 + (s_F + 1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0} \right)^2 \right] - \\ - kT \left[X'_B - \log \zeta(s_B) + \log \left(\frac{T_\lambda}{T} \right)^{s_B} + b_2 \zeta(s_B) (1 + X'_F) \left(\frac{T_\lambda}{T} \right)^{s_B} \right. \\ \left. + \frac{b_3}{2} \{ \zeta(s_B) \}^2 (1 + 2X'_F) \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right]. \quad \dots \quad (69)$$

Combining, as before, with the expressions for the pure liquids we have

$$\frac{p_B}{p_B^0} = X'_B \exp \left[-X'_B - \frac{X'_F}{s_F + 1} \frac{V_B^0}{V_F^0} \frac{\xi_0}{kT} \left\{ 1 + (1 + s_F) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0} \right)^2 \right\} + \frac{\zeta(s_B + 1)}{\zeta(s_B)} \left(\frac{T_\lambda}{T} \right)^{s_B} - \right. \\ \left. - \left\{ b_2 \zeta(s_B) (1 + X'_F) \left(\frac{T_\lambda}{T} \right)^{s_B} + \frac{b_3}{2} [\zeta(s_B)]^2 (1 + 2X'_F) \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right\} \right] \\ \text{for } T_\lambda < T \leq T_0, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (70)$$

$$\frac{p_B}{p_B^0} = X'_B \exp \left[X'_F - \frac{X'_F}{s_F + 1} \frac{V_B^0}{V_F^0} \frac{\xi_0}{kT} \left\{ 1 + (s_F + 1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0} \right)^2 \right\} - \right. \\ \left. - \left\{ b_2 \zeta(s_B) \left(1 + X'_F - \frac{1}{X'_B} \right) \left(\frac{T_\lambda}{T} \right)^{s_B} + \right. \right. \\ \left. \left. + \frac{b_3}{2} [\zeta(s_B)]^2 \left(1 + 2X'_F - \frac{1}{(X'_B)^2} \right) \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right\} \right], \quad (70')$$

for $T > T_0$ and,

$$\log \frac{p_F}{p_F^0} = \frac{s_F + X'_B}{1 + s_F} \frac{\xi_0}{kT} \left[1 - \frac{\pi^2}{6} (s_F + 1) \left(\frac{kT}{\xi_0} \right)^2 \right] - \frac{s_F}{s_F + 1} \frac{\xi_0^0}{kT} \left[1 - (s_F + 1) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0^0} \right)^2 \right] - \\ - X'_B \frac{V_F^0}{V_B^0} \left[1 - b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} - b_3 [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} - \dots \right]. \quad \dots \quad (71)$$

Further, eq. (66) gives for U , S and C (all in terms of R) the expressions

$$U = -\frac{1}{R} (N_F \chi_F^0 + N_B \chi_B^0) + \frac{s_F}{1 + s_F} X'_F \frac{\xi_0}{k} \left[1 + (1 + s_F) \frac{\pi^2}{6} \left(\frac{kT}{\xi_0} \right)^2 \right] + \\ + X'_B s_B T \left[1 - b_2 \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} - b_3 [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} - \dots \right], \quad \dots \quad (72)$$

$$S = \frac{\pi^2}{3} s_F X_F \frac{kT}{\xi_0} + X_B \left[s_B + \log \left(\frac{T}{T_\lambda} \right)^{s_B} - b_2 (s_B - 1) \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} - \frac{b_3}{2} (2s_B - 1) [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} - \dots \right], \quad \dots \quad (73)$$

$$C = \frac{\pi^2}{3} s_F X_F \frac{kT}{\xi_0} + X_B s_B \left[1 + b_2 (s_B - 1) \zeta(s_B) \left(\frac{T_\lambda}{T} \right)^{s_B} + b_3 (2s_B - 1) [\zeta(s_B)]^2 \left(\frac{T_\lambda}{T} \right)^{2s_B} + \dots \right]. \quad \dots \quad (74)$$

The last equation together with eq. (65) leads to the following expression for the jump in the specific heat of the mixture at its λ -point

$$\Delta C = X_B s_B \left[1 + b_2 (s_B - 1) \zeta(s_B) + b_3 (2s_B - 1) [\zeta(s_B)]^2 + \dots \right] - C_B \Gamma(s_B + 2) \zeta(s_B + 1) (X_F V_F^0 + X_B V_B^0) (kT_\lambda)^{s_B}. \quad \dots \quad (75)$$

As expected, eqs. (33), (53) and (75) show that the specific heat jump at the λ -temperature of the mixture is the same whether Liq. F is Boltzmannian, non-degenerate or degenerate. The discontinuity in the specific heat is a property of the Bose assembly alone and therefore there is no question of it being affected by the nature of the Fermi assembly. The difference in the ΔC for pure Liq. B and the ΔC for Liq. B in the mixture arises purely from the difference in the volumes of the Bose assembly in the two cases.

8. DISCUSSION OF RESULTS

Recently, Bhagat and Katti (1954b) have calculated the saturated vapour pressures of Liq. He^4 and Liq. He^3 on the basis of smoothed potential models of ideal Bose and Fermi gases. They find that the theoretical results are in good agreement with the observed pressures. On the other hand, there are many pronounced discrepancies between the theoretical predictions for an ideal Bose gas and the observed properties of Liq. He^4 . For instance, we have for the specific heat of an ideal Bose gas the expression $C \propto T^{1.5}$ as opposed to the observed relation $C \propto T^0$ ($T > 0.6^\circ\text{K.}$) for liquid He^4 . This discrepancy may be due to the complete neglect of the interactions between the molecules in the liquid state. As pointed out by Dingle (1952) a considerable improvement in the theory is obtained by the introduction of a 'self-consistent' field, that is, by assuming that each molecule is moving in a field due to its neighbours. Thus, the ideal-gas spectrum, $\epsilon = \frac{p^2}{2m}$, is replaced by an ad hoc spectrum, $\epsilon = Bp^{3/2}$. This modification is assumed to take account of the interactions between the molecules. We do not wish to insist here on the details of this concept, which is certainly open to much criticism. It may, however, lead to a somewhat better description of the properties of mixtures of the liquids. The modification in the energy spectrum leads to distribution laws of the more general type:

$$N(\epsilon) d\epsilon = \frac{VC\epsilon^{s-1}}{\frac{1}{A} \exp\left(\frac{\epsilon}{kT}\right) \pm 1}.$$

Next, following Heer and Daunt, we introduce constant or smoothed potential wells. In other words, we consider the 'liquid' as being a Bose or a Fermi gas in a constant potential well, each particle of the assembly being subject to a self-consistent field due to its neighbours. Thus we have written the distribution laws in the form:

$$N(\epsilon) d\epsilon = \frac{VC\epsilon^{s-1}}{\frac{1}{A} \exp\left(\frac{\epsilon}{kT} - \frac{\chi^0}{kT}\right) \pm 1}.$$

This single-particle approximation to the liquids stresses the rôle of statistics, obeyed by them, in the behaviour of their mixtures. It may, therefore, be expected that when a large amount of experimental data become available our discussion would be of great importance for testing the influence of statistics of the two components on the properties of their mixtures.

Below we discuss some of the important results obtained in the previous sections:

(a) *Variation of the λ -temperature with Concentration.*

Three values of s_B , namely, 3/2, 5 and 6, were used for computing $\frac{T_\lambda}{T_0}$ from eq. (9). The molar volumes, as usual, were taken to be 27.6 c.c. and 37.6 c.c. for He⁴ (Liq. B) and He³ (Liq. F), respectively. The results are plotted in Fig. 1 as a function of the He³ concentration, X_F . Measurements of the λ -temperature of He³-He⁴ mixtures have been carried out by Daunt and Heer (1950) and by Abraham *et al.* (1949). Their results are also shown in Fig. 1. It is seen that the

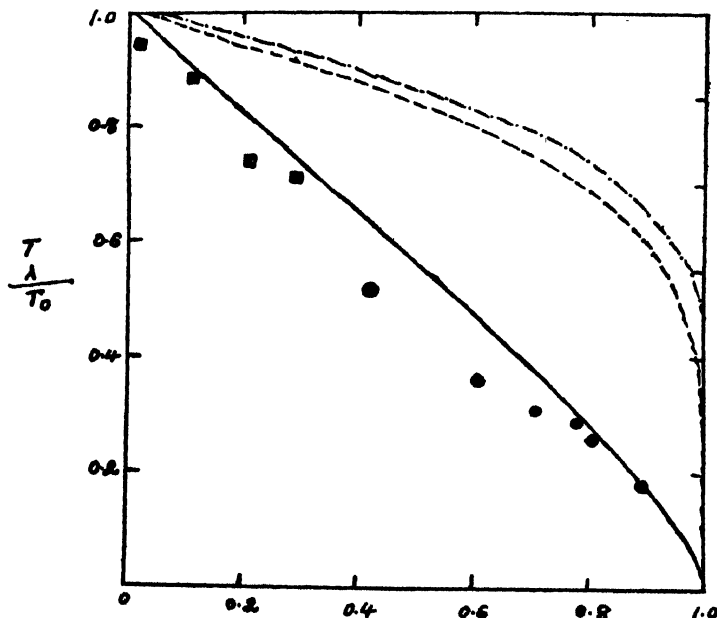


FIG. 1. Plots of the λ -temperature, T_λ , against concentration, X_F , of He³ in He⁴ mixtures. The curves are computed from eq. (9) with $s_B = 3/2$ (—), $s_B = 5$ (---) and $s_B = 6$ (-.-.-). Experimental results are indicated by dots (Daunt *et al.*) and by squares (Abraham *et al.*).

experimental points are represented fairly closely by the curve for $S_B = 3/2$, (which corresponds to the ideal gas case). For higher values of S_B the theoretical curve deviates more and more from the experimental values.

The slope of the T_λ vs. X_F curve for very low concentrations of He (i.e. $X_F \rightarrow 0$) becomes, from eq. (10),

$$\begin{aligned}\frac{\partial T_\lambda}{\partial X_F} &= -\frac{T_0}{s_B} \frac{V_F^0}{V_B^0} \\ &= -1.986 \text{ for } s_B = \frac{3}{2}, \\ &= -0.496 \text{ for } s_B = 6.\end{aligned}$$

These values may be compared with those obtained by de Boer and Gorter (1950), -2.78 , and by Stout (1949), -3.50 , whereas the observed value is ~ -2.8 .

For very high concentrations (i.e. $X_F \rightarrow 1$)

$$\frac{\partial T_\lambda}{\partial X_F} \rightarrow \infty \text{ since } s_B > 1.$$

Hence the curve should approach the abscissae vertically in each case. The experimental points also indicate such a behaviour.

(b) The Vapour Pressures

Using observed values for the saturated pressures p_F^0 and p_B^0 , the total vapour pressure ($p_F + p_B$) over a 20 per cent mixture was computed from the theoretical expressions in the following cases:—

- (i) when Liq. F is Boltzmannian, with $s_B = 3/2$,
- (ii) when Liq. F is non-degenerate, with $s_B = 3/2$,
- (iii) when Liq. F is degenerate and $s_B = 3/2$, and
- (iv) when Liq. F is Boltzmannian and $s_B = 6$.

In calculating the values of p_B for temperatures above T_λ no distinction was made between the regions $T_\lambda < T < T_0$ and $T > T_0$, in fact, the expression for the latter region were used throughout the range $T > T_\lambda$. The results are plotted in Fig. 2 as a function of temperature. Weinstock *et al.* (1950) have observed the total pressure over a 20.3 per cent He^3 - He^4 mixture. We have included their results in Fig. 2 for the sake of comparison. Clearly there is very little difference between the curves I_A and II, corresponding, respectively, to cases (i) and (ii). However, curve I_B (case iv) has a discontinuity at the λ -point, whereas curve III (case iii) is much lower than the experimental values. The agreement between the theoretical predictions in the first two cases and the observed pressures may not, however, be stressed. As pointed out in Sec. 4 the assumption $G \approx F$ entails a considerable error (Bhagat and Katti, 1954a).

In Fig. 2a are shown the plots of eqs. (28) and (28'), curves I and II respectively. In obtaining Fig. 2, as already mentioned, we used for p_B , throughout the range $T > T_\lambda$, an expression which is valid only for $T > T_0$. Fig. 2a shows, in the representative Boltzmannian case, how far this approximation is valid.

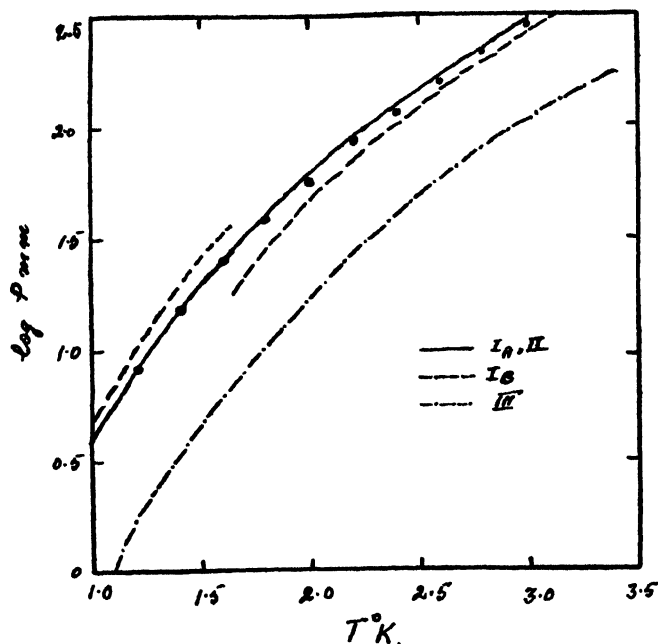


FIG. 2. Plots of the computed values of the total vapour pressure over a 20% He^3 in He^4 mixture. Curve I_A is for Liq. F Boltzmannian and $s_B = 3/2$, curve I_B for Liq. F Boltzmannian and $s_B = 6$, curve II for Liq. F non-degenerate and $s_B = 3/2$, and curve III for Liq. F degenerate and $s_B = 3/2$. Experimental points (indicated by circles) are due to Weinstock *et al.*

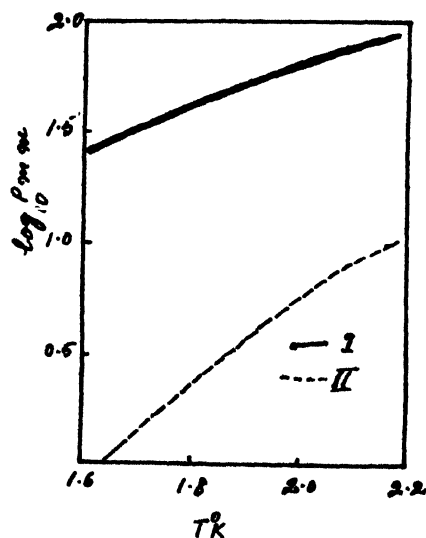


FIG. 2a. Plots of the total vapour pressures above a 20% He^3 in He^4 mixture in the temperature range $T_\lambda < T < T_0$, as computed from eq. (28), curve I, and eq. (28'), curve II,

(c) *The Distribution Coefficient.*

From eqs. (17) and (18) and (27) and (28') we have, for dilute solutions (i.e. $N_F \ll N_B$),

$$T < T_\lambda$$

$$\frac{X_V}{X_L} = \frac{V_F^0}{V_B^0} \cdot \frac{p_F^0}{p_B^0} \exp \left[1 - \left(\frac{T}{2.186} \right)^{s_B} \frac{\zeta(s_B + 1)}{\zeta(s_B)} \cdot \frac{V_F^0}{V_B^0} \right], \quad \dots \quad (76)$$

$$T > T_\lambda$$

$$\frac{X_V}{X_L} = \frac{V_F^0}{V_B^0} \cdot \frac{p_F^0}{p_B^0} \left[\exp \left\{ 1 - \frac{V_F^0}{V_B^0} \left(1 - b_2 \zeta(s_B) \left(\frac{2.186}{T} \right)^{s_B} - b_3 [\zeta(s_B)]^2 \left(\frac{2.186}{T} \right)^{2s_B} \right) \right\} \right], \quad \dots \quad (77)$$

where X_V and X_L are the relative concentrations in the vapour and the liquid phases and for $T_\lambda \approx T_0$ we have taken the observed value of 2.186°K. In obtaining

(76) and (77) it has been assumed that $X_V = \frac{p_F}{p_B}$. This relation would hold also in the non-perfect gas approximation to the vapours provided that the partial

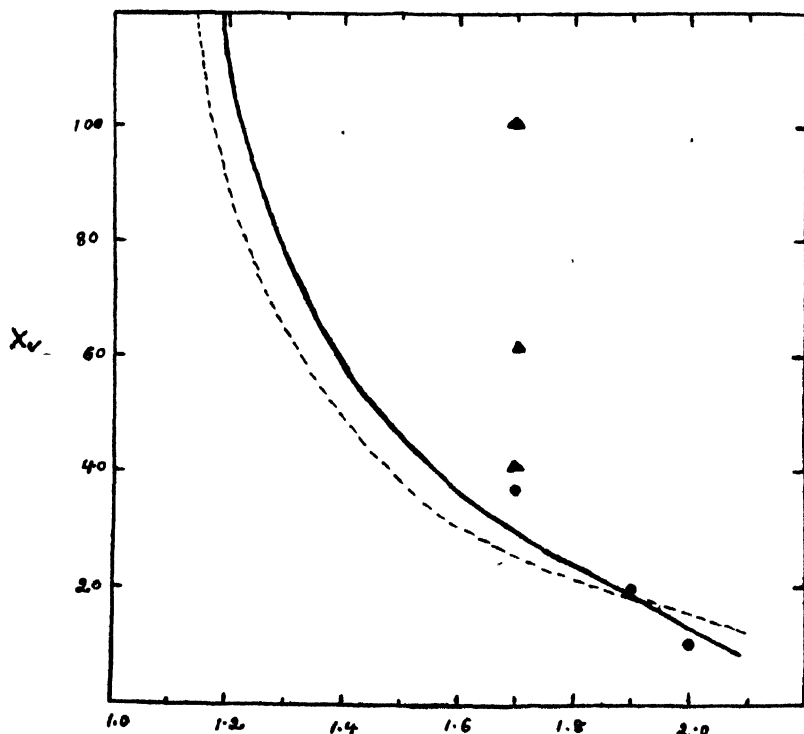


FIG. 3. Plots of the relative concentration $\frac{X_V}{X_L}$ for very low concentrations ($X_F \ll 1$) as computed from eqs. (76) and (77). Experimental points are due to Taconis *et al.* (circles) and to Lane *et al.* (triangles).

pressures are thought of as fugacities. Values of $\left(\frac{X_V}{X_L}\right)$ have been calculated from eqs. (76) and (77) for two values of S_B , namely, $3/2$ and 6 . The results are shown in Fig. 3. Measurements of the distribution coefficient of dilute solutions of He^3 in He^4 have been carried out by Taconis *et al.* (1949) and by Lane and his co-workers (1949). Their results have also been shown in Fig. 3. The percentage error in the experimental results is so large that no decision can be made as to which law is preferable. However, considering more concentrated solutions Daunt, Tseng and Heer (1952) have reported good agreement between the $3/2$ law and the experimental results.

(d) *The Specific Heat.*

In this case no experimental data are available for comparison. Theoretical values of the specific heat (in units of $R = k(N_F + N_B)$) have been computed for a 20 per cent mixture in the following cases:—

- (i) Liq. F Boltzmannian,
- (ii) Liq. F non-degenerate Fermi liquid, and
- (iii) Liq. F degenerate Fermi liquid.

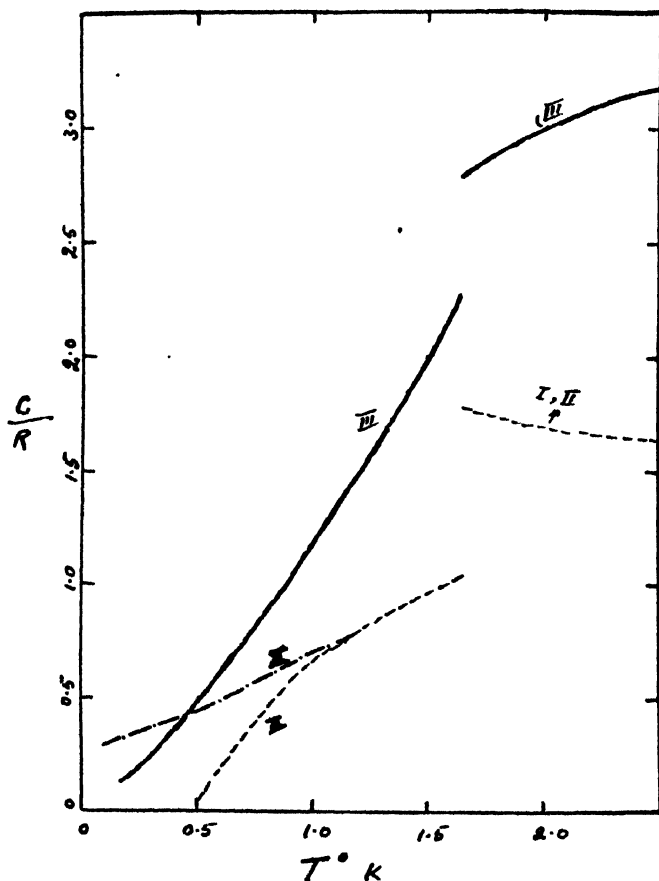


FIG. 4. Plots of the computed values of the specific heat, C/R , of a 20% He^3 in He^4 mixture against temperature. Curve I is for Liq. F Boltzmannian, Curve III for Liq. F degenerate and Curve II for Liq. F non-degenerate.

In each case we have taken $s_B = s_F = 3/2$. The results are represented by curves I, II and III, respectively, in Fig. 4.

In case (i) as the temperature decreases from higher values towards the λ -temperature ($\sim 1.63^\circ\text{K.}$) C increases slowly from 1.5 to 1.6. At 1.63°K. it drops discontinuously to the value 1.15 and decreases slowly to 0.3 as the absolute zero is approached. For temperatures above about 1°K. the curve for case (ii) follows very closely the curve for case (i). Below 1°K. they deviate appreciably from each other and the former drops to zero nearabout 0.5°K. For case (iii) C tends to zero as the absolute zero is approached and above the λ -temperature it increases slowly with the rise of temperature.

Table I below shows the various relations for C at different concentrations. The relations clearly show that in the Boltzmannian case the specific heat increases with concentration at low temperatures. At temperatures above the λ -point, however, it has, more or less, a constant value, $\sim 1.5 R$, the temperature terms having little contribution. In the degenerate case $C \rightarrow 0$ as $T \rightarrow 0$ for temperatures below the λ -point whereas in the non-degenerate case C rapidly decreases to zero at $T \sim 0.5^\circ\text{K.}$ Thus the behaviour of C for temperature below 1°K. is very sensitive to the nature of Liq. F being considerably different according as Liq. F is Boltzmannian, non-degenerate or degenerate. Experimental measurements of the specific heat in this region would therefore give an indication as to the nature of He^3 .

In passing, we may point out that recent experiments of Fairbank *et al.* (1953) on the magnetic susceptibility of He^3 indicate that He^3 behaves as a classical liquid even down to the lowest temperatures.

(e) The Entropy of Mixing.

In the case where Liq. F is taken to be a degenerate Fermi liquid we obtain for the entropy of mixing, from eq. (58), the expression

$$\Delta S = -\frac{\partial(\Delta F)}{\partial T} = -\frac{\partial}{\partial T} \left[\frac{s_F}{s_F + 1} (\xi_0 - \xi_0^0) + \text{terms involving temperature} \right]. \quad (78)$$

As already pointed out by Daunt and Heer (1951) we note that $\Delta S \rightarrow 0$ as the absolute zero is approached. This leads us to expect an unmixing of the liquids as the absolute zero is approached. This may be in accordance with Taconis' hypothesis that He^3 mixes only with the normal part of He^4 so that at the absolute zero, where no normal He^4 atoms are present, there would be a complete unmixing of the liquids. Also, it is clear that eq. (78) is in accordance with Nernst's theorem. The classical expression for ΔS , however, does not show such a behaviour, that is, ΔS does not vanish at the absolute zero.

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SUMMARY

Smoothed potential models of the type used by Heer and Daunt are considered when the distribution laws have the generalized form introduced by Kothari and Singh. In the mixture the liquids are assumed to form statistically independent systems. The thermodynamic properties of mixtures (both above and below their λ -points) are derived when the 'Fermi liquid' is (i) Boltzmannian, (ii) Non-degenerate and (iii) Degenerate. Comparison with observed properties of He^3 - He^4 mixtures is carried out. The behaviour of the specific heat, below 1°K. , is found to be very sensitive to the nature of the Fermi liquid.

TABLE I
Theoretical expressions for the specific heat C/R .

Concentration $X\%$	Liq. F Boltzmannian		Liq. F Degenerate	
	Below λ -point	Above λ -point	Below λ -point	Above λ -point
0.5	$0.0076 + 0.3555 T$	$1.5 + \frac{0.0049}{T^{\frac{1}{3}}} + \frac{0.0030}{T^3}$	$0.001 T + 0.3555 T^{\frac{1}{3}}$	$1.49 + 0.0100 T + \frac{0.0049}{T^{\frac{1}{3}}} + \frac{0.0030}{T^3}$
3.0	$0.045 + 0.3589 T$	$1.5 + \frac{0.02789}{T^{\frac{1}{3}}} + \frac{0.0166}{T^3}$	$0.0605 T + 0.3589 T^{\frac{1}{3}}$	$1.44 + 0.0605 T + \frac{0.02789}{T^{\frac{1}{3}}} + \frac{0.0166}{T^3}$
20.0	$0.300 + 0.3806 T$	$1.5 + \frac{0.5352}{T^{\frac{1}{3}}} + \frac{0.2287}{T^3}$	$0.403 T + 0.3808 T^{\frac{1}{3}}$	$1.20 + 0.403 T + \frac{0.5352}{T^{\frac{1}{3}}} + \frac{0.2287}{T^3}$
60.0	$0.900 + 0.4320 T$	$1.5 + \frac{0.0946}{T^{\frac{1}{3}}} + \frac{0.0143}{T^3}$	$1.010 T + 0.4320 T^{\frac{1}{3}}$	$0.60 + 1.010 T + \frac{0.0946}{T^{\frac{1}{3}}} + \frac{0.0140}{T^3}$

REFERENCES

- Abraham, B. M.; Weinstock, B. and Osborne, D. W. (1949b). Lambda-temperatures of He³-He⁴ Mixtures. *Phys. Rev.*, **76**, 864.
- Bhagat, S. M. and Katti, P. K. (1954a). The Vapour Pressure of Mixtures of He³ and He⁴. *Science and Culture*, **20**, 43-45.
- (1954b). A Note on the Vapour Pressure of Liquid He II and Liquid He³. *Ibid.*, **19**, 461-462.
- Daunt, J. G. and Heer, C. V. (1950). Lambda-temperatures of Solutions of He³ in He⁴ below 1° K. *Phys. Rev.*, **79**, 46-51.
- Daunt, J. G., Tseng, T. P. and Heer, C. V. (1952). Further Considerations Regarding the Theory of Bose-Einstein Liquids. *Ibid.*, **86**, 911-915.
- de Boer, J. and Gorter, C. J. (1950). Statistical Mechanics of Liquid Mixtures of He³ and He⁴. *Physica*, **16**, 225-238.
- Dingle, R. B. (1952). Theories of He II. *Advances in Physics*, **1**, 111-168.
- Fairbank, W. M., Ard, W. B., Dehmelt, H. G., Gordy, W. and Williams, S. R. (1953). Temperature Dependence of the Nuclear Susceptibility of He³ between 1.2° K. and 4.2° K. *Phys. Rev.*, **92**, 208.
- Heer, C. V. and Daunt, J. G. (1951). A Contribution to the Theory of Bose-Einstein Liquids. *Ibid.*, **81**, 447-454.
- Kothari, D. S. and Singh, B. N. (1941). Bose-Einstein Statistics and Degeneracy. *Proc. Roy. Soc. (London)*, **A178**, 135-152.
- (1942). Thermodynamics of a relativistic Fermi-Dirac Gas. *Ibid.*, **A180**, 414.
- Lane, C. T., Fairbank, H. A., Aldrich, L. T. and Nier, A. O. (1949). On the He³-He⁴ Equilibrium below the Lambda-Point. *Ibid.*, **75**, 46.
- London, F. (1938). On the Bose-Einstein Condensation. *Ibid.*, **54**, 947-954.
- Osborne, D. W., Weinstock, B. and Abraham, B. M. (1949a). Comparison of the Flow of Isotopically Pure Liquid He³ and He⁴. *Ibid.*, **75**, 988.
- Stout, J. W. (1949). Solutions of He³ in He⁴. *Ibid.*, **76**, 864.
- Taconis, K. W., Beenakker, J. J. M., Nier, A. O. C., and Aldrich, L. T. (1949). Measurements Concerning the Vapour-Liquid Equilibrium of Solutions of He³ in He⁴ below 2.19° K. *Physica*, **15**, 733-739.
- Weinstock, D., Osborne, D. W., and Abraham, B. M. (1950). Phase Relations in He³-He⁴ Solutions. *Phys. Rev.*, **77**, 400.

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STUDY OF THE MINERAL PHYSICS OF THE FERRUGINOUS MANGANESE ORES OF THE VISAKHAPATNAM DISTRICT

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INTRODUCTION

In a series of articles, R. S. Dean and others (1934) have pointed out the importance of the study of mineral physics and showed that such studies can greatly help in the magnetic separation of economically important minerals. They have also announced the remarkable discovery that the hysteretic characteristics of minerals with even very feeble magnetic properties are susceptible to development, modification, and control through the development of interfacial area and mechanical and heat treatment. Thus today, in all advanced countries the beneficiation by magnetic separation of low grade minerals by alternating current treatment, preceded by modification of its magnetic properties by sizing, heating and packing to different sizes, temperatures and densities has now been adopted as a standard method of separating useful economic minerals.

The application of these ideas to a similar study of the physics of minerals occurring in our country has been engaging the attention of the authors for some time. They have confined their attention in this paper to the studies on the manganese ores of the Visakhapatnam district only.

The purpose of the present paper is thus a preliminary study of the mineral physics of the ferruginous manganese ores of the Visakhapatnam district. The hysteretic parameters determined in the present studies will be coupled with the results of heat treatment and application of alternating current magnetic separation in the later investigations and then it will be possible to show how far these studies are useful in actual beneficiation.

Accordingly ferruginous manganese ores from two different localities (a) Garividi, and (b) Garbham (Visakhapatnam district) are collected. Of these, two representative samples are powdered and washed in running water. They are then chemically analysed for the total iron and manganese percentages by the usual methods (Low, 1927).

The average of five analyses for Garividi ore and six analyses for the Garbham ore are given in the table below, along with the average values of the previous workers.

				Present analyses	Average values of (Krishnan, 1951) the previous workers
(a) Garividi ore—					
Iron	18%		11.22%
Manganese	41.2%		42.96%
(b) Garbham ore—					
Iron	5.3%		9.99%
Manganese	46.9%		45.39%

The powdered samples are passed through sieves of various meshes and the mean sieve size is taken as being representative of the grain size for each fraction. Each fraction is labelled according to its grain size. In all, four different grain sizes for the ore from Garividi and five grain sizes for the ore from Garbham are studied.

APPARATUS FOR THE MEASUREMENT OF HYSTERESIS

The apparatus of Bruckshaw and Rao (1950) is employed for the measurement of the magnetic properties of these ore powders. This is an inductive method employing the oscillographic technique. The specimen to be examined is placed in a uniform magnetic field, and the secondary field induced into it, which is proportional to the energising field is then investigated.

Fig. 1 shows the principal parts of the apparatus. The two energising coils *B* and *D* are designed to give a magnetic field of 250 oersteds per ampere. The maximum safe current that can be passed is 1.5 amperes giving a field of 375 oersteds at a point midway between the coils *B* and *D*. Alternating current is employed and this induces e.m.f.s in the three nearly identical coils *A*, *C* and *E*. The specimen in the form of a two-centimeter cube is placed within the pick-up coil *C* with its centre coinciding with the geometrical centre of the coil system. Without the specimen in place, the number of turns of these detecting coils is adjusted so that the e.m.f. in *C* is balanced by the sum of the e.m.f.s in *A* and *E*. When the specimen is introduced, the flux through *C* is modified accordingly and an out-of-balance e.m.f. is set up, its magnitude depending on the magnetic characteristics of the specimen.

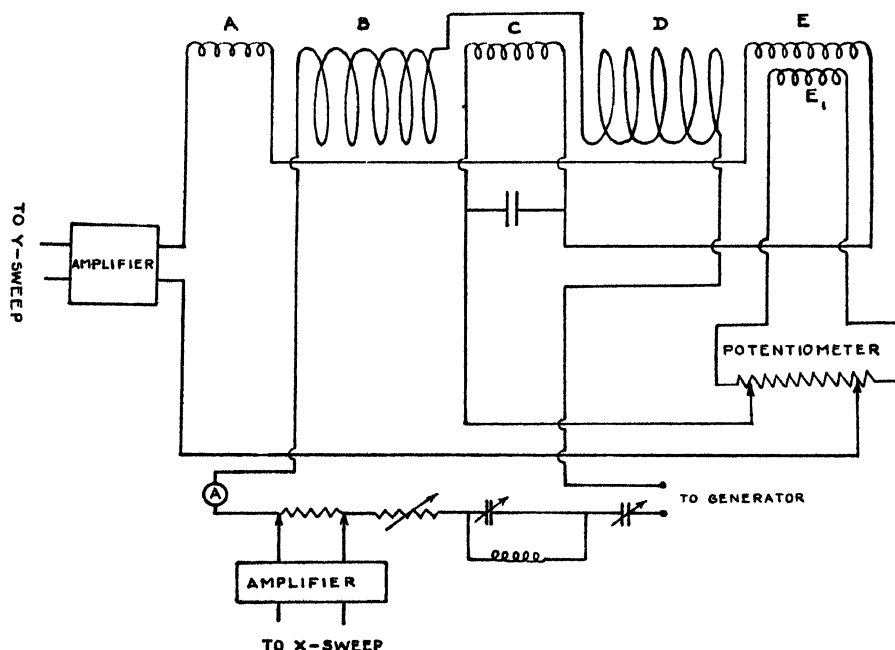


FIG. 1.

The induction at any instant in the specimen is given by $B = H + 4\pi I$. The contribution arising from the term H is identical with that obtained without the specimen and is compensated by the initial balance. The out-of-balance e.m.f. is proportional to $4\pi \frac{dI}{dt}$ or $4\pi \frac{dI}{dH} \times \frac{dH}{dt}$. This e.m.f. after amplification is impressed on the vertical deflection plates of the cathode-ray oscillograph, the horizontal plates of which are joined to the ends of the resistor in the energising circuit. The instantaneous deflection, is then proportional to the field. Thus a combination of the vertical and the horizontal forces gives a graph on the oscillograph between $\frac{dI}{dH} H_0 \cos \omega t$ and $H_0 \sin \omega t$, where H_0 is the peak field.

This graph on the oscillograph is photographed on a 36 mm. film. The following exposures are taken for each specimen:—

- (a) with the specimen inside the coil C ,
- (b) without the specimen,
- (c) with the Y -sweep removed, giving a horizontal reference axis, and
- (d) with the X -sweep removed, giving the Y axis. ●

METHOD OF MEASUREMENT

The photographs are enlarged to ten times their original size and the X axis of the enlarged picture is divided into twenty equal parts. This X axis is taken as representing the field axis and the twenty equal parts are named as 0, 0.1 H_0 , 0.2 H_0 , H_0 . At each of these points ordinates are erected and the difference in ordinates between the trace with the specimen and the trace without the specimen, after correcting for the non-linear magnification in the oscillograph, is a measure of $\frac{dI}{dt}$ or $\frac{dI}{dH} \times \frac{dH}{dt}$ or $\frac{dI}{dH} \cdot H_0 \cos \omega t$. So, these values divided by $\cos \omega t$ are proportional to $\frac{dI}{dH}$ and are plotted on the Y axis against the field. The curve is then integrated from 0 to H and this gives I at various values of H . The I and H curves are drawn from these values and the arbitrary units of I are transferred into absolute units from a previous calibration of the oscillographic deflections.

The chief advantages of this method are:

1. The apparatus readily gives the hysteresis loop at any desired field strength and from this all the magnetic properties can be read.
2. The laborious calculations are very much simplified.
3. The packing density of the powder on which the magnetic measurements mostly depend is readily determined and can be varied over a considerable range by mixing with a magnetically inert material.
4. This method can conveniently be used in the case of rock specimens as well as in the study of mineral powders.
5. This requires comparatively small amounts of powder (10 to 15 gms.) and can be used with advantage for minerals available in small quantities.

The ore fractions of the various grain sizes are weighed and mixed with known quantities of plaster of Paris, used here as a magnetically inert material and also as a cementing agent. This mixture is wetted slightly and the paste is put in a two-centimeter cube mould. By weighing the prepared cube and by knowing the amounts of the ore and the plaster of Paris used, the packing density is calculated.

By varying the amount of plaster of Paris, various packing densities are obtained. Apart from the nine fractions of varying grain sizes mentioned previously, six fractions with different packing densities for Garividi ore and four for Garbham ore are also studied.

EXPERIMENTAL RESULTS

Employing the above method, I and H curves for all the nineteen fractions of the ferruginous manganese ores from Garividi and Garbham, are drawn at a peak field of 375 oersteds. The I - H curves obtained by the above method have been omitted from the list of figures given here, but the magnetic properties read from those curves have been tabulated.

Tables 1(a) and 1(b) give the values of $4\pi I$ and permeability at various field strengths differing by 37.5 oersteds, for various grain sizes. Table 2(a) gives the values of $4\pi I$ and permeability for $(-20+40)$ mesh at six packing densities for the Garividi ore and Table 2(b) gives the same data for $(-40+60)$ mesh Garbham ore, at four packing densities. All the values for the magnetisation per unit area of cross-section in the above four tables are expressed as 4π times the intensity of magnetisation, as has been done by Gottschalk and Wartman (1935). $4\pi I$ has been called by Gottschalk and Wartman as magnetisation density corresponding to the intensity of magnetisation I .

The curves in Figs. 2 to 6 pertain to the ore from Garividi area and the curves for Garbham ore can be obtained from the data available in Tables 1(b), 2(b) and 3(b). They are found to be similar to the curves in Figs. 2 to 6 and so are omitted. Fig. 2 represents the variation of permeability with grain size. These values of permeability are corrected for a standard value of packing density. Fig. 3 shows the variation of 'magnetisation density' with packing density at different field strengths. Examination of Fig. 3 shows that the variation of the magnetisation density with packing density is more linear for higher fields than for lower fields.

TABLE 1(a)

*Variation of magnetisation density and permeability with field strength
(Garividi Ore)*

H Oersteds	Fraction 1. - 20 + 40 mesh		Fraction 2. - 40 + 60 mesh		Fraction 3. - 50 + 80 mesh		Fraction 4. - 120 + 140 mesh	
	$4\pi I$ Gausses	Perme- ability	$4\pi I$ Gausses	Perme- ability	$4\pi I$ Gausses	Perme- ability	$4\pi I$ Gausses	Perme- ability
0	52.3	..	60.98	..	60.98	..	21.79	..
37.5	91.6	3.4	91.47	3.2	91.47	3.44	39.42	2.05
75.0	117.6	3.2	113.20	2.55	113.20	2.51	63.17	1.82
112.5	148.1	2.32	135.10	2.21	135.10	2.21	82.77	1.75
150.0	174.3	2.16	156.80	2.05	156.80	2.05	104.60	1.70
187.5	196.0	2.05	174.30	2.00	174.30	2.00	119.80	1.64
225.0	217.8	1.98	191.70	1.90	191.70	1.90	135.10	1.60
262.5	235.3	1.90	209.00	1.81	204.70	1.78	150.30	1.57
300.0	252.6	1.84	222.20	1.74	215.60	1.71	161.20	1.54
337.5	270.1	1.80	235.30	1.70	226.60	1.67	169.90	1.50
375.0	283.1	1.75	239.60	1.64	235.30	1.63	174.30	1.47

Fe gms./cm.³

0.415

0.320

0.1806

Mn gms./cm.³

0.9443

0.7283

0.4111

Grain size in microns

476

264

211

TABLE 1(b)

*Variation of magnetisation density and permeability with field strength
(Garbham Ore)*

<i>H</i> Oerstedes	Fraction 1. -20+40 mesh		Fraction 2. -40+60 mesh		Fraction 3. -60+80 mesh		Fraction 4. -80+100 mesh		Fraction 5. -100+120 mesh	
	$4\pi I$ Gaus- ses	Perme- ability	$4\pi I$ Gaus- ses	Perme- ability	$4\pi I$ Gaus- ses	Perme- ability	$4\pi I$ Gaus- ses	Perme- ability	$4\pi I$ Gaus- ses	Perme- ability
0	16.2	..	22.7	..	19.80	..	20.5	..	17.0	..
37.5	44.3	2.18	42.6	2.14	32.40	1.86	30.7	1.82	23.9	1.64
75.0	54.5	1.72	52.8	1.70	34.10	1.46	37.5	1.50	28.97	1.39
112.5	64.8	1.57	63.1	1.56	40.90	1.36	47.7	1.42	34.08	1.30
150.0	75.0	1.50	71.6	1.48	49.40	1.33	52.8	1.35	39.20	1.26
187.5	83.5	1.45	78.4	1.42	56.20	1.30	59.6	1.32	42.60	1.23
225.0	90.3	1.40	85.2	1.38	61.30	1.27	64.8	1.29	44.30	1.20
262.5	97.2	1.38	90.3	1.34	66.50	1.25	68.2	1.26	49.40	1.19
300.0	102.0	1.34	95.4	1.32	71.60	1.24	71.6	1.24	51.10	1.17
337.5	107.4	1.32	98.8	1.30	74.98	1.22	75.0	1.22	54.58	1.16
375.0	110.8	1.295	102.2	1.27	80.10	1.21	77.5	1.21	57.90	1.15

Fe gms./cm.³ 0.0848

0.1056

0.0829

0.0798

0.0694

Mn gms./cm.³ 0.7500

0.9344

0.7336

0.7062

0.6142

Grain size in
microns

476

264

185

143

114

TABLE 2(a)

*Variation of magnetisation density and permeability with field strength at various packing densities
(Garividi ore - 20+40 mesh)*

<i>H</i> Oer- stedes	Fraction 1.		Fraction 2.		Fraction 3.		Fraction 4.		Fraction 5.		Fraction 6.	
	$4\pi I$ Gausses	Perme- ability	$4\pi I$ Gausses	Perme- ability	$4\pi I$ Gausses	Perme- ability	$4\pi I$ Gausses	Perme- ability	$4\pi I$ Gausses	Perme- ability	$4\pi I$ Gausses	Perme- ability
0	69.7	..	52.3	..	34.9	..	22.2	..	15.05	..	10.00	..
37.5	100.9	3.90	91.6	3.4	65.3	2.74	47.9	2.3	45.08	2.15	30.85	1.82
75.0	148.1	2.97	117.6	3.20	91.5	2.22	69.7	1.93	69.70	1.92	47.95	1.64
112.5	182.9	2.63	148.1	2.32	113.2	2.09	82.8	1.78	82.80	1.74	60.98	1.54
150.0	217.8	2.45	174.3	2.16	139.4	1.93	101.6	1.72	101.90	1.67	69.69	1.48
187.5	248.3	2.32	196.0	2.05	156.8	1.84	122.2	1.65	113.2	1.60	82.77	1.44
225.0	274.5	2.23	217.8	1.98	178.6	1.82	139.4	1.63	126.30	1.577	91.47	1.41
262.5	300.5	2.15	235.3	1.90	191.7	1.73	156.8	1.60	139.40	1.53	101.90	1.38
300.0	318.5	2.09	252.6	1.84	209.0	1.70	173.8	1.58	148.10	1.49	108.90	1.36
337.5	335.8	1.99	270.1	1.80	217.8	1.65	182.9	1.54	161.20	1.48	117.60	1.35
375.0	339.8	1.89	283.1	1.75	226.6	1.60	196.0	1.52	169.90	1.45	122.00	1.33

Fe gms./cm.³ 0.35

0.32

0.2740

0.2125

0.175

0.134

Mn gms./cm.³ 0.7968

0.73

0.6236

0.4837

0.3982

0.3050

TABLE 2(b)

*Variation of magnetisation density and permeability with field strength at various packing densities
(Garbham ore — 40+60 mesh)*

H Oersteds	Fraction 1.		Fraction 2.		Fraction 3.		Fraction 4.	
	$4\pi I$ Gausses	Perme- ability	$4\pi I$ Gausses	Perme- ability	$4\pi I$ Gausses	Perme- ability	$4\pi I$ Gausses	Perme- ability
0	39.08	..	30.7	..	27.3	..	17.04	..
37.5	47.70	2.27	42.6	2.14	37.5	2.00	25.60	1.68
75.0	61.30	1.82	52.8	1.70	46.0	1.61	34.08	1.45
112.5	73.30	1.65	63.1	1.56	52.8	1.47	40.90	1.36
150.0	83.50	1.56	71.6	1.48	59.6	1.40	46.00	1.31
187.5	94.02	1.51	78.4	1.42	66.5	1.35	51.10	1.27
225.0	102.20	1.45	85.2	1.38	71.6	1.32	56.20	1.25
262.5	108.10	1.41	90.3	1.34	76.7	1.29	59.60	1.23
300.0	115.90	1.39	95.4	1.32	81.8	1.27	64.80	1.22
337.5	120.8	1.36	98.8	1.30	85.2	1.25	69.90	1.21
375.0	124.40	1.33	102.2	1.272	92.6	1.24	73.30	1.195

Fe gms./cm.³

0.0825

0.0787

0.0742

0.0636

Mn gms./cm.³

0.7000

0.6965

0.6566

0.5628

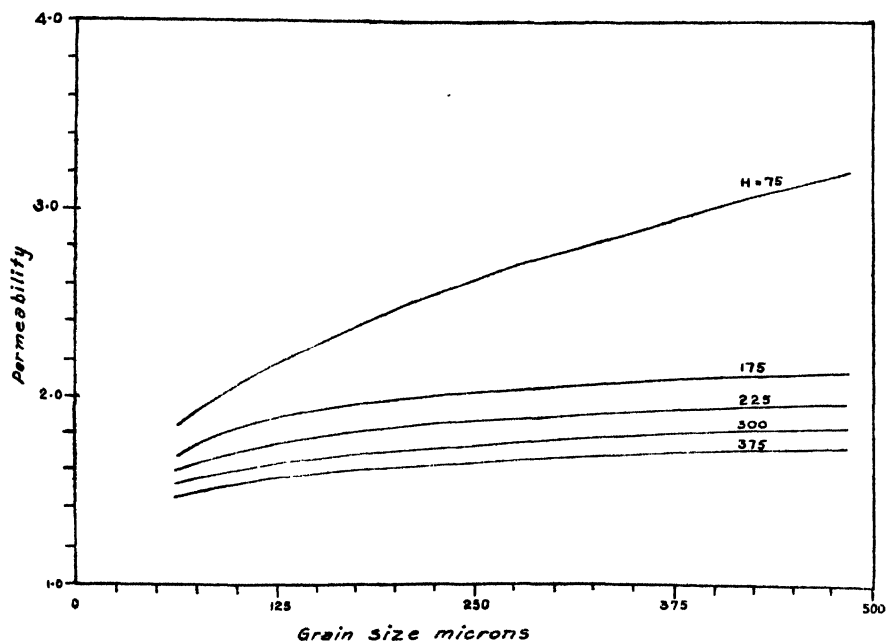


FIG. 2.

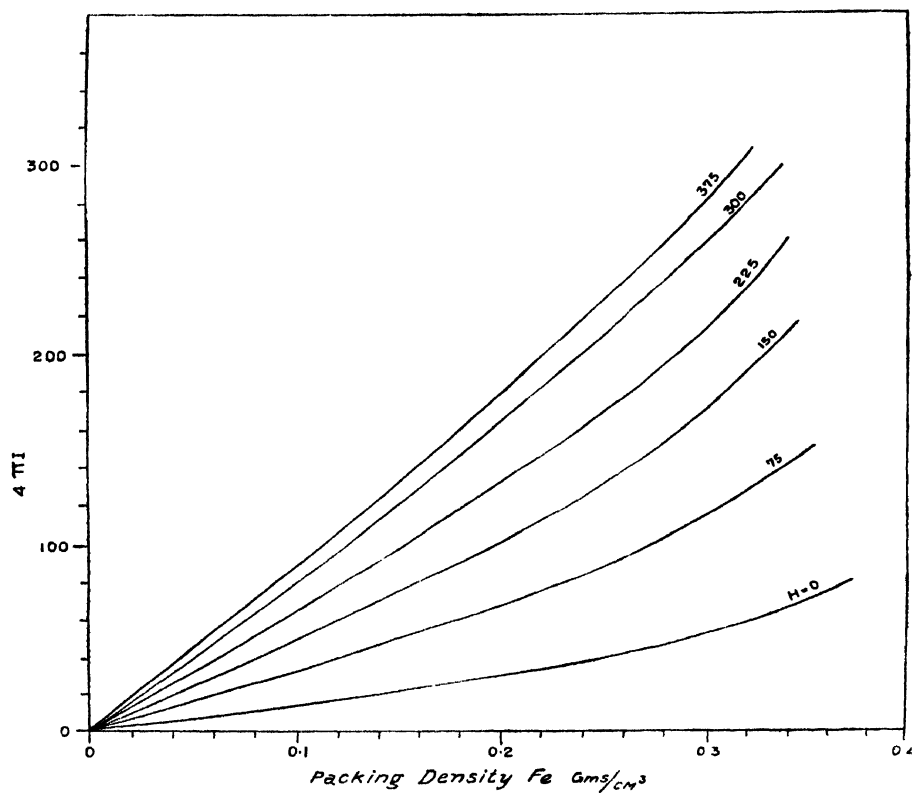


FIG. 3.

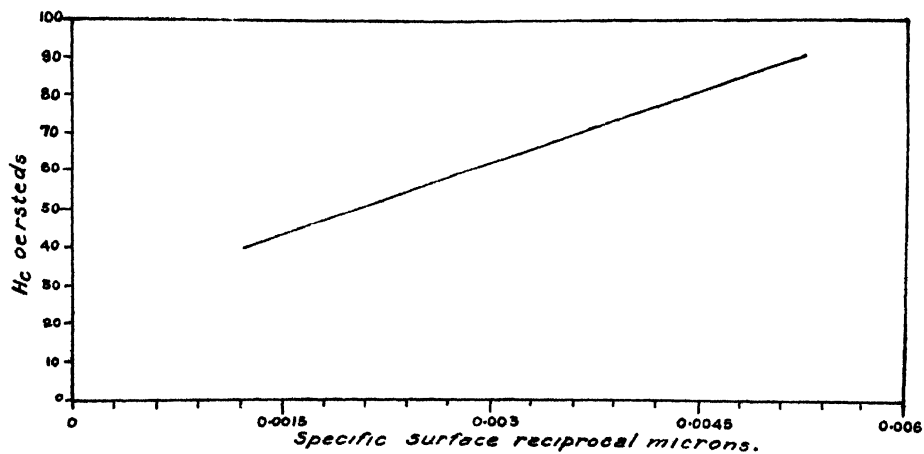


FIG. 4.

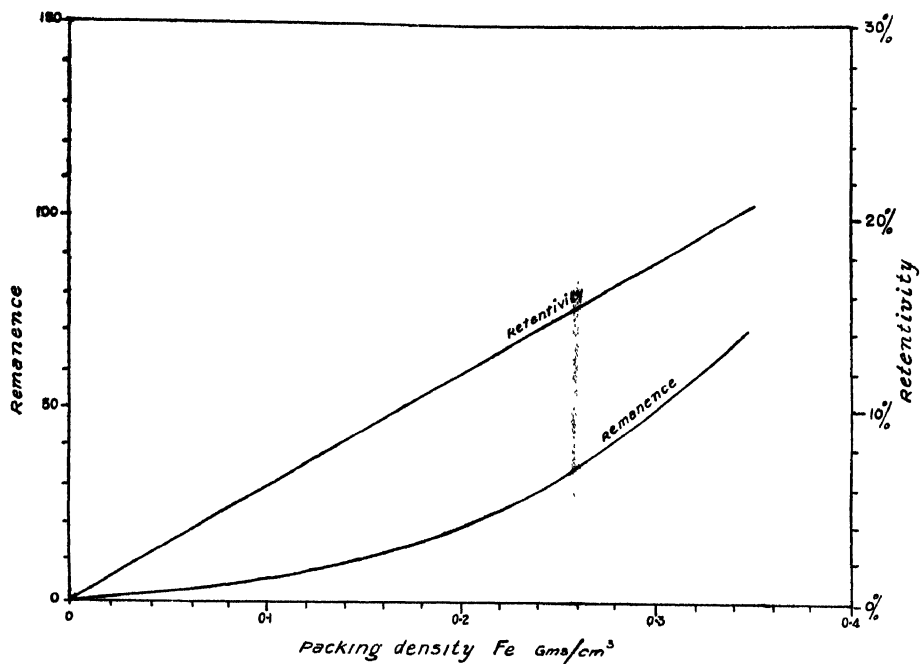


FIG. 5.

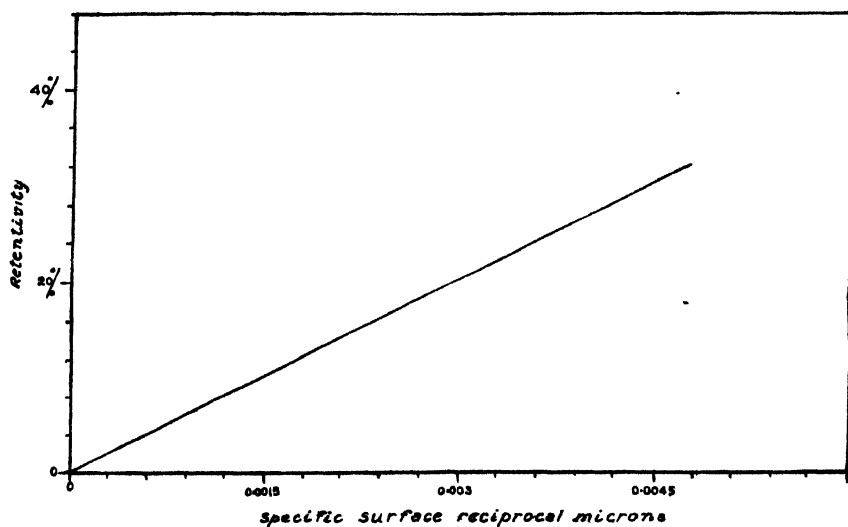


FIG. 6.

Table 3(a) gives the coercive force and remanence values for four fractions from Garividi and Table 3(b) for five fractions from Garbham. The average grain size in microns and the packing densities at which the measurements are made, are also shown in these tables.

TABLE 3(a)

Variation of coercive force, remanence and retentivity with grain size and packing density (Garividi ore samples)

Grain size in microns	Coercive force in Oersteds	Remanence in Gaussess	Retentivity* %	Packing density Fe gms./cm. ³
476	50.0	52.3	13.0	0.4150
264	72.5	60.98	25.45	0.3200
211	82.0	60.98	32.0	0.1806
63	..	21.79

TABLE 3(b)

Variation of coercive force, remanence and retentivity with grain size and packing density (Garbham ore samples)

Grain size in microns	Coercive force in Oersteds	Remanence in Gaussess	Retentivity* %	Packing density Fe gms./cm. ³
476	70.75	16.2	8.3	0.0848
264	71.25	22.7	12.5	0.1056
185	78.75	19.8	17.56	0.0829
143	84.40	20.5	23.1	0.0798
114	86.25	17.0	29.25	0.0694

As can be seen from the Tables 3(a) and 3(b), the coercive force increases as the grain size decreases and that this relation is linear can be seen by plotting the values of H_c against grain size. However, it is found more convenient (Gottschalk, 1935) to plot the coercive force against the specific surface of the powder. Specific surface is the area of the surface divided by weight. For spherical particles, it is $\pi d^2 / \frac{m(\pi d^3)}{6} = \frac{6}{md}$ and for cubic particles $6d^2/md^3 = 6/md$ where m is the density of the particle, d the diameter or the length of the side, as the case may be. The specific surface can thus generally be taken to be inversely proportional to the grain size (k/d).

The nature of the relationship between the coercive force and specific surface can be seen from the fig. 4 where the coercive force is plotted as ordinate against specific surface expressed in reciprocal microns. In fig. 5 is plotted the remanence against the packing density for the (-20+40) mesh fraction from Garividi. The remanence increases with packing density and it is found that the retentivity expressed as $4\pi I$ at zero field divided by $4\pi I$ (max.) expressed per cent, is more linear with the packing density. The retentivity per cent is also plotted in Fig. 5.

A relation between specific surface (expressed in reciprocal microns) and retentivity is shown in fig. 6 after correcting for the variation of packing density. It can be seen from figure 6 that retentivity is a linear function of the specific surface.

* This column gives the retentivity per cent after reducing for fixed packing density (Fe is 0.32 gms./cm.³ for Garividi ore and 0.0694 for the Garbham ore).

DISCUSSION ON RESULTS

The results obtained are summarised as follows :—

- (1) The permeability decreases as the field strength increases ;
- (2) The coercive force increases with decrease in particle size ;
- (3) The remanence increases with decrease in particle size ; and
- (4) With the packing density, the remanence and the intensity of magnetisation both show an increase.

The above results are in conformity with those of earlier workers (Gottschalk and Wartman, 1935).

In the curve showing the variation of permeability with grain size at 40 microns a sudden increase in permeability is observed by Gottschalk and Wartman (1935). This sudden kink is predominant between thirty to hundred oersteds and above that field, the permeability curve is almost horizontal to the grain size axis. In the present investigation the minimum grain size obtained is only 114 microns in the case of the Garividi ore and 63 microns in the case of the Garbham ore. Smaller grain sizes could not be obtained as the proper sieves are not available. So the knee in the permeability grain size curve, which is predominant in Gottschalk's measurements cannot be expected to appear in the curves of the present investigation.

The dependence of the coercive force on the grain size has been proved by the dispersion theory (Dean, Gottschalk and Davis, 1934) of magnetic hardening and also by the magnetic measurements of Nagaoka (1896) on iron amalgam. The observed decrease in coercive force with increasing grain size constitutes additional evidence in favour of the above-mentioned dispersion theory.

The linear variation between retentivity and specific surface is significant and is in conformity with the results of Gottschalk (1935).

These values of magnetic hysteresis measured at low fields represent the lower limits, since the saturation has not been obtained. Another apparatus working on the same principle and suitable for measurements at saturation fields of approximately 1,500 oersteds has been constructed and, it is hoped, it will yield more interesting results.

The nature of the relation of heat treatment and chemical composition to magnetic properties of these ores is also under investigation and the results will be published shortly.

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ABSTRACT

The magnetisation properties of ferruginous manganese ores of Garividi and Garbham mines of Visakhapatnam district have been examined initially, to investigate the possibilities of developing a method for separation of these ores by alternating current magnetic techniques. The method of measurement used here is the oscillographic technique of Bruckshaw and Rao. The results obtained show, in conformity with those of earlier workers, that the hysteresis properties of the mineral powders are susceptible to development, modification and control through the development of the interfacial area of the powders.

REFERENCES

- Bruckshaw, J. Mc. G., and Rao, B. S. R. (1950). Magnetic Hysteresis of Igneous rocks. *Proc. Phy. Soc.*, 63 (B), 931.
- Dean, R. S., Gottschalk, V. H., and Davis, C. W. (1934). Magnetic Separation of minerals. *U.S. Bureau of Mines, R.I. 3223*, 3.
- Gottschalk, V. H., and Wartman, F. S. (1935). Magnetisation curves for magnetite powders. *Ibid.*, *R.I. 3268*, 67.
- Gottschalk, V. H. (1935). Coercive force of magnetite powders. *Ibid.*, *R.I. 3268*, 83.
- Krishnan, M. S. (1951). Mineral Resources of Madras. *Mem. Geol. Surv., Ind.*, 80, 213.
- Low, A. H. (1927). Technical Methods of ore analyses for colleges and Chemists. John Wiley & Sons, Inc., 109 and 146.
- Nagaoka (1896). *Wied. Ann.*, 66, 59.

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ON BALLISTICS OF COMPOSITE CHARGES FOR POWER LAW OF BURNING

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1. INTRODUCTION

The composite charge, which consists of a mixture of grains of two or more nominal sizes with the same or different composition, is of considerable importance in designing successive charges of a gun. The theory of composite charge has been discussed by Corner (1951), Hunt, Hinds and Clemmow (1950) and Venkatesan and Patni (1953).

An approximate solution has been obtained by Corner and Hunt, Hinds and Clemmow by reducing the problem to a single equivalent charge with adjusted propellant parameters. Venkatesan and Patni have given a direct treatment of the problem based on the Hunt-Hinds system of internal ballistics on the following assumptions :

- (i) the covolume of the gases equals the specific volume of each propellant,
- (ii) $\gamma_1 = \gamma_2 = \gamma$, since γ is practically same for all propellants,
- (iii) a linear law of burning : $r = \beta p$.

In this paper, making the above assumptions about covolume and γ and taking a power law of burning, the authors have given a direct treatment of the problem for two tubular ($\theta = 0$) propellants having the same index of rate of burning. Three sets of propellants having the same index of rate of burning, are given below :

Propellant	α	β
W	0.97	1.16
HSC	0.97	1.63
MD	0.91	1.50
NFQ	0.91	0.79
WM	1.05	1.03
ASN	1.05	0.69

The treatment is obviously applicable to a number of cases in which composite charge of the same propellant is used.

2. FUNDAMENTAL EQUATIONS

The fundamental equations with a single charge for a power law of burning have been derived in H.M.S.O. Publication (2) and in case of composite charge they become

$$F_1 C_1 Z_1 + F_2 C_2 Z_2 = p \{A(x+l)\} + \frac{1}{2} (\gamma - 1) w_1 v^2,$$

where

$$w_1 = 1.05w + \frac{C_1}{3} + \frac{C_2}{3}$$

$$w_1 v \frac{dv}{dx} = Ap$$

$$Z_1 = (1-f_1)(1+\theta_1 f_1)$$

$$Z_2 = (1-f_2)(1+\theta_2 f_2)$$

$$D \frac{df_1}{dt} = -\beta_1 p^\alpha$$

$$D \frac{df_2}{dt} = -\beta_2 p^\alpha$$

where

$C_1, F_1, \beta_1, D_1, \theta_1, f_1, Z_1$ refer to the first charge

and

$C_2, F_2, \beta_2, D_2, \theta_2, f_2, Z_2$ refer to the second charge.

Making the following dimensionless transformations,

$$\xi = 1 + \frac{x}{l}$$

$$\eta_1 = \frac{vAD_1}{F_1 C_1 \beta_1} \left(\frac{F_1 C_1}{Al} \right)^{1-\alpha}$$

$$\eta_2 = \frac{vAD_2}{F_2 C_2 \beta_2} \left(\frac{F_2 C_2}{Al} \right)^{1-\alpha}$$

$$\zeta_1 = p \frac{Al}{F_1 C_1}$$

$$\zeta_2 = p \frac{Al}{F_2 C_2}$$

$$M_1 = \frac{A^2 D_1^2}{F_1 C_1 \beta_1^2 w_1} \left(\frac{F_1 C_1}{Al} \right)^{2-2\alpha}$$

$$M_2 = \frac{A^2 D_2^2}{F_2 C_2 \beta_2^2 w_1} \left(\frac{F_2 C_2}{Al} \right)^{2-2\alpha},$$

the fundamental equations in dimensionless variables become

$$Z_1 + \frac{F_2 C_2}{F_1 C_1} Z_2 = \zeta_1 \xi + \frac{\gamma-1}{2M_1} \eta_1^2 \quad \dots \quad \dots \quad \dots \quad (1)$$

$$\eta_1 \frac{d\eta_1}{d\xi} = M_1 \zeta_1 \quad \dots \quad \dots \quad \dots \quad (2a)$$

$$\eta_2 \frac{d\eta_2}{d\xi} = M_2 \zeta_2 \quad \dots \quad \dots \quad \dots \quad (2b)$$

$$Z_1 = (1-f_1)(1+\theta_1 f_1) \quad \dots \quad \dots \quad \dots \quad (3a)$$

$$Z_2 = (1-f_2)(1+\theta_2 f_2) \quad \dots \quad \dots \quad \dots \quad (3b)$$

$$\eta_1 \frac{df_1}{d\xi} = -\zeta_1^\alpha \quad \dots \quad (4a)$$

$$\eta_2 \frac{df_2}{d\xi} = -\zeta_2^\alpha \quad \dots \quad (4b)$$

The variable η_1 can be eliminated quite easily.

Differentiating (1) and using (2a), we have

$$d(Z_1 + RZ_2) = \xi d\zeta_1 + \gamma \zeta_1 d\xi \quad \dots \quad (5)$$

where

$$R = \frac{F_2 C_2}{F_1 C_1}$$

Dividing (2) by (4) we get

$$\frac{d\eta_1}{df_1} = -M_1 \zeta_1^{1-\alpha} \quad \dots \quad (6a)$$

$$\frac{d\eta_2}{df_2} = -M_2 \zeta_2^{1-\alpha} \quad \dots \quad (6b)$$

Equations (4) can be written as,

$$\eta_1 = -\zeta_1^\alpha \frac{d\xi}{df_1} \quad \dots \quad (7a)$$

$$\eta_2 = -\zeta_2^\alpha \frac{d\xi}{df_2} \quad \dots \quad (7b)$$

Equations (6a) and (7a), and (6b) and (7b), then give

$$\frac{d}{df_1} \left(\zeta_1^\alpha \frac{d\xi}{df_1} \right) = M_1 \zeta_1^{1-\alpha} \quad \dots \quad (8a)$$

$$\frac{d}{df_2} \left(\zeta_2^\alpha \frac{d\xi}{df_2} \right) = M_2 \zeta_2^{1-\alpha} \quad \dots \quad (8b)$$

3. CONSTANT BURNING SURFACE WITH COVOLUME COEFFICIENT NEGLECTED.

The energy equation (5) can be written as

$$dZ_1 + RdZ_2 = \xi^{1-\gamma} d(\zeta_1 \xi^\gamma) \quad \dots \quad (9)$$

Also since

$$\theta_1 = \theta_2 = 0$$

$$Z_1 = 1 - f_1$$

$$Z_2 = 1 - f_2$$

so that equations (8) become

$$-\frac{d}{dZ_1} \left(\zeta_1^\alpha \frac{d\xi}{dZ_1} \right) = M_1 \zeta_1^{1-\alpha} \quad \dots \quad (10a)$$

$$-\frac{d}{dZ_2} \left(\zeta_2^\alpha \frac{d\xi}{dZ_2} \right) = M_2 \zeta_2^{1-\alpha} \quad \dots \quad (10b)$$

To find the relation between Z_1 and Z_2 , divide (4a) by (4b) to get

$$\frac{df_1}{df_2} = \frac{D_2\beta_1}{D_1\beta_2} = T \text{ (say).}$$

This at once gives

$$\frac{dZ_1}{dZ_2} = T'$$

$$\text{or } dZ_2 = \frac{1}{T'} dZ_1$$

Equation (9) can be re-written as

$$Q dZ_1 = \xi^{1-\gamma} d(\xi_1 \xi^\gamma)$$

where

$$Q = 1 + \frac{R}{T} = 1 + \frac{F_2 C_2 \beta_2 D_1}{F_1 C_1 \beta_1 D_2}$$

Now make the substitution

$$\xi = \left(\frac{X}{Q}\right)^m \text{ and } \xi_1 \xi^\gamma = \left(\frac{Y}{M_1}\right)^n \quad \dots \quad \dots \quad \dots \quad (11)$$

wherein the values of m and n have yet to be chosen.

Then

$$\xi_1 = \left(\frac{Y}{M_1}\right)^n \left(\frac{X}{Q}\right)^{-\gamma}$$

and

$$Q dZ_1 = n \left(\frac{X}{Q}\right)^{(1-\gamma)m} Y^{n-1} M_1^{-n} dY$$

or

$$dZ_1 = \frac{n \left(\frac{X}{Q}\right)^{(1-\gamma)m}}{Q} Y^{n-1} M_1^{-n} dY \quad \dots \quad \dots \quad \dots \quad (12)$$

On substituting in (10a), and choosing $n = \frac{1}{(3-2\alpha)}$, $m = \frac{2n}{\gamma-n}$, we obtain, after simplifying,

$$XY \frac{d^2 X}{dY^2} - \frac{(\gamma-1)nY}{(\gamma-n)} \left(\frac{dX}{dY}\right)^2 + \frac{1}{2}(1+n) X \frac{dX}{dY} = \frac{1}{2}n(\gamma-n).$$

If further we put

$$Y = \frac{(1+n)}{n(\gamma-n)} Z$$

the resulting equation becomes,

$$\frac{2XZ}{1+n} \frac{d^2 X}{dZ^2} - \frac{2n(\gamma-1)}{(1+n)(\gamma-n)} Z \left(\frac{dX}{dZ}\right)^2 + X \frac{dX}{dZ} = 1 \quad \dots \quad \dots \quad (14)$$

Equation (14) is valid so long as both the charges are burning. The equation is not solvable analytically and only a numerical solution can be obtained by step-by-step methods such as that of Runge and Kutta. The solution depends upon γ and α and the initial conditions, which depend upon Q , representing the

loading conditions. In the solution one tabulates X and $\frac{dX}{dZ}$ corresponding to different values of Z .

We can consider two cases according as the shot-start pressure is finite or zero.

For the first case

$$X_0 = Q$$

and

$$Z_0 = \frac{n(\gamma-n)M}{(1-n)} \zeta_0^{\frac{1}{n}}$$

where the suffix zero refers to the conditions obtaining at the instant of shot-start.

The initial value of $\frac{dX}{dZ}$ is determined from the condition that initially the velocity is zero. By substitution in (4a) we obtain

$$\eta_1 = 2 \left(\frac{X}{Q} \right)^{-\frac{1}{2}n(\gamma-1)} \left[\frac{Z}{\gamma-n} \right]^{\frac{1}{2}(1+n)} \left[\frac{nM}{1+n} \right]^{\frac{1}{2}(1-n)} \frac{dX}{dZ} \dots \dots (15)$$

so that $\frac{dX}{dZ}$ is zero initially.

In the case of zero shot-start pressure, which is considered hereinafter,

$$X_0 = Q \text{ and } Z_0 = 0.$$

The initial value of $\frac{dX}{dZ}$ as obtained from (14) is $\frac{1}{Q}$ and satisfies (15).

The extension of this treatment to the case of finite shot-start pressure presents no difficulty, simply the solution of equation (14), depending as it does on the initial condition, will be different accordingly.

A series solution of (14) for $Z_0 = 0$ can be developed. The expansion up to first four terms is

$$X = Q + \frac{Z}{Q} - \frac{(1-n)(\gamma+n)Z^2}{2(3+n)(\gamma-n)Q^3} \left[1 - \frac{\{(1-n)(7+3n) + (\gamma-1)(7-5n)\}Z}{3(5+n)(\gamma-n)Q^2} \right] + \dots\dots$$

This series helps us in checking the early stages of the calculation of X and $\frac{dX}{dZ}$ in terms of Z and also in finding the value of $\frac{d^2X}{dZ^2}$ for the initial conditions.

4. CONDITIONS FOR SIMULTANEOUS AND NON-SIMULTANEOUS BURNING OUT OF THE TWO CHARGES.

Now two cases arise:

- (i) when both the charges burn out simultaneously,
- (ii) when the two charges burn out at different times.

To determine the conditions for the two cases, we consider the equation

$$dZ_1 = T dZ_2$$

which on integration gives

$$Z_1 = TZ_2$$

(initially, when the burning starts, both Z_1 , and Z_2 are zero).

If both the charges burn out simultaneously then at that instant $Z_1 = Z_2 = 1$ so that the condition for simultaneous burning out is $T = 1$.

When the two charges burn out at different times, suppose, for the sake of definiteness that C_1 burns out first. Then at that instant t_1 , $Z_1 = 1$ and $Z_2 = Z_2, t_1$ so that

$$Z_2, t_1 = \frac{1}{T}$$

since Z_2, t_1 is a positive proper fraction, T must be greater than one. Similarly if C_2 burns out first,

$$T < 1.$$

5. CONSIDERATION OF CASE (i), i.e., $T = 1$

We now proceed to find out the expressions for maximum pressure, muzzle velocity, and the velocity and shot travel at 'all burnt' position.

Maximum Pressure.

We have

$$\zeta_1 = \left[\frac{(1+n)Z}{n(\gamma-n)\eta_1} \right]^n \left(\frac{X}{Q} \right)^{-m};$$

$$\text{for } \zeta_1 \text{ to be maximum, } \frac{d\zeta_1}{dZ} = 0,$$

$$\text{i.e., } \frac{dX}{dZ} = \frac{nX}{\gamma m Z} = \frac{\gamma-n}{2\gamma} \frac{X}{Z}. \quad \dots \dots \dots (16)$$

Equation (16) can be solved numerically. If X_1 , Z_1 be the solution of equation (16), then the maximum pressure is given by

$$p_{\max} = \left[\frac{F_1^2 C_1^2 \rho_1^2 w_1}{A^2 D_1^2 l} \right] \left[\frac{(1+n)Z_1}{n(\gamma-n)} \right]^n \cdot Q^{\gamma m} \cdot 1^{-\gamma m}$$

All-burnt.

From (12) we have

$$\begin{aligned} M_1^n Q^{m(1-\gamma)+1} \cdot Z_1 &= \left[\frac{n+1}{n(\gamma-n)} \right]^n \int_0^{Z^n} X^{-m(\gamma-1)} \cdot d(Z^n) \quad \dots \quad (17) \\ &= I(Z) \end{aligned}$$

where $I(Z)$ is a function which can be tabulated for a specific case.

Equation (17) expresses the fraction Z_1 , of the charge C_1 , burnt. Since $Z_1 = TZ_2 = Z_2$ (in this case) the fraction of the charge C_2 burnt out will also be the same. This fact is otherwise also obvious.

At all-burnt,

$$M_1^n Q^{m(1-\gamma)+1} = I(Z_2)$$

Hence the shot-travel at all-burnt is $l \left\{ \left(\frac{X_2}{Q} \right)^m - 1 \right\}.$

Velocity

$$\begin{aligned}
 v &= \frac{F_1 C_1 \beta_1}{A D_1} \left(\frac{A l}{F_1 C_1} \right)^{1-\alpha} \cdot \eta_1 \\
 &= \frac{F_1 C_1 \beta_1}{A D_1} \left[\frac{A^3 l D_1^2}{F_1^2 C_1^2 \beta_1^2 w_1} \right]^{\frac{1}{2}(1-n)} \cdot Q^{\frac{1}{2}m(\gamma-1)} \cdot V(Z)
 \end{aligned}$$

where

$$V(Z) = \frac{2}{X^{\frac{1}{2}m(\gamma-1)}} \left[\frac{Z}{\gamma-n} \right]^{\frac{1}{2}(1+n)} \left[\frac{n}{1+n} \right]^{\frac{1}{2}(1-n)} \frac{dX}{dZ}$$

and can be tabulated for a specific case.

After all-burnt, $d(Z_1 + RZ_2) = 0$ and from (9) $\zeta_1 \xi^\gamma$ is constant and equal to its value at X_2, Z_2 . The energy equation then gives

$$\frac{1}{2}(\gamma-1) \frac{\eta_1^2}{M_1} = 1 - \zeta_1 \xi = 1 - \left[\frac{(1+n)Z_2}{n(\gamma-n)M_1} \right] \cdot \xi^{1-\gamma}$$

which determines the muzzle velocity.

The solution in this case is thus seen to be based on three numerical tables viz, $X(Z)$, $I(Z)$, $V(Z)$.

6. CONSIDERATION OF CASE (ii), i.e., $T > 1$

When C_1 burns out first, $Z_1 = 1$ and (16) gives

$$M_1^n Q^{m(1-\gamma)+1} = I(Z_{t_1})$$

Corresponding to Z_{t_1} , we can find X_{t_1} and $\left(\frac{dX}{dZ} \right)_{t_1}$ from the tables for case (i).

X_{t_1} , Z_{t_1} , and $\left(\frac{dX}{dZ} \right)_{t_1}$ determine the distance travelled (x_{t_1}), pressure behind (p_{t_1}) and the velocity of the shot (v_{t_1}) at that instant. Also from the relation $Z_1 = TZ_2$, we find that the value of Z_2 at that instant is $\frac{1}{T}$ so that the remaining charge (C_2, t_1)

and web (D_2, t_1) of C_2 are $C_2 \left(1 - \frac{1}{T} \right)$ and $D_2 \left(1 - \frac{1}{T} \right)$ respectively.

In order to obtain the final solution we have to consider the single charge C_2 with the above initial conditions. We get the differential equation (14) with the following transformation for X and Z :—

$$\begin{aligned}
 X &= \xi^{\frac{\gamma-n}{2n}} \\
 Z &= \frac{n(\gamma-n)}{1+n} M(\xi^\gamma)^{\frac{1}{n}},
 \end{aligned}$$

where

$$\begin{aligned}\xi &= \left(1 + \frac{x}{l}\right) \\ \zeta &= \frac{pAl}{F_2 C_2, t_1} \\ \eta &= \frac{vAD_{2, t_1}}{F_2 C_2, t_1 \beta_2} \left(\frac{F_2 C_2, t_1}{Al}\right)^{1-\alpha} \\ M &= \frac{A^2 D_{2, t_1}^2}{F_2 C_2, t_1 \beta_2^2 w_1} \left(\frac{F_2 C_2, t_1}{Al}\right)^{2-2\alpha}\end{aligned}$$

Equation (14) is solved numerically with given initial values of X , Z , and $\frac{dX}{dZ}$, known from x_{t_1} , p_{t_1} and v_{t_1} . The solution is tabulated as before and in order to find pressure, shot travel and velocity at any instant after C_1 is burnt out, we have to proceed in the same way as indicated earlier.

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ABSTRACT

This communication presents a direct treatment, based on Hunt-Hinds system of internal ballistics, for composite charges consisting of two tubular propellants with the same index of rate of burning.

REFERENCES

- Corner, J. (1950). *Theory of Internal Ballistics of Guns*. John Wiley, New York.
 H. M. Stationery Office (1951). *Internal Ballistics*, 1951.
 Venkatesan N. S. and Patni, G. C. (1953). Influence of composite charges on maximum pressure and muzzle velocity. *Def. Sci. Jour.*, 3, Jan.

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EFFECT OF SHOT-START PRESSURE ON MUZZLE VELOCITY, MAXIMUM PRESSURE AND ALLIED QUANTITIES FOR COMPOSITE CHARGES

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1. INTRODUCTION

The main problem of Internal Ballistics is the calculation of muzzle velocity and maximum pressure for given loading conditions in a gun. Various methods have been devised for the solution of the equations of internal ballistics. Hunt-Hinds system (1951) is based on the characteristic assumptions (i) the linear rate of burning, and (ii) the circumstances of band engraving may be represented by the use of shot-start pressure (the pressure at which the shot starts its motion). Venkatesan (1952) obtained explicit expression for the maximum pressure in terms of shot-start pressure for $\theta = 0$, considering a single charge, which the author of this paper generalised for all values of θ (Aggarwal, 1954). Further Mehta and Aggarwal have recently given the 'Dependence of Muzzle Velocity and Allied Quantities on Shot-start Pressure'.

In this paper the author has derived explicit relations between muzzle velocity and shot-start pressure; and, maximum pressure and shot-start pressure for the more complicated case of composite charges. The shot-velocity, pressure and shot-travel at the point where one charge is burnt out, when two charges burn out at different times, at all burnt and also for any position of the shot after all burnt, have been expressed as explicit functions of shot-start pressure. These relations have been obtained for tubular propellants for a linear law of burning with the usual assumption of neglecting covolume correction.

2. BASIC EQUATIONS

The basic equations of the internal ballistic, for tubular composite charges, are

$$\frac{F_1 C_1 Z_1 + F_2 C_2 Z_2}{Al} = p \left(1 + \frac{x}{l} \right) + \frac{\gamma - 1}{2Al} w_1 v^2 \quad \dots \quad (1)$$

where

$$w_1 = 1.05w + \frac{C_1}{3} + \frac{C_2}{3}$$

$$w_1 \frac{dv}{dt} = Ap \quad \dots \quad (2)$$

$$Z_1 = (1 - f_1) \quad \dots \quad (3a)$$

$$Z_2 = (1 - f_2) \quad \dots \quad (3b)$$

$$D_1 \frac{df_1}{dt} = -\beta_1 p \quad \dots \quad (4a)$$

$$D_2 \frac{df_2}{dt} = -\beta_2 p \quad \dots \quad (4b)$$

where $C_1, F_1, \beta_1, D_1, f_1, Z_1$ refer to first charge, and $C_2, F_2, \beta_2, D_2, f_2, Z_2$ refer to second charge.

The initial conditions at the shot-start are

$$x = 0; v = 0; f_1 = f_{10}; f_2 = f_{20}; Z_1 = Z_{10}; Z_2 = Z_{20}$$

Hence from equation (i) we see that,

$$\frac{F_1 C_1 Z_{10} + F_2 C_2 Z_{20}}{Al} = p_0,$$

where p_0 is the shot-start pressure

$$\text{or} \quad Z_{10} + \frac{F_2 C_2}{F_1 C_1} Z_{20} = \frac{Al}{F_1 C_1} p_0 \quad \dots \quad (5)$$

From (4a) and (4b) we get,

$$\frac{df_1}{df_2} = \frac{\beta'}{\beta''} \quad \text{where } \beta' = \frac{\beta_1}{D_1}, \quad \beta'' = \frac{\beta_2}{D_2}.$$

Also from (3a) and (3b) we get,

$$\frac{dZ_1}{dZ_2} = \frac{df_1}{df_2} = \frac{\beta'}{\beta''} = T \text{ (say)}$$

$$\therefore dZ_1 = T dZ_2.$$

Integrating this equation and applying the initial conditions we get,

$$Z_1 = T Z_2.$$

Equation (5) becomes,

$$Z_{10} + \frac{R}{T} Z_{10} = \frac{Al}{F_1 C_1} p_0$$

where

$$R = \frac{F_2 C_2}{F_1 C_1}$$

or

$$Q Z_{10} = \frac{Al}{F_1 C_1} p_0, \quad Q = 1 + \frac{R}{T}$$

or

$$Z_{10} = \frac{Al}{Q F_1 C_1} p_0$$

This shows that Z_{10} represents the shot-start pressure.

We can also transform Z_{10} in terms of Z_{20} by the relation

$$Z_{10} = T Z_{20}$$

Now we will express muzzle velocity, maximum pressure and other quantities in terms of Z_{10} or Z_{20}

3. CASE I. WHEN BOTH THE CHARGES BURN OUT SIMULTANEOUSLY, I.E., $T = 1$

Dividing equation (2) by (4a) and integrating we get,

$$\bullet \quad v = \frac{A}{\beta' w_1} (f_{10} - f_1) \quad \dots \quad (8)$$

$$\text{or} \quad v = \frac{A}{\beta' w_1} [Z_1 - Z_{10}]$$

At the position of all burnt $Z_1 = 1$

$$\therefore v_b = \frac{A}{\beta' w_1} [1 - Z_{10}] \quad \dots \quad (8a)$$

This expresses all burnt velocity as an explicit function of Z_{10} the shot-start pressure.

Venkatesan has given the relation between shot-travel and velocity as,

$$\xi = \left[\left(\frac{a}{a-v} \right)^a \left(\frac{b}{b+v} \right)^b \right]^{\frac{1}{k(a+b)}} \quad \dots \quad (9)$$

where

$$k = \frac{\gamma-1}{2}$$

$$(a-b) = \frac{2}{\gamma-1} \left[\frac{F_1 C_1 \beta'}{A} + \frac{F_2 C_2 \beta''}{A} \right]$$

and

$$ab = \frac{2}{\gamma-1} \frac{F_1 C_1 Z_{10} + F_2 C_2 Z_{20}}{w_1}$$

At all burnt, equation (9) becomes

$$\xi_b = \left[\left(\frac{a}{a-v_b} \right)^a \left(\frac{b}{b+v_b} \right)^b \right]^{\frac{1}{k(a+b)}} \quad \dots \quad (10)$$

Substituting the value of v_b from (8) we get an explicit expression between shot-travel at all burnt and shot-start pressure.

$$\text{Also} \quad p = \frac{w_1 k(a-v)(b+v)}{A l \xi}$$

This becomes at all burnt,

$$p_b = \frac{w_1 k(a-v_b)(b+v_b)}{A l \xi_b} \quad \dots \quad (11)$$

Substituting the value of v_b and ξ_b we get pressure at all burnt as an explicit function of shot-start pressure.

4. MUZZLE-VELOCITY IN TERMS OF SHOT-START PRESSURE

We have expressed x_b , p_b , v_b earlier as explicit functions of shot-start pressure Z_{10} . After all burnt, the motion of the shot is governed by the propellant gases behind it. Let x , v , b be the shot-travel, velocity and pressure at any instant after all burnt. Since we are neglecting heat losses the expansion of the gas will be adiabatic.

$$\therefore p_b (A \xi_b)^\gamma = p (A \xi)^\gamma$$

$$\text{or} \quad p = \frac{p_b \xi_b^\gamma}{\xi^\gamma} \quad \dots \quad (12)$$

The dynamical equation of the shot is,

$$v \frac{dv}{d\xi} = \frac{Al}{w_1} p.$$

$$\text{or } v dv = \frac{Al p_b \xi_b^\gamma}{w_1} \frac{d\xi}{\xi^\gamma}, \text{ from (12)}$$

which gives on integration,

$$v^2 - v_b^2 = \frac{2Al p_b \xi_b^\gamma}{(1-\gamma)w_1} (\xi^{1-\gamma} - \xi_b^{1-\gamma}) \quad \dots \quad \dots \quad \dots \quad (13)$$

At the muzzle

$$\xi = \xi_s = 1 + \frac{x_s}{l} \text{ and } v = v_s.$$

We have,

$$v_s^2 = v_b^2 + \frac{2Al p_b \xi_b^\gamma}{(1-\gamma)w_1} (\xi_s^{1-\gamma} - \xi_b^{1-\gamma}) \quad \dots \quad \dots \quad \dots \quad (14)$$

Substituting the value of v_b , ξ_b , p_b which have been expressed as explicit functions of shot-start pressure, we get muzzle velocity v_s as an explicit function of short-start pressure.

5. CASE II. WHEN THE TWO CHARGES BURN OUT AT DIFFERENT TIMES

In this case let us suppose that charge C_1 burns out first and let us represent with suffix (2, 1) the position at which C_1 is just burnt out.

The velocity at the point where C_1 is just burnt out is given by (8), as

$$\begin{aligned} v_{2,1} &= \frac{A}{\beta' w_1} f_{10} \\ &= \frac{A}{\beta' w_1} (1 - Z_{10}) \quad \dots \quad \dots \quad \dots \quad (15) \end{aligned}$$

This equation (15) gives $v_{2,1}$ as explicit function of shot-start pressure. The shot-travel $\xi_{2,1}$ is given by equation (9) as,

$$\xi_{2,1} = \left[\left(\frac{a}{a - v_{2,1}} \right)^a \left(\frac{b}{b + v_{2,1}} \right)^b \right]^{\frac{1}{k(a+b)}} \quad \dots \quad \dots \quad \dots \quad (16)$$

This equation gives $\xi_{2,1}$ as an explicit function of shot-start pressure. The pressure $p_{2,1}$ is given by equation (11) as,

$$p_{2,1} = \frac{w_1 k(a - v_{2,1})(b + v_{2,1})}{Al \xi_{2,1}} \quad \dots \quad \dots \quad \dots \quad (17)$$

This equation gives $p_{2,1}$ as an explicit function of shot-start pressure.

So we have expressed $v_{2,1}$, $\xi_{2,1}$ and $p_{2,1}$ as explicit functions of shot-start pressure.

When C_1 is burnt out and only C_2 is burning we have from (2) and (4b),

$$v = \frac{A}{\beta' w_1} (f_{20} - f_2) \quad \dots \quad \dots \quad \dots \quad (18)$$

Let us denote the position when C_2 is also burnt out by suffix (2) and dashes. The velocity at all burnt is given by,

$$\begin{aligned} v'_2 &= \frac{A}{\beta'' w_1} f_{20} \\ &= \frac{A}{\beta'' w_1} [1 - Z_{20}] \\ &= \frac{A}{\beta'' w_1} \left[1 - \frac{Z_{10}}{T} \right] \quad \dots \quad \dots \quad \dots \quad (19) \end{aligned}$$

This represents v'_2 explicitly in terms of shot-start pressure Z_{10} . During the time when only charge C_2 is burning, Venkatesan has shown,

$$\xi = \xi_{2,1} \left[\left(\frac{a_1 - v_{2,1}}{a_1 - v} \right)^{a_1} \left(\frac{b_1 + v_{2,1}}{b_1 + v} \right)^{b_1} \right]^{\frac{1}{k_1(a_1 + b_1)}} \quad \dots \quad \dots \quad (20)$$

where

$$\begin{aligned} k_1 &= \frac{\gamma - 1}{2} \\ a_1 - b_1 &= \frac{1}{k_1} \frac{F_2 C_2 \beta''}{A} \\ a_1 b_1 &= \frac{F_1 C_1 + F_2 C_2 Z_{20}}{w_1 k_1} \end{aligned}$$

At all burnt position,

$$\xi'_2 = \xi_{2,1} \left[\left(\frac{a_1 - v_{2,1}}{a_1 - v'_2} \right)^{a_1} \left(\frac{b_1 + v_{2,1}}{b_1 + v'_2} \right)^{b_1} \right]^{\frac{1}{k_1(a_1 + b_1)}} \quad \dots \quad \dots \quad (20a)$$

Equation (20a) represents ξ'_2 in terms of $\xi_{2,1}$, $v_{2,1}$ which are all explicitly expressed in terms of shot-start pressure. Hence ξ'_2 can be represented explicitly as a function of short-start pressure.

Also,

$$p = \frac{w_1 k_1 (a_1 - v)(b_1 + v)}{A l \xi}$$

At all burnt,

$$p'_2 = \frac{w_1 k_1 (a_1 - v'_2)(b_1 + v'_2)}{A l \xi'_2} \quad \dots \quad \dots \quad \dots \quad (21)$$

This represents p'_2 explicitly as functions of shot-start pressure.

After all burnt we have to proceed exactly as in section 4, and we get, velocity, pressure and shot-travel after all burnt explicitly as functions of shot-start pressure, and

$$v_3^2 = v_2^2 + \frac{2 A l p'_2 \xi_2^{\gamma}}{(1 - \gamma) w_1} (\xi_3^{1 - \gamma} - \xi_2^{1 - \gamma}) \quad \dots \quad \dots \quad \dots \quad (22)$$

Since on the right hand side v'_2 , p'_2 and ξ'_2 are explicit functions of shot-start pressure, this expresses muzzle velocity as an explicit function of shot-start pressure.

6. MAXIMUM PRESSURE

The following cases may arise when maximum pressure occurs.

- (a) Both the charges are burning.
- (b) C_1 is burnt out and C_2 is burning.
- (c) At the position of 'all burnt'.

Case (a)

Venkatesan has given that,

$$p_1 = \frac{w_1 k(a-v_1)(b+v_1)}{Al \xi_1} \text{ where } v_1 = \frac{k(a-b)}{2k+1}$$

and ξ_1 is the value of ξ obtained by putting $v = v_1$ in equation (9).

Now

$$\begin{aligned} v_1 &= \frac{k(a-b)}{2k+1} \\ &= \frac{\gamma-1}{2\gamma} (a-b) \\ &= \frac{1}{\gamma} \left[\frac{F_1 C_1 \beta'}{A} + \frac{F_2 C_2 \beta''}{A} \right] \\ &= \frac{F_1 C_1 \beta'}{A\gamma} \left[1 + \frac{F_2 C_2 \beta''}{F_1 C_1 \beta'} \right] \\ &= \frac{Q F_1 C_1 \beta'}{A\gamma} \quad (\text{a constant}). \end{aligned}$$

ξ_1 is an explicit function of shot-start pressure. Here p_1 is expressed as an explicit function of shot-start pressure.

Case (b)

$$p_1 = \frac{w_1 k(a_1-v_1)(b_1+v_1)}{Al \xi_1}, \text{ where } v_1 = \frac{k_1(a_1-b_1)}{2k_1+1} \text{ which is a constant but } \xi_1 \text{ is}$$

a function of shot-start pressure. ξ_1 is obtained by putting $v = v_1$ in equation (20).

Hence p_1 is expressed as an explicit function of shot-start pressure.

Case (c)

In this case the maximum pressure can occur (i) when both the propellants are burning or (ii) at the point of all burnt.

Case (i) has already been dealt in case (a).

Case (ii) The condition for this case are,

$$f_{11} = 0, \quad f_{21} = 0$$

$$\text{which reduce to } f_{10} = \frac{\beta' w_1 k(a-b)}{A(2k+1)}$$

$$f_{20} = \frac{\beta'' w_1 k(a-b)}{A(2k+1)}$$

and the maximum pressure in this case can be expressed easily in terms of shot-start pressure.

7. SUMMARY

In this paper, the author has derived explicit relations, between muzzle velocity and shot-start pressure; and, maximum pressure and shot-start pressure. The shot-velocity, pressure and shot-travel at the point where one charge is burnt out, when two charges burn out at different times, at all burnt and also for any position of the shot after all burnt have been expressed as explicit functions of shot-start pressure. These relations have been obtained for tubular propellant for a linear law of burning with the usual assumption of neglecting covolume correction.

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9. REFERENCES

- Aggarwal, S. P. (1954). Relation between Maximum Pressure and Shot-start Pressure. *Proc. Nat. Inst. Sci.*, **20**, 307-317.
- Aggarwal, S. P. and Mehta, A. K. Dependence of Muzzle-Velocity and Allied quantities on Shot-start Pressure. (*In press*).
- Corner, J. (1950) Theory of Internal Ballistics of Guns. John Willy, New York.
- H. M. Stationery Office. (1951). Internal Ballistics.
- Venkatesan, N. S. and Patni G. C. (1953). Influence of Composite Charges on Maximum Pressure and Muzzle Velocity. *Def. Sci. Journ.*, **3**, Jan.
- Venkatesan, N. S. (1952). A Note on the Relation between Maximum Pressure and Shot-start Pressure. *Proc. Nat. Inst. Sci.*, **18**, 265-272.

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THE EFFECT OF EXCHANGE POTENTIAL ON THE MASS-RADIUS RELATION FOR COLD BODIES

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1. In recent years, the internal constitution of the planets and white dwarf stars has received considerable attention. An excellent review has been made by Wildt (1947). The work has been further extended by Ramsey (1948, 1950 and 1951), Bullen (1949*a*, 1949*b*, and 1950), Brown (1950), Mestel (1952*a* and 1952*b*) and others.

We, in this paper, propose to investigate the mass-radius relationship of spherical *cold** bodies, taking into account the electron-exchange potential.

Kothari (1938), on the basis of the theory of pressure-ionization, calculated the mass-radius relationship of cold bodies. He assumed that the material is composed of atoms of one kind only. Regarding the chemical composition of the material, the following alternative assumptions were made:

- (i) The material is hydrogen: referred to as assumption *H*.
- (ii) The material is iron: referred to as assumption *F*.

It is satisfactory to note that the known mass-radius values of the planets are in the region bounded by the curves drawn on the assumptions *H* and *F*. The theory led to a maximum radius for cold bodies and it also seemed to indicate that the two major planets, Saturn and Jupiter, are predominantly hydrogen planets. However, Kothari's theory does not take into account the exchange interaction between the free electrons and it also neglects the internal temperatures of the cold bodies.

2. Let us consider a completely degenerate (i.e. temperature is zero) electron gas with nV electrons in volume V , half of them of each spin. It is assumed that the volume is filled with a uniform distribution of positive charge to make it electrically neutral. The maximum momentum of the Fermi-distribution is given by:

$$P_0 = h \left(\frac{3n}{8\pi} \right)^{\frac{1}{3}}$$

where h is Planck's constant.

The exchange potential can be conveniently stated in terms of the ratio $\eta = P/P_0$, P being the magnitude of the momentum of the electron. Dirac (1930) found that:

$$\text{the exchange potential energy} = -e^2(4P_0/h) F(\eta) \quad \dots \quad (1)$$

$$\text{where} \quad F(\eta) = \frac{1}{2} + \frac{1-\eta^2}{4\eta} \ln \left[(1+\eta)/(1-\eta) \right] \quad \dots \quad (1')$$

* Matter will be considered *cold* or *degenerate* whenever the free electrons present form a degenerate gas.

Slater (1951) suggested that we may use this result to produce a good approximation to the exchange term in the Hartree-Fock equation by averaging over the momenta P of the various electrons. This gives,

$$\text{the exchange potential energy} = -3e^2 \left(\frac{3n}{8\pi} \right)^{\frac{1}{2}}. \quad \dots \quad (2)$$

For a degenerate electron gas at temperature θ , Zirin (1953) found that the exchange potential energy is given by:

$$\text{the exchange potential} = -3e^2 \beta \left(\frac{3n}{8\pi} \right)^{\frac{1}{2}}, \quad \dots \quad (3)$$

where β is a function of $(k\theta/\epsilon_0)$, k being the Boltzmann constant and ϵ_0 the energy of the top of the Fermi band.

We will use the expressions given by equations (2) and (3) later on.

3. The Virial Theorem states that for an assembly of particles interacting according to the inverse square law of force,

$$2T + W = 3pV, \quad \dots \quad (4)$$

where T is the total kinetic energy of all the particles, W the total potential energy, V the volume and p the external pressure to which the assembly is subject.

Let us consider materials composed of atomic weight A and atomic number Z , compressed to such an extent that pressure-ionization takes place. Dividing the material into similar spherical cells, each containing a nucleus of charge Ze at its centre and Z electrons uniformly distributed throughout the cell, we have the radius of the cell a in terms of the density ρ ,

$$a = \left(\frac{\gamma_1 A m_H}{\rho} \right)^{\frac{1}{3}}, \quad \dots \quad (5)$$

where m_H is the mass of an hydrogen atom, and γ_1 is a factor of the order of unity. Its exact value depends on many factors—particularly on lattice structure.

Now, we will derive, first the mass-radius relationship of a spherical aggregate of cold matter, taking into account exchange energy at zero temperature and then applying second approximation, we will derive the above relationship for higher temperatures.

4. *Mass-Radius relation at zero-temperature.*—Let us estimate the total kinetic energy T . The kinetic energy of a degenerate gas, for zero-temperature, is given by the usual relation:

$$E_0 = \frac{3}{10} \frac{h^2}{m} \left(\frac{3n^*}{8\pi} \right)^{\frac{2}{3}} N, \quad \dots \quad (6)$$

where n^* is the total electron concentration, m the mass of an electron, and N the total number of electrons. Substituting $Z/\gamma_2 a^3$ for n^* , Z for N and eliminating a with the help of equation (5), the kinetic energy per cell is given by:

$$T' = \frac{3}{10} \frac{h^2}{m} \left(\frac{3}{8\pi} \frac{Z\rho}{\gamma_1 \gamma_2 A m_H} \right)^{\frac{2}{3}} Z. \quad \dots \quad (7)$$

Here γ_2 is a factor of the order of unity. Multiplying T' by $(\rho V/A m_H)$, the total number of cells, the total kinetic energy is given by:

$$T = \frac{3}{10} \frac{h^2}{m} \left(\frac{3}{8\pi} \frac{Z\rho}{\gamma_1 \gamma_2 A m_H} \right)^{\frac{2}{3}} Z \frac{\rho V}{A m_H}. \quad \dots \quad (8)$$

The total potential energy W is the sum of two parts— W_P being the electrostatic potential energy and W_E the potential energy due to the exchange of electrons. We will assume that the total potential energy is obtained by multiplying the energy per cell by the total number of cells. It is not easy to calculate W_P accurately. Approximately, we get:

$$\begin{aligned} -W'_P &= \int_0^a \frac{\left(Ze - \frac{4\pi}{3} x^3 n^* e \right)}{x} 4\pi x^2 n^* e \, dx \\ &= \frac{9}{10} \frac{Z^2 e^2}{a}, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (9) \end{aligned}$$

and hence, eliminating a with the help of equation (5) and multiplying by $(\rho V / Am_H)$, we get,

$$-W_P = \frac{9}{10} Z^2 e^2 \left(\frac{\rho}{\gamma_1 Am_H} \right)^{\frac{1}{3}} \frac{\rho V}{Am_H} \quad \dots \quad \dots \quad \dots \quad (10)$$

The expression for the electron exchange potential energy at zero temperature, as calculated by Slater, is given by equation (2):

$$\text{exchange potential energy} = -3e^2 \left(\frac{3n}{8\pi} \right)^{\frac{1}{3}} \quad \dots \quad \dots \quad (2)$$

For the sake of simplicity, we will take $n^* = Z / \frac{4\pi}{3} a^3$, the concentration of total electrons in place of n , the concentration of free electrons, in the above expression. Substituting this in the above expression and multiplying it with $(\rho V / Am_H)$, the total exchange energy is given by

$$-W_E = 3e^2 \left(\frac{9}{32\pi^2 \gamma_1 Am_H} \right)^{\frac{1}{3}} \frac{\rho V}{Am_H} \quad \dots \quad \dots \quad \dots \quad (11)$$

Comparing W_P and W_E we find that,

$$\frac{W_E}{W_P} = \frac{5}{(12\pi^2)^{\frac{1}{3}}} \frac{1}{Z^{\frac{1}{3}}} \quad \dots \quad \dots \quad \dots \quad \dots \quad (12)$$

Thus, we see that in the case of hydrogen the ratio W_E/W_P will be maximum, i.e. it will be nearly equal to unity and will be smaller the higher the atomic number of the element is. In the case of iron, it will be equal to about 1/228. Hence, exchange potential energy cannot be neglected for hydrogen and other low atomic number elements.

The pressure in degenerate matter effectively depends on the free-electron concentration and is given, at zero temperature, by the expression:

$$p = \frac{8\pi}{15} \frac{\hbar^2}{m} \left(\frac{3n}{8\pi} \right)^{\frac{5}{3}} \quad \dots \quad \dots \quad \dots \quad \dots \quad (13)$$

where n is the number of free-electrons per unit volume.

Let us define μ , the mean molecular weight per free electron, by the relation:

$$\mu = \frac{\rho}{\mu m_H} \quad \dots \quad \dots \quad \dots \quad \dots \quad (14)$$

If the material is r times ionized,

$$n = \frac{r\rho}{Am_H}, \text{ and } \mu = \frac{A}{r}. \quad \dots \dots \dots (15)$$

Thus, the value of μ gives a measure of the degree of ionization. If the matter is fully ionized, $\mu = \mu_0 = A/Z$. Eliminating n between equations (13) and (14) we have:

$$p = K \frac{\rho^{\frac{5}{3}}}{\mu^{\frac{5}{3}}} \quad \dots \dots \dots (16)$$

where

$$K = \frac{8\pi h^2}{15 m} \left(\frac{3}{8\pi m_H} \right)^{\frac{5}{3}}. \quad \dots \dots \dots (16')$$

Substituting equations (8), (10), (11) and (16) in equation (4), we get:

$$\mu = \frac{\mu_0(\gamma_1\gamma_2)^{\frac{2}{3}}}{\left[1 - \frac{(\gamma_1\gamma_2)^{\frac{1}{3}}}{\gamma_1^{\frac{1}{3}}} \left(\frac{\Delta AZ}{\rho} \right)^{\frac{1}{3}} \eta_z \right]^{\frac{3}{2}}}, \quad \dots \dots \dots (17)$$

where

$$\Delta = \left[2 \cdot 3^{\frac{1}{3}} \pi^{\frac{1}{3}} m e^2 m_H^{\frac{1}{3}} / h^2 \right]^{\frac{3}{2}}, \quad \dots \dots \dots (17')$$

$$\mu_0 = A/Z, \quad \dots \dots \dots (17'')$$

and

$$\eta_z = 1 + \frac{5}{(12\pi^2 Z^5)^{\frac{1}{3}}}. \quad \dots \dots \dots (17''')$$

As a rough approximation, put γ_1 and γ_2 equal to unity. Then equation (17) reduces to:

$$\mu = \frac{\mu_0}{\left[1 - \left(\frac{\Delta AZ}{\rho} \right)^{\frac{1}{3}} \eta_z \right]^{\frac{3}{2}}}. \quad \dots \dots \dots (18)$$

If ρ^* denotes the density of the material in a singly ionized state, then

$$\rho^* = \frac{(\Delta AZ)\eta_z^{\frac{3}{2}}}{(1 - 1/Z^{\frac{1}{3}})^{\frac{3}{2}}}. \quad \dots \dots \dots (19)$$

Hence, we may write the equation (18) as,

$$\mu = \frac{\mu_0}{\left[1 - \left(\frac{\rho^*}{\rho} \right)^{\frac{1}{3}} \left(1 - \frac{1}{Z^{\frac{1}{3}}} \right) \right]^{\frac{3}{2}}}. \quad \dots \dots \dots (20)$$

Now we will consider the case of pressure-ionization in hydrogen and in iron. For a given value of μ the density ρ can be calculated by the help of equation (18), and knowing the value of μ and ρ , the pressure p can easily be found out by equation (16). Figure 1 gives the plot of $\log \rho - \log p$ in the case of hydrogen and iron. The dotted curve gives the plot of $\log \rho - \log p$ in the case of hydrogen, when exchange potential is neglected, i.e. when $\eta_z = 1$; the corresponding curve for iron does not differ appreciably from the case when exchange potential is taken into account as is evident from Table I. The ladder of μ values has been marked on the curves. The straight lines $\mu = 1$ and $\mu = 56/26$, would give the relation between $\log \rho - \log p$ at complete ionization, i.e., when $\mu = A/Z$, for hydrogen and iron, respectively. The $\log p - \log \rho$ values for *atomic* or *metallic* hydrogen as obtained by Ramsey (1951) are indicated by crosses (\times).

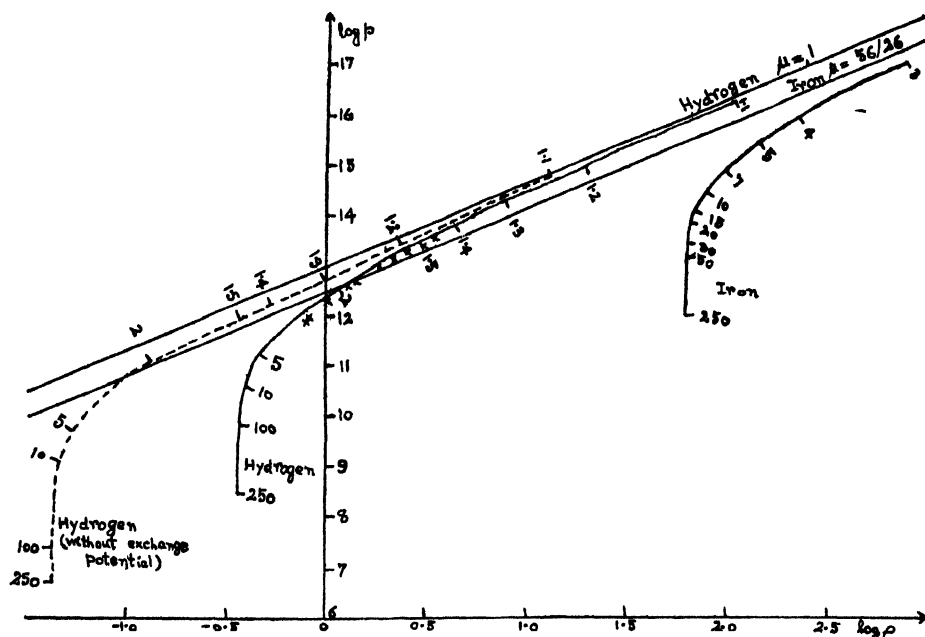


FIG. 1. The continuous curves show the relation between pressure and density of hydrogen and of iron, when exchange potential energy is taken into account. The dotted curve gives pressure-density relation of hydrogen, when exchange term is neglected. The ladder of μ values has been marked along the curves.

The straight lines in the figure would represent the pressure-density relation if the ionization did not vary with density but was complete (i.e. $\mu = A/Z$) at all densities, for the case of hydrogen and iron.

Ramsey's (1951) calculated values of pressure-density for metallic hydrogen, have been marked by crosses (X).

TABLE I

Hydrogen					Iron				
μ	Without exchange potential		With exchange potential		μ	Without exchange potential		With exchange potential	
	$\log \rho$	$\log p$	$\log \rho$	$\log p$		$\log \rho$	$\log p$	$\log \rho$	$\log p$
250	2.637	6.728	1.552	8.252	250	1.800	12.000	1.806	12.010
100	2.637	7.392	1.552	8.916	50	1.807	13.176	1.813	13.185
10	2.665	9.105	1.580	10.629	30	1.816	13.561	1.822	13.571
5	2.729	9.713	1.644	11.238	20	1.832	13.881	1.838	13.891
2	1.130	11.044	0.044	12.568	15	1.852	14.123	1.858	14.132
1.5	1.563	11.974	0.478	13.499	10	1.905	14.504	1.911	14.514
1.4	1.739	12.151	0.654	13.842	7	1.997	14.916	2.003	14.925
1.3	1.990	12.789	0.905	14.314	5	2.167	15.443	2.173	15.453
1.2	0.361	13.500	1.296	15.024	4	2.374	15.949	2.380	15.959
1.1	1.136	14.820	2.050	16.344	3	2.917	17.062	2.923	17.072

We may approximately regard,

$$\rho = \frac{3M}{4\pi R^3}, \quad \dots \dots \dots (21)$$

where M is the mass and R the radius of the spherical aggregate of matter.

Then equation (18) reduces to

$$\dots = \frac{\mu_0}{\left[1 - \left(\frac{4\pi}{3\odot}\right)^{\frac{1}{2}} (\Delta AZ)^{\frac{1}{2}} \left(\frac{\odot}{M}\right)^{\frac{1}{2}} R \eta_z\right]^{\frac{1}{2}}}, \quad \dots \dots (22)$$

where \odot is the mass of the sun.

The radius of a spherical aggregate of cold matter of mass M in equilibrium under its own gravitational forces is given by the well-known relation due to Milne (1932), (neglecting the relativistic mechanics effect, which is justified so long as $M < \odot$),

$$R = \frac{l}{\mu_0^{\frac{1}{2}}} \left(\frac{\odot}{M}\right)^{\frac{1}{2}}, \quad \dots \dots (23)$$

where

$$l = \frac{5(\omega_{3/2}^0)^{\frac{1}{2}} K}{2^{\frac{1}{2}} \pi^{\frac{1}{2}} G \odot} = 2.79 \times 10^9 \text{ cm.} \quad \dots \dots (23')$$

Here $\omega_{3/2}^0$ is a constant (2.1219) characteristic of Emden's solution of Emden's equation of index $3/2$ and G is the constant of Gravitation.

Eliminating μ from equations (22) and (23) we get

$$R = \frac{\frac{l}{\mu_0^{\frac{1}{2}}} \left(\frac{\odot}{M}\right)^{\frac{1}{2}}}{1 + \frac{l}{\mu_0^{\frac{1}{2}}} \left(\frac{4\pi}{3\odot}\right)^{\frac{1}{2}} (\Delta AZ)^{\frac{1}{2}} \left(\frac{\odot}{M}\right)^{\frac{1}{2}} \eta_z}, \quad \dots \dots (24a)$$

$$= \frac{\frac{l}{\mu_0^{\frac{1}{2}}} \left(\frac{\odot}{M}\right)^{\frac{1}{2}}}{1 + \frac{l}{\mu_0^{\frac{1}{2}}} \left(\frac{4\pi}{3\odot}\right)^{\frac{1}{2}} \rho^{\frac{1}{2}} \left(1 - \frac{1}{Z^{\frac{1}{2}}}\right) \left(\frac{\odot}{M}\right)^{\frac{1}{2}}}. \quad \dots \dots (24b)$$

Equation (24) shows that there is a maximum radius R_{\max} at a particular mass M_0 . By differentiating with respect to M , we get,

$$R_{\max} = \frac{l^{\frac{1}{2}}}{2\mu_0^{\frac{1}{2}} \eta_z^{\frac{1}{2}}} \left(\frac{3\odot}{4\pi}\right)^{\frac{1}{2}} \frac{1}{(\Delta AZ)^{\frac{1}{2}}}, \quad \dots \dots (25a)$$

$$= \frac{l^{\frac{1}{2}}}{2\mu_0^{\frac{1}{2}} \left(1 - \frac{1}{Z^{\frac{1}{2}}}\right)^{\frac{1}{2}}} \left(\frac{3\odot}{4\pi}\right)^{\frac{1}{2}} \frac{1}{\rho^{\frac{1}{2}}}, \quad \dots \dots (25b)$$

and

$$\frac{M_0}{\odot} = \frac{l^{\frac{1}{2}}}{\mu_0^{\frac{1}{2}}} \left(\frac{4\pi}{3\odot}\right)^{\frac{1}{2}} (\Delta AZ)^{\frac{1}{2}} \eta_z^{\frac{1}{2}}, \quad \dots \dots (26a)$$

$$= \frac{l^{\frac{1}{2}}}{\mu_0^{\frac{1}{2}}} \left(\frac{4\pi}{3\odot}\right)^{\frac{1}{2}} \rho^{\frac{1}{2}} \left(1 - \frac{1}{Z^{\frac{1}{2}}}\right)^{\frac{1}{2}}. \quad \dots \dots (26b)$$

We now proceed to numerical work. We have to make some assumptions regarding the chemical composition of the cold bodies. We make the following alternative assumptions:

- (i) The cold bodies are composed of hydrogen: referred to as assumption *H*.
- (ii) The cold bodies are composed of iron: referred to as assumption *F*.

In the assumption *F*, we can either take ρ^* as given by equation (19) or can identify it with the density of the ordinary metal (7.86 g./cm.³); we will refer to them as assumptions *F_a* and *F_b* respectively. Figure 2 gives the calculated values of the radii at different masses. If we neglect the exchange potential (as done by Kothari, 1938) we get the dotted curve *KH-H* in the case of hydrogen, but there is no appreciable difference in the case of assumption *F_a*, whereas in the case of *F_b*, the results are identical as is evident from Table II. This is what we had concluded earlier as well. The observed mass-radius values of the planets and white dwarf stars are indicated by crosses (×).

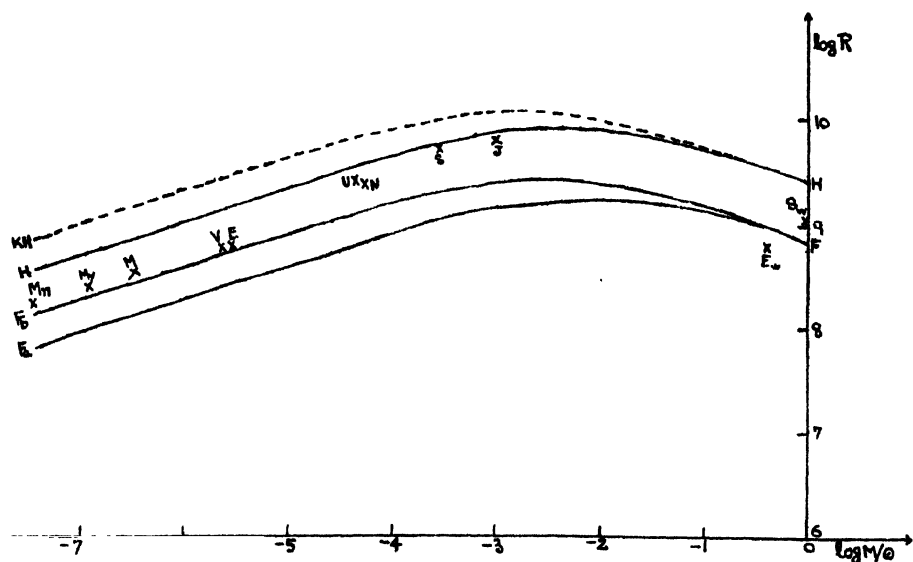


FIG. 2. The plot shows the theoretical relation between mass and radius. The continuous curves *H-H*, *F_a-F*, and *F_b-F* are for hydrogen, iron-assumption *F_a*, and iron-assumption *F_b* respectively, when exchange potential energy is taken into account. The dotted curve *KH-H* is for hydrogen, when exchange potential energy is neglected.

<i>M_n</i> = Moon	<i>E</i> = Earth	<i>J</i> = Jupiter
<i>M_y</i> = Mercury	<i>U</i> = Uranus	<i>E_w</i> = O ₃ Eridani B
<i>M</i> = Mars	<i>N</i> = Neptune	<i>S_w</i> = Sirius B
<i>V</i> = Venus	<i>S</i> = Saturn	

It will be observed that the two major planets, Saturn and Jupiter, are predominantly hydrogen planets. Uranus and Neptune lies nearer to hydrogen curve than to iron curve. The terrestrial planets lie near to the iron curve, indicating that they are composed of heavier substances like iron.

The values of R_{\max} and M_0/\odot on different assumptions can be readily obtained from equation (25) and equation (26) respectively, and are given below. The values under the heading 'observed' have been estimated from the run of the observed mass-radius values. The values of R_{\max} and M_0/\odot when exchange potential is not taken into account, is also given for the sake of comparison.

TABLE II

			Calculated $\log R$					
			Without exchange potential			With exchange potential		
			H	F_a	F_b	H	F_a	F_b
		$\log\left(\frac{M}{\odot}\right)$						
Moon		8.558	8.865	7.811	8.114	8.561	7.809	8.114
Mercury		7.081	9.039	7.985	8.288	8.734	7.983	8.288
Mars		7.520	9.185	8.131	8.434	8.881	8.130	8.434
Venus		6.387	9.469	8.419	8.721	9.168	8.147	8.721
Earth		6.479	9.499	8.450	8.750	9.198	8.447	8.750
Uranus		5.646	9.854	8.827	9.117	9.569	8.825	9.117
Neptune		5.714	9.872	8.848	9.137	9.590	8.847	9.137
Saturn		4.456	10.036	9.066	9.328	9.792	9.064	9.328
Jupiter		4.980	10.091	9.190	9.415	9.894	9.189	9.415
O ₂ Eridani B		1.644	9.555	8.980	8.995	9.546	8.980	8.995
Sirius B		1.982	9.446	8.879	8.888	9.440	8.879	8.888

TABLE III

		Without exchange potential		With exchange potential	
		R_{\max} in cm.	M_0/\odot	R_{\max} in cm.	M_0/\odot
'Observed' ..		8×10^9	1.6×10^{-3}	8×10^9	1.6×10^{-3}
Assumption H ..		12.5×10^9	1.41×10^{-3}	8.76×10^9	4.04×10^{-3}
Assumption F_a ..		1.95×10^9	7.90×10^{-3}	1.95×10^9	7.95×10^{-3}
Assumption F_b ..		2.77×10^9	2.77×10^{-3}	2.77×10^9	2.77×10^{-3}

5. *Mass-Radius relation at Non-Zero Temperature.*—Now we will apply the second approximation. We will denote the quantities in the case of non-zero temperature by a bar over the symbols, that differ from zero-temperature case.

Let us calculate the total kinetic energy. The kinetic energy of a degenerate electron gas to the second approximation is given by

$$E_1 = \frac{3}{8} N \zeta_0 \left[1 + \frac{5\pi^2}{12} \left(\frac{k\theta}{\zeta_0} \right)^2 \right], \quad \dots \dots \dots (27)$$

where
$$\zeta_0 = \frac{\hbar^2}{2m} \left(\frac{3n^*}{8\pi} \right)^{\frac{1}{3}}. \quad \dots \dots \dots (27')$$

Here θ is the temperature in absolute units. Substituting $Z/\gamma_2 a^3$ for n^* , Z for N and eliminating a by the help of equation (5) and multiplying it by $(\rho \bar{V}/Am_H)$, the total number of cells, the total kinetic energy is given by the following expression:

$$\bar{T} = \frac{3}{10} \frac{\hbar^2}{m} \left(\frac{3}{8\pi} \frac{Z\bar{\rho}}{\gamma_1\gamma_2 Am_H} \right)^{\frac{1}{3}} Z \left\{ 1 + \left(\frac{A\Gamma}{Z\bar{\rho}} \right)^{\frac{1}{3}} (\gamma_1\gamma_2)^{\frac{1}{3}} \bar{\theta}^2 \right\} \frac{\bar{\rho}\bar{V}}{Am_H}, \quad \dots \quad (28)$$

$$\text{where} \quad \Gamma = \frac{8\pi}{3} \left[\frac{5}{3} \frac{\pi^2 k^2 m^2 m_H^{\frac{1}{3}}}{\hbar^4} \right]^{\frac{1}{3}}. \quad \dots \quad (28')$$

The total electrostatic potential, as before, is given by:

$$-\bar{W}_P = \frac{9}{10} Z^2 e^2 \left(\frac{\bar{\rho}}{\gamma_1 Am_H} \right)^{\frac{1}{3}} \frac{\bar{\rho}\bar{V}}{Am_H}. \quad \dots \quad (29)$$

Introducing Zirin's modification in the Slater's formula for exchange potential, the total exchange potential energy at non-zero temperatures is given by:

$$-\bar{W}_E = 3e^2\beta \left(\frac{9}{32\pi^2} \frac{Z\bar{\rho}}{\gamma Am_H} \right)^{\frac{1}{3}} \frac{\bar{\rho}\bar{V}}{Am_H}, \quad \dots \quad (30)$$

where β is a function of $k\bar{\theta}/\epsilon_0$, ϵ_0 being the energy of the top of the Fermi band.

The pressure for a Fermi-Dirac electron gas, to the second approximation, is given by:

$$\bar{p} = \frac{8\pi}{15} \frac{\hbar^2}{m} \left(\frac{3n}{8\pi} \right)^{\frac{2}{3}} \left[1 + \frac{5\pi^2}{12} \left(\frac{k\bar{\theta}}{\zeta_0} \right)^2 \right], \quad \dots \quad (31)$$

$$\text{where} \quad \zeta_0 = \frac{\hbar^2}{2m} \left(\frac{3\eta}{8\pi} \right)^{\frac{2}{3}}. \quad \dots \quad (31')$$

Here we are taking n instead of n^* , because in degenerate matter the pressure effectively depends on free electron concentration.

Eliminating n from equation (31) by the help of equation (14), we get,

$$\bar{p} = K \frac{\bar{\rho}^{\frac{2}{3}}}{\mu^{\frac{2}{3}}} + L \frac{\bar{\rho}^{\frac{1}{3}}}{\mu^{\frac{1}{3}}} \bar{\theta}^2, \quad \dots \quad (32)$$

$$\text{where} \quad K = \frac{8\pi}{15} \frac{\hbar^2}{m} \left(\frac{3}{8\pi m_H} \right)^{\frac{2}{3}}, \quad \dots \quad (32')$$

$$\text{and} \quad L = \frac{4\pi^{\frac{1}{2}} k^2 m}{3^{\frac{1}{2}} \hbar^2 m_H^{\frac{1}{3}}}, \quad \dots \quad (32'')$$

Substituting equations (28), (29), (30) and (32) in equation (4), we obtain

$$\bar{\mu} = \frac{\mu_0(\gamma_1\gamma_2)^{\frac{1}{3}} \left/ \left\{ 1 + \left(\frac{\mu_0\Gamma}{\bar{\rho}} \right)^{\frac{1}{3}} (\gamma_1\gamma_2)^{\frac{1}{3}} \bar{\theta}^2 \right\}^{\frac{1}{3}} \right.}{\left[1 - \frac{\frac{(\gamma_1\gamma_2)^{\frac{1}{3}} \left(\frac{\Delta AZ}{\bar{\rho}} \right)^{\frac{1}{3}} \bar{\eta}_Z + (\gamma_1\gamma_2)^{\frac{1}{3}} \left(\frac{\mu_0\Omega}{\bar{\rho}} \right)^{\frac{1}{3}} \left(\frac{\bar{\rho}}{\bar{\mu}} \right)^{\frac{1}{3}} \bar{\theta}^2}{1 + \left(\frac{\mu_0\Gamma}{\bar{\rho}} \right)^{\frac{1}{3}} (\gamma_1\gamma_2)^{\frac{1}{3}} \bar{\theta}^2} \right]^{\frac{1}{3}}}, \quad \dots \quad (33)$$

$$\text{where} \quad \Omega = \left(\frac{80\pi^{\frac{10}{3}} k^2 m^2 m_H^{\frac{1}{3}}}{3^{\frac{1}{2}} \hbar^4} \right)^{\frac{1}{3}}, \quad \dots \quad (33')$$

$$\text{and} \quad \bar{\eta}_Z = 1 + \frac{5\beta}{(12\pi^2 Z^5)^{\frac{1}{3}}}. \quad \dots \quad (33'')$$

As a rough approximation, we replace γ_1 and γ_2 by unity. Then we get,

$$\bar{\mu} = \frac{\mu_0 / \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}} \right)^{\frac{1}{3}} \bar{\theta}^2 \right\}^{\frac{1}{3}}}{\left[\frac{\left(\frac{\Delta A Z}{\bar{\rho}} \right)^{\frac{1}{3}} \bar{\eta}_Z + \left(\frac{\mu_0 \Omega}{\bar{\rho}} \right)^{\frac{1}{3}} \left(\frac{\bar{\rho}}{\bar{\mu}} \right)^{\frac{1}{3}} \bar{\theta}^2}{1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}} \right)^{\frac{1}{3}} \bar{\theta}^2} \right]^{\frac{1}{3}}} \quad \dots \quad (34)$$

If $\bar{\rho}^*$ denotes the density of the material in singly ionized state, then,

$$\bar{\rho}^* = \frac{(\Delta A Z) \bar{\eta}_Z^3}{\left[\left\{ 1 - \frac{1}{Z^{\frac{1}{3}} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}^*} \right)^{\frac{1}{3}} \bar{\theta}^2 \right\}} \right\} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}^*} \right)^{\frac{1}{3}} \bar{\theta}^2 \right\} - \left(\frac{\mu_0 \Omega}{\bar{\rho}^*} \right)^{\frac{1}{3}} \left(\frac{\bar{\rho}^*}{A} \right)^{\frac{1}{3}} \bar{\theta}^2 \right]^3} \quad \dots \quad (35)$$

As this equation cannot be exactly solved for $\bar{\rho}^*$, we replace $\bar{\rho}^*$ by ρ^* as given by equation (19), on the right-hand-side of equation (35), as an approximation. Then,

$$\bar{\rho}^* = \frac{(\Delta A Z) \bar{\eta}_Z^3}{\left[\left\{ 1 - \frac{1}{Z^{\frac{1}{3}} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\rho^*} \right)^{\frac{1}{3}} \bar{\theta}^2 \right\}} \right\} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\rho^*} \right)^{\frac{1}{3}} \bar{\theta}^2 \right\} - \left(\frac{\mu_0 \Omega}{\rho^*} \right)^{\frac{1}{3}} \left(\frac{\rho^*}{A} \right)^{\frac{1}{3}} \bar{\theta}^2 \right]^3} \quad \dots \quad (36)$$

Hence we may write equation (34) as:

$$\bar{\mu} = \frac{\mu_0 / \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}} \right)^{\frac{1}{3}} \bar{\theta}^2 \right\}^{\frac{1}{3}}}{\left[1 - \frac{\left(\frac{\bar{\rho}^*}{\bar{\rho}} \right)^{\frac{1}{3}} \left[\left\{ 1 - \frac{1}{Z^{\frac{1}{3}} \left\{ 1 + \left(\mu_0 \Gamma / \bar{\rho}^* \right)^{\frac{1}{3}} \bar{\theta}^2 \right\}} \right\} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}^*} \right)^{\frac{1}{3}} \bar{\theta}^2 \right\} + \left(\frac{\mu_0 \Omega}{\bar{\rho}} \right)^{\frac{1}{3}} \left(\frac{\bar{\rho}}{\bar{\mu}} \right)^{\frac{1}{3}} \bar{\theta}^2 \right]^{\frac{1}{3}}}{1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}} \right)^{\frac{1}{3}} \bar{\theta}^2} \right]^{\frac{1}{3}}} \quad \dots \quad (37)$$

We may regard, approximately that,

$$\bar{\rho} = \frac{3M}{4\pi \bar{R}^3} \quad \dots \quad (38)$$

The radius \bar{R} of a spherical aggregate composed of cold matter is given by (Milne, 1932; see equation (23)):

$$\bar{R} = \frac{l}{\bar{\mu}^{\frac{1}{3}}} \left(\frac{\odot}{\bar{M}} \right)^{\frac{1}{3}} \quad \dots \quad (39)$$

Substituting the value of $\bar{\rho}$ from equation (38) and then eliminating $\bar{\mu}$ from equation (34) by the help of equation (39) we get,

$$\bar{R} = \frac{\frac{l}{\mu_0^{\frac{1}{3}}} \left(\frac{\odot}{\bar{M}} \right)^{\frac{1}{3}} \left\{ 1 + \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{3}} (\mu_0 \Gamma)^{\frac{1}{3}} \left(\frac{\odot}{\bar{M}} \right)^{\frac{1}{3}} \bar{R}^4 \bar{\theta}^2 \right\}}{1 + \frac{l}{\mu_0^{\frac{1}{3}}} \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{3}} (\Delta A Z)^{\frac{1}{3}} \left(\frac{\odot}{\bar{M}} \right)^{\frac{1}{3}} \bar{\eta}_Z + l^{\frac{1}{3}} \Omega^{\frac{1}{3}} \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{3}} \left(\frac{\odot}{\bar{M}} \right)^{\frac{1}{3}} \bar{R}^{\frac{1}{3}} \bar{\theta}^2} \quad \dots \quad (40a)$$

$$\bar{R} = \frac{\frac{l}{\mu_0^{\frac{1}{2}}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} \left\{ 1 + \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{2}} (\mu_0 \Gamma)^{\frac{1}{2}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} \bar{R}^4 \bar{\theta}^2 \right\}}{1 + \frac{l}{\mu_0^{\frac{1}{2}}} \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{2}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} \bar{\rho}^{*1} \left[\left\{ 1 - \frac{1}{Z^{\frac{1}{2}}} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}^*} \right)^{\frac{1}{2}} \bar{\theta}^2 \right\} \right\} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}^*} \right)^{\frac{1}{2}} \bar{\theta}^2 \right\} - \left(\frac{\mu_0 \Omega}{\bar{\rho}^*} \right)^{\frac{1}{2}} \left(\frac{\bar{\rho}^*}{A} \right)^{\frac{1}{2}} \bar{\theta}^2 \right] + l^{\frac{1}{2}} \Omega^{\frac{1}{2}} \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{2}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} \bar{R}^{\frac{1}{2}} \bar{\theta}^2}. \quad (40b)$$

Since equation (40) cannot be exactly solved for \bar{R} , we, as an approximation, replace \bar{R} by R , as given by equation (24) on the right-hand-side of the above equation. Then we get:

$$\bar{R} = \frac{\frac{l}{\mu_0^{\frac{1}{2}}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} \left\{ 1 + \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{2}} (\mu_0 \Gamma)^{\frac{1}{2}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} R^4 \bar{\theta}^2 \right\}}{1 + \frac{l}{\mu_0^{\frac{1}{2}}} \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{2}} (\Delta A Z)^{\frac{1}{2}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} \bar{\eta}_z + l^{\frac{1}{2}} \Omega^{\frac{1}{2}} \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{2}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} R^{\frac{1}{2}} \bar{\theta}^2}, \quad (41a)$$

$$= \frac{\frac{l}{\mu_0^{\frac{1}{2}}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} \left\{ 1 + \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{2}} (\mu_0 \Gamma)^{\frac{1}{2}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} R^4 \bar{\theta}^2 \right\}}{1 + \frac{l}{\mu_0^{\frac{1}{2}}} \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{2}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} \bar{\rho}^{*1} \left[\left\{ 1 - \frac{1}{Z^{\frac{1}{2}}} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}^*} \right)^{\frac{1}{2}} \bar{\theta}^2 \right\} \right\} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}^*} \right)^{\frac{1}{2}} \bar{\theta}^2 \right\} - \left(\frac{\mu_0 \Omega}{\bar{\rho}^*} \right)^{\frac{1}{2}} \left(\frac{\bar{\rho}^*}{A} \right)^{\frac{1}{2}} \bar{\theta}^2 \right] + l^{\frac{1}{2}} \Omega^{\frac{1}{2}} \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{2}} \left(\frac{\odot}{M} \right)^{\frac{1}{2}} R^{\frac{1}{2}} \bar{\theta}^2}. \quad (41b)$$

We now proceed to numerical work. We will make similar assumptions as in the zero-temperature case, regarding the chemical composition of the cold bodies. The calculated values of radii of cold bodies at temperatures 0°A ., 10^4°A . and 10^5°A are tabulated below:

TABLE IV

	$\log \frac{M}{\odot}$	Assumption H			Assumption F_a			Assumption F_b		
		$\bar{\theta} = 0^\circ\text{A}$	$\bar{\theta} = 10^4^\circ\text{A}$	$\bar{\theta} = 10^5^\circ\text{A}$	$\bar{\theta} = 0^\circ\text{A}$	$\bar{\theta} = 10^4^\circ\text{A}$	$\bar{\theta} = 10^5^\circ\text{A}$	$\bar{\theta} = 0^\circ\text{A}$	$\bar{\theta} = 10^4^\circ\text{A}$	$\bar{\theta} = 10^5^\circ\text{A}$
Moon ..	8.558	8.561	8.578	8.975	7.809	7.809	7.811	8.114	8.114	8.119
Mercury ..	7.081	8.734	8.751	9.122	7.983	7.983	7.985	8.288	8.288	8.291
Mars ..	7.520	8.881	8.897	9.244	8.130	8.130	8.131	8.434	8.434	8.436
Venus ..	6.387	9.168	9.183	9.477	8.417	8.417	8.419	8.721	8.721	8.718
Earth ..	6.479	9.198	9.213	9.501	8.447	8.447	8.449	8.750	8.750	8.748
Uranus ..	5.646	9.569	9.581	9.777	8.825	8.825	8.826	9.117	9.117	9.108
Neptune ..	5.714	9.590	9.601	9.792	8.847	8.847	8.847	9.137	9.137	9.128
Saturn ..	4.456	9.792	9.798	9.938	9.064	9.064	9.065	9.328	9.328	9.315
Jupiter ..	4.980	8.894	8.898	9.967	9.189	9.189	9.189	9.415	9.415	9.401
O ₃ Eridani B ..	1.644	9.546	9.546	9.546	8.980	8.980	8.980	8.995	8.995	8.994
Sirius B ..	1.982	9.440	9.440	9.440	8.879	8.879	8.879	8.888	8.888	8.887

Thus we see that the radii of the cold bodies increase with increase in temperature (excluding the case of assumption F_b which we will consider presently). The increase in the radii of cold bodies is appreciable in the case of assumption H but in the case of assumption F_a the increase is very small, in the temperature-range considered. This is due to the fact that the value of β (see equation (30)) varies appreciably at the three temperatures considered in the case of assumption H , whereas it remains more or less constant (i.e. equal to unity) in the case of assumption F_a at these temperatures. Coming to the case of assumption F_b , we notice that the radii of the cold bodies, taken in order of their increasing mass, increases for smaller bodies but decreases for heavier bodies, with increase in temperature. This is due to the fact that whereas the value $\bar{\rho}^*$ in the case of assumption F_a assumes different values at different temperatures, it is constant in the case of assumption F_b .

Concluding, I take this opportunity to express my sincere thanks to Prof. D. S. Kothari and Dr. F. C. Auluck for suggesting the problem and for constant help and encouragement throughout the work. I am also thankful to the Government of India for the award of a scholarship.

ABSTRACT

The mass-radius relationship for cold bodies, has been calculated, based on the theory of pressure-ionization, taking into account the exchange potential of the electron. The theory has been developed for zero-temperature as well as for non-zero temperatures.

REFERENCES

- Brown, H. (1950). On the Constitutions and Structures of the Planets. *Ap. J.*, **111**, 641-653.
 Bullen, K. E. (1949a). On the Constitution of Venus. *Ibid.*, *R.A.S.*, **109**, 457-461.
 ——— (1949b). On the Constitution of Mars. *Mon. Not., R.A.S.*, **109**, 688-692.
 ——— (1950). Venus and Earth's Inner Core. *Ibid.*, **110**, 256-259.
 Dirac, P. A. M. (1930). Note on Exchange Phenomena in the Thomas Atom. *Proc. Camb. Phil. Soc.*, **26**, 376-387.
 Kothari, D. S. (1938). The Theory of Pressure-Ionization and its Applications. *Proc. Roy. Soc., Ser. A.*, **165**, 486-500.
 Mestel, L. (1952a). On the Theory of White Dwarfs: I. The Energy sources of White Dwarfs. *Mon. Not., R.A.S.*, **112**, 583-597.
 ——— (1952b). On the Theory of White Dwarfs: II. The Accretion of Interstellar Matter by White Dwarfs. *Ibid.*, **112**, 598-605.
 Milne, E. A. (1932). The Problem of Stellar Structure. *Ibid.*, **92**, 611-643.
 Ramsey, W. H. (1948). On the Constitution of the Terrestrial Planets. *Ibid.*, **108**, 406-416.
 ——— (1950). The Planets and the White Dwarfs. *Ibid.*, **110**, 444-454.
 ——— (1951). On the Constitution of Major Planets. *Ibid.*, **111**, 427-447.
 Slater, J. C. (1951). A simplification of the Hartree-Fock Method. *Phys. Rev.*, **81**, 385-390.
 Wildt, R. (1947). The Constitution of the Planets. *Mon. Not., R.A.S.*, **107**, 84-102.
 Zirin, H. (1953). The Exchange Potential in an Electron Gas at Non-zero Temperature. *Phys. Rev.*, **90**, 795.

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INTEGRALS ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

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In a recent paper (Shanti Saran, 1954) I have defined the hypergeometric functions of three variables and deduced the important properties of these functions including some integral representations. In this paper I obtain the single, double and triple integral representations of these functions. The integrals are either of Pochhammer's double loop type or of Mellin-Barne's type. The first type of integrals is useful in the integration of the system of partial differential equations satisfied by these functions and the latter type in Mellin's investigation of the hypergeometric functions.

In Pochhammer's double loop type of integrals $[a_1, a_2, \dots a_m; b_1, b_2, \dots b_n]$ is the notation for the double loop which contains $a_1, a_2, \dots a_m$ within one loop and $b_1, b_2, \dots b_n$ within the other. It is understood that all other singularities of the integrand are outside the contour of integration. z^α is interpreted as $\exp(\alpha \log z)$, where $\log z$ is real when z is positive, and continuous on the contour of integration.

It will be assumed that x, y and z have such values that the infinite series occurring in the analysis converges. Exceptional values of the parameters which would make some of the gamma functions become infinite are tacitly excluded. The general validity of the results follows by analytic continuation.

The integrals deduced are either of the form

$$\int (-t)^{\mu-1} (t-1)^{\nu-1} f(u) g(v) dt$$

where u is a function of x and t , and v of y, z and t ;

or

$$\int (-t)^{\mu-1} (t-1)^{\nu-1} f(u) dt$$

where u is a function of x, y, z and t and the contour is $[1+, 0+; 1-, 0-]$.

In the first, one can say that Euler's transformation factorizes our hypergeometric equations f and g , each of which satisfies the ordinary differential equation (or Appell's hypergeometric differential equation in the case of $g(v)$).

In § 2 single integrals of Pochhammer's type have been deduced and lastly in § 3 single, double and treble integrals of Mellin-Barne's type have been investigated for these functions.

Following the notation given in (Shanti Saran, 1954), we define the hypergeometric functions of three variables as below :

$$F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\ = \sum \frac{(\alpha_1, m+n+p) (\beta_1, m) (\beta_2, n+p) x^m y^n z^p}{(1, m) (1, n) (1, p) (\gamma_1, m) (\gamma_2, n) (\gamma_3, p)} \dots \dots (1.1)$$

$$F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) \\ = \sum \frac{(\alpha_1, m+n+p)(\beta_1, m+p)(\beta_2, n)x^m y^n z^p}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} \quad \dots \quad (1.2)$$

$$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) \\ = \sum \frac{(\alpha_1, m+n+p)(\beta_1, m)(\beta_2, n)(\beta_3, p)x^m y^n z^p}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} \quad \dots \quad (1.3)$$

$$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\ = \sum \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)x^m y^n z^p}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n)(\gamma_3, p)} \quad \dots \quad (1.4)$$

$$F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) \\ = \sum \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)x^m y^n z^p}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} \quad \dots \quad (1.5)$$

$$F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) \\ = \sum \frac{(\alpha_1, m)(\alpha_2, n)(\alpha_3, p)(\beta_1, m+p)(\beta_2, n)x^m y^n z^p}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} \quad \dots \quad (1.6)$$

$$F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) \\ = \sum \frac{(\alpha_1, m+p)(\alpha_2, n)(\beta_1, m+n)(\beta_2, n)x^m y^n z^p}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} \quad \dots \quad (1.7)$$

$$F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) \\ = \sum \frac{(\alpha_1, m+p)(\alpha_2, n)(\beta_1, m+p)(\beta_2, n)x^m y^n z^p}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} \quad \dots \quad (1.8)$$

$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) \\ = \sum \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m)(\beta_2, n)(\beta_3, p)x^m y^n z^p}{(1, m)(1, n)(1, p)(\gamma_1, m+n+p)} \quad \dots \quad (1.9)$$

and

$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z) \\ = \sum \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)x^m y^n z^p}{(1, m)(1, n)(1, p)(\gamma_1, m+n+p)} \quad \dots \quad (1.10)$$

The triple summation in the above series extends over all positive integral values of m, n, p from zero to infinity.

As proved in (Shanti Saran, 1954), the domains of convergence of the above series are given by

$$\begin{aligned} F_E: r + (\sqrt{s} + \sqrt{t})^2 &= 1 & F_F: rs &= (1-s)(s-t) \\ F_G: r+s &= 1 \} & F_K: t &= (1-r)(1-s) \\ r+t &= 1 \} \\ F_M: r+t &= 1 \} & F_N: s(1-r) + t(1-s) &= 0 \\ s &= 1 \} \end{aligned}$$

$$F_P: 4rst$$

$$= (st - s - t)^2$$

$$F_R: s(1 - \sqrt{r})^2 + t(1 - s) = 0$$

$$F_S: \frac{1}{s} + \frac{1}{t} = 1,$$

$$F_T: t = r - rs + s$$

where

$$|x| < r, \quad |y| < s \quad \text{and} \quad |z| < t.$$

§ 2. We know the following integral representations (Erdélyi, 1953) for F_2 , F_3 and F_4 , viz.

$$F_2(\rho + \rho_1 - 1; \beta, \beta_1; \gamma, \gamma_1; x, y) = \frac{\Gamma(\rho) \Gamma(\rho_1) \Gamma(2 - \rho - \rho_1)}{(2\pi i)^2} \times \int (-t)^{-\rho} (t-1)^{-\rho_1} {}_2F_1\left(\rho, \beta; \gamma; \frac{x}{t}\right) {}_2F_1\left(\rho_1, \beta_1; \gamma_1; \frac{y}{1-t}\right) dt \quad \dots \quad (2.1)$$

where $|t| > |x|, |1-t| > |y|$ along the contour.*

$$F_3(\alpha, \alpha_1, \beta, \beta_1; \rho + \rho_1; x, y) = \frac{\Gamma(1 - \rho) \Gamma(1 - \rho_1) \Gamma(\rho + \rho_1)}{(2\pi i)^2} \times \int (-t)^{\rho-1} (t-1)^{\rho_1-1} {}_2F_1(\alpha, \beta; \rho; tx) {}_2F_1(\alpha_1, \beta_1; \rho_1; (1-t)y) dt \quad (2.2)$$

and

$$F_4(\alpha, \beta; \gamma, \gamma_1; x, y) = \frac{\Gamma(\gamma) \Gamma(\gamma_1) \Gamma(2 - \gamma - \gamma_1)}{(2\pi i)^2} \times \int (-t)^{-\gamma} (t-1)^{-\gamma_1} {}_2F_1\left(\alpha, \beta; \gamma + \gamma_1 - 1; \frac{x}{t} + \frac{y}{1-t}\right) dt \quad \dots \quad (2.3)$$

where $\left|\frac{x}{t} + \frac{y}{1-t}\right| < 1$ along the contour.*

From the definition of F_E , we get

$$F_E = \sum_{m=0}^{\infty} \frac{(\alpha_1, m) (\beta_1, m)}{(1, m) (\gamma_1, m)} F_4(\alpha_1 + m, \beta_2; \gamma_2, \gamma_3; y, z) x^m.$$

Using (2.3), we get

$$\begin{aligned} F_E &= \sum_{m=0}^{\infty} \frac{(\alpha_1, m) (\beta_1, m) x^m}{(1, m) (\gamma_1, m)} \frac{\Gamma(\gamma_2) \Gamma(\gamma_3) \Gamma(2 - \gamma_2 - \gamma_3)}{(2\pi i)^2} \times \\ &\times \int (-t)^{-\gamma_2} (t-1)^{-\gamma_3} {}_2F_1\left(\alpha_1 + m, \beta_2; \gamma_2 + \gamma_3 - 1; \frac{y}{t} + \frac{z}{1-t}\right) dt \\ &= \frac{\Gamma(\gamma_2) \Gamma(\gamma_3) \Gamma(2 - \gamma_2 - \gamma_3)}{(2\pi i)^2} \int (-t)^{-\gamma_2} (t-1)^{-\gamma_3} \times \\ &\times \sum_{m=0}^{\infty} \frac{(\alpha_1, m) (\beta_1, m) x^m}{(1, m) (\gamma_1, m)} \times {}_2F_1\left(\alpha_1 + m, \beta_2; \gamma_2 + \gamma_3 - 1; \frac{y}{t} + \frac{z}{1-t}\right) dt. \end{aligned}$$

* The contour of integration is a Pochhammer's double loop type $(1+, 0+, 1-, 0-)$ and t^p , etc., have their principal values.

Since the series for m is absolutely convergent for $|x| + \left| \frac{y}{t} + \frac{z}{1-t} \right| < 1$, the change in the order of integration and summation is permissible and we have

$$F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \frac{\Gamma(\gamma_2)\Gamma(\gamma_3)\Gamma(2-\gamma_2-\gamma_3)}{(2\pi i)^2} \times \\ \times \int (-t)^{-\gamma_2} (t-1)^{-\gamma_3} F_2\left(\alpha_1; \beta_1, \beta_2; \gamma_1, \gamma_2+\gamma_3-1; x, \frac{y}{t} + \frac{z}{1-t}\right) dt \quad (2.4)$$

where $|x| + \left| \frac{y}{t} + \frac{z}{1-t} \right| < 1$ along the contour.*

Also, rewriting F_E as

$$\sum_{n=0}^{\infty} \frac{(\alpha_1, n) (\beta_2, n)}{(1, n) (\gamma_2, n)} F_2(\alpha_1+n; \beta_1, \beta_2+n; \gamma_1, \gamma_3; x, z)y^n,$$

and using (2.1), we get

$$F_E = \sum_{n=0}^{\infty} \frac{(\alpha_1, n) (\beta_2, n)y^n}{(1, n) (\gamma_2, n)} \times \frac{\Gamma(\rho) \Gamma(\rho_1) \Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \times \\ \times \int (-t)^{-\rho} (t-1)^{-\rho_1} {}_2F_1\left(\rho, \beta_1; \gamma_1; \frac{x}{t}\right) {}_2F_1\left(\rho_1, \beta_2+n; \gamma_3; \frac{z}{1-t}\right) dt$$

where $|t| > |x|$, $|1-t| > |z|$ along the contour and $\alpha_1 = \rho + \rho_1 - 1$.

As before,

$$F_E = \frac{\Gamma(\rho)\Gamma(\rho_1)\Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \times \\ \times \int (-t)^{-\rho} (t-1)^{-\rho_1} {}_2F_1\left(\rho, \beta_1; \gamma_1; \frac{x}{t}\right) {}_2F_1\left(\rho_1, \beta_2; \gamma_2, \gamma_3; \frac{y}{1-t}, \frac{z}{1-t}\right) dt \quad (2.5)$$

where $\alpha_1 = \rho + \rho_1 - 1$ and $|t| > |x|$, and $\left| \sqrt{\frac{y}{1-t}} \right| + \left| \sqrt{\frac{z}{1-t}} \right| < 1$ along the contour.

Using similar methods we can easily prove the following formulae:—

$$F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(2-\gamma_1-\gamma_2)}{(2\pi i)^2} \times \\ \times \int (-t)^{-\gamma_1} (t-1)^{-\gamma_2} F_1\left(\alpha_1; \beta_2, \beta_1; \gamma_1+\gamma_2-1; \frac{y}{1-t}, \frac{x}{t} + \frac{z}{1-t}\right) dt \quad (2.6)$$

where $|1-t| > |y|$ and $\left| \frac{x}{t} + \frac{z}{1-t} \right| < 1$ along the contour.

$$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \frac{\Gamma(\rho) \Gamma(\rho_1) \Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \times \\ \times \int (-t)^{-\rho} (t-1)^{-\rho_1} {}_2F_1\left(\rho, \beta_1; \gamma_1; \frac{x}{t}\right) {}_2F_1\left(\rho_1; \beta_2, \beta_3; \gamma_2; \frac{y}{1-t}, \frac{z}{1-t}\right) dt \quad (2.7)$$

where $\alpha_1 = \rho + \rho_1 - 1$, $|t| > |x|$, $|1-t| > |y|$ and $|1-t| > |z|$ along the contour.

* The contour of integration is a Pochhammer's double loop type $(1+, 0+, 1-, 0-)$ and t^{ρ} , etc., have their principal values.

$$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \frac{\Gamma(\rho) \Gamma(\rho_1) \Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \times \\ \times \int (-t)^{-\rho} (t-1)^{-\rho_1} {}_2F_1\left(\rho, \alpha_1; \gamma_1; \frac{x}{t}\right) F_2\left(\alpha_2; \beta_2, \rho_1; \gamma_2, \gamma_3; y, \frac{z}{1-t}\right) dt \quad (2.8)$$

where $|t| > |x|$ and $\frac{1}{1-t} < 1 - |y|$ along the contour and $\beta_1 = \rho + \rho_1 - 1$.

$$F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \frac{\Gamma(\rho) \Gamma(\rho_1) \Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \times \\ \times \int (-t)^{-\rho} (t-1)^{-\rho_1} {}_2F_1\left(\rho, \alpha_1; \gamma_1; \frac{x}{t}\right) F_1\left(\alpha_2; \beta_2, \rho_1; \gamma_2; y, \frac{z}{1-t}\right) dt \quad (2.9)$$

where $\beta_1 = \rho + \rho_1 - 1$ and $|t| > |x|$ and $|1-t| > |z|$ along the contour.

$$F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \frac{\Gamma(1-\rho) \Gamma(1-\rho_1) \Gamma(\rho+\rho_1)}{(2\pi i)^2} \times \\ \times \int (-t)^{\rho-1} (t-1)^{\rho_1-1} {}_2F_1(\alpha_2, \beta_2; \rho; ty) F_2(\beta_1; \alpha_1, \alpha_3; \gamma_1, \rho_1; x, (1-t)z) dt \quad \dots (2.10)$$

where $\gamma_2 = \rho + \rho_1$ and $|x| + |(1-t)z| < 1$ and $|yt| < 1$ along the contour.

Also,

$$F_N = \frac{\Gamma(\rho) \Gamma(\rho_1) \Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \int (-t)^{-\rho} (t-1)^{-\rho_1} {}_2F_1\left(\rho, \alpha_1; \gamma_1; \frac{x}{t}\right) \times \\ \times F_3\left(\alpha_2, \alpha_3; \beta_2, \rho_1; \gamma_2; y, \frac{z}{1-t}\right) dt \quad \dots (2.11)$$

where $\beta_1 = \rho + \rho_1 - 1$ and $|t| > |x|$ and $|1-t| > |z|$ along the contour.

$$F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \frac{\Gamma(1-\rho) \Gamma(1-\rho_1) \Gamma(\rho+\rho_1)}{(2\pi i)^2} \times \\ \times \int (-t)^{\rho-1} (t-1)^{\rho_1-1} F_K(\beta_2, \beta_1, \beta_1, \alpha_1, \alpha_2, \alpha_1; \rho_1, \rho, \gamma_1; (1-t)z, ty, x) dt \quad \dots (2.12)$$

where $\gamma_2 = \rho + \rho_1$ and $u = (1-r)(1-s)$ with $|x| < u, |ty| < s, |z(1-t)| < r$ along the contour.

Also,

$$F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \frac{\Gamma(\rho) \Gamma(\rho_1) \Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \times \\ \times \int (-t)^{-\rho} (t-1)^{-\rho_1} F_N\left(\rho, \alpha_2, \rho_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; \frac{x}{t}, y, \frac{z}{1-t}\right) dt \quad \dots (2.13)$$

where $\alpha_1 = \rho + \rho_1 - 1, s(1-r) + u(1-s) = 0$ with $|x| < r, \frac{1}{1-t} < u, |y| < s$ along the contour.

$$F_R = \frac{\Gamma(\gamma_1) \Gamma(\gamma_2) \Gamma(2-\gamma_1-\gamma_2)}{(2\pi i)^2} \times \\ \times \int (-t)^{\gamma_1} (t-1)^{-\gamma_2} F_3\left(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma_1+\gamma_2-1; \frac{y}{1-t}, \frac{x}{t} + \frac{z}{1-t}\right) dt \quad \dots (2.14)$$

where $\left|\frac{y}{1-t}\right| < 1$ and $\left|\frac{x}{t} + \frac{z}{1-t}\right| < 1$ along the contour.

Also,

$$F_R = \frac{\Gamma(1-\rho) \Gamma(1-\rho_1) \Gamma(\rho+\rho_1)}{(2\pi i)^2} \times \int (-t)^{\rho-1} (t-1)^{\rho_1-1} {}_2F_1(\alpha_2, \beta_2; \rho; ty) \times \\ \times F_4(\alpha_1, \beta_1; \gamma_1, \rho_1; x, (1-t)z) dt \quad \dots (2.15)$$

where $\gamma_2 = \rho + \rho_1$ and $|\sqrt{x}| + |\sqrt{(1-t)z}| < 1$ and $|yt| < 1$ along the contour.

$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \frac{\Gamma(1-\rho) \Gamma(1-\rho_1) \Gamma(\rho+\rho_1)}{(2\pi i)^2} \times \\ \times \int (-t)^{\rho-1} (t-1)^{\rho_1-1} {}_2F_1(\alpha_1, \beta_1; \rho; tx) F_1(\alpha_2; \beta_2, \beta_3; \rho_1; y(1-t), z(1-t)) dt \quad \dots (2.16)$$

where $\gamma_1 = \rho + \rho_1$ and $|tx| < 1$, $|y(1-t)| < 1$ and $|z(1-t)| < 1$ along the contour.

$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \frac{\Gamma(1-\rho) \Gamma(1-\rho_1) \Gamma(\rho+\rho_1)}{(2\pi i)^2} \times \\ \times \int (-t)^{\rho-1} (t-1)^{\rho_1-1} F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \rho, \rho_1, \rho_1; tx, (1-t)y, (1-t)z) dt \quad \dots (2.17)$$

where $\gamma_1 = \rho + \rho_1$, $r+u=1$, $s=1$; and $|tx| < r$, $|(1-t)y| < s$, $|(1-t)z| < t$ along the contour.

We know that *

$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z) \\ = (1-z)^{-\alpha_2} F_S\left(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \gamma_1-\beta_1-\beta_2; \gamma_1, \gamma_1, \gamma_1; x, \frac{y-z}{1-z}, \frac{-z}{1-z}\right).$$

Using (2.16) we get

$$F_T = (1-z)^{-\alpha_2} \frac{\Gamma(1-\rho) \Gamma(1-\rho_1) \Gamma(\rho+\rho_1)}{(2\pi i)^2} \times \\ \times \int (-t)^{\rho-1} (t-1)^{\rho_1-1} {}_2F_1(\alpha_1, \beta_1; \rho; tx) \times \\ F_1\left(\alpha_2; \beta_2, \rho+\rho_1-\beta_1-\beta_2, \rho_1; \frac{y-z}{1-z} (1-t), \frac{-z}{1-z} (1-t)\right) dt$$

where $|tx| < 1$ and $\left|\frac{y-z}{1-z} (1-t)\right| < 1$, $\left|\frac{-z}{1-z} (1-t)\right| < 1$ along the contour and $\gamma_1 = \rho + \rho_1$.

* (Shanti Saran, 1954). Use (5.15).

§ 3. MELLIN-BARNE'S CONTOUR INTEGRALS

In this section I have deduced single, double and treble integral representations for these functions. I give the detailed deductions for F_E only and similar methods give the integral representations for the other nine functions also.

(1) *Single Integral representation*

From the definition of F_E , we have

$$F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3, x, y, z) \\ = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m (\beta_1)_m}{(1)_m (\gamma_1)_m} F_4(\alpha_1+m, \beta_2; \gamma_2, \gamma_3; y, z)x^m.$$

Using the relation (Appell, P. et Kampè, J. de Feriet, 1926), namely

$$\frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\gamma_1)} F_4(\alpha, \beta; \gamma, \gamma_1; x, y) \\ = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} {}_2F_1(\alpha+t, \beta+t; \gamma; x) \frac{\Gamma(\alpha+t) \Gamma(\beta+t)}{\Gamma(\gamma_1+t)} \Gamma(-t) (-y)^t dt$$

we obtain

$$\frac{\Gamma(\alpha_1) \Gamma(\beta_2)}{\Gamma(\gamma_2)} F_E = \frac{1}{2\pi i} \sum_{m=0}^{\infty} \frac{(\beta_1)_m x^m}{(1)_m (\gamma_1)_m} \int_{-\infty}^{+\infty} \frac{\Gamma(\alpha_1+m+t) \Gamma(\beta_2+t)}{\Gamma(\gamma_3+t)} \times \\ \times {}_2F_1(\alpha_1+m+t, \beta_2+t; \gamma_2; y) \Gamma(-t) (-z)^t dt.$$

Changing the order of integration and summation which is easily justifiable for $|x|+|y|<1$, we get

$$\frac{\Gamma(\alpha_1) \Gamma(\beta_2)}{\Gamma(\gamma_2)} F_E = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} F_2(\alpha_1+t, \beta_1, \beta_2+t; \gamma_1, \gamma_2; x, y) \times \\ \times \frac{\Gamma(\alpha_1+t) \Gamma(\beta_2+t)}{\Gamma(\gamma_2+t)} \Gamma(-t) (-z)^t dt \quad \dots \quad (3.1)$$

Similarly, by rewriting F_E as

$$\sum_{m=0}^{\infty} \frac{(\alpha_1)_m (\beta_2)_m}{(1)_m (\gamma_2)_m} F_2(\alpha_1+m; \beta_1, \beta_2+m; \gamma_1, \gamma_3; x, z)y^m$$

and using the relation (Appell, P. et Kampè, J. de Feriet, 1926)

$$\frac{\Gamma(\alpha) \Gamma(\beta_1)}{\Gamma(\gamma_1)} F_2(\alpha; \beta, \beta_1; \gamma, \gamma_1; x, y) \\ = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} {}_2F_1(\alpha+t, \beta; \gamma; x) \frac{\Gamma(\alpha+t) \Gamma(\beta_1+t)}{\Gamma(\gamma_1+t)} \Gamma(-t) (-y)^t dt,$$

we obtain

$$\frac{\Gamma(\alpha_1) \Gamma(\beta_1)}{\Gamma(\gamma_1)} F_E = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} F_4(\alpha_1+t, \beta_2; \gamma_1, \gamma_3; x, z) \times \\ \times \frac{\Gamma(\alpha_1+t) \Gamma(\beta_1+t)}{\Gamma(\gamma_1+t)} \Gamma(-t) (-y)^t dt \quad \dots (3.2)$$

(2) Double Integral representation

Using the above integral for F_2 in (3.1) we get

$$\frac{\Gamma(\alpha_1) \Gamma(\beta_1) \Gamma(\beta_2)}{\Gamma(\gamma_2) \Gamma(\gamma_3)} F_E = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} {}_2F_1(\alpha_1+s+t, \beta_1+t; \gamma_1; x) \times \\ \times \frac{\Gamma(\alpha_1+s+t) \Gamma(\beta_1+s) \Gamma(\beta_2+t)}{\Gamma(\gamma_1+s) \Gamma(\gamma_3+t)} \Gamma(-s) \Gamma(-t) (-x)^s (-z)^t dt \quad \dots (3.3)$$

A similar double integral follows by using the relation of F_4 in (3.1).

(3) Triple Integral representation

From (3.3) we get

$$\frac{\Gamma(\alpha_1) \Gamma(\beta_1) \Gamma(\beta_2)}{\Gamma(\gamma_2) \Gamma(\gamma_3)} F_E = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} {}_2F_1(\alpha_1+s+t, \beta_2+t; \gamma_2; y) \times \\ \times \frac{\Gamma(\alpha_1+s+t) \Gamma(\beta_1+s) \Gamma(\beta_2+t)}{\Gamma(\gamma_1+s) \Gamma(\gamma_3+t)} \Gamma(-s) \Gamma(-t) (-x)^s (-z)^t ds dt.$$

Using the relation (Appell, P. et Kampé, J. de Feriet)

$$\frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\gamma)} {}_2F_1(\alpha, \beta; \gamma; x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha+s) \Gamma(\beta+s)}{\Gamma(\gamma+s)} \Gamma(-s) (-x)^s ds$$

we get

$$\frac{\Gamma(\alpha_1) \Gamma(\beta_1) \Gamma(\beta_2)}{\Gamma(\gamma_1) \Gamma(\gamma_2) \Gamma(\gamma_3)} F_E \\ \frac{1}{(2\pi i)^3} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha_1+r+s+t) \Gamma(\beta_1+r) \Gamma(\beta_2+s+t)}{\Gamma(\gamma_1+r) \Gamma(\gamma_2+s) \Gamma(\gamma_3+t)} \times \\ \times \Gamma(-r) \Gamma(-s) \Gamma(-t) (-x)^r (-y)^s (-z)^t dr ds dt.$$

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REFERENCES

- Appell, P. et Kampé, J. de Feriet (1926). Fonctions hypergéométriques et hypersphériques (Gauthier Villars).
 Erdélyi, A. (1953). Higher Transcendental Functions, 1, 231-32.
 Shanti Saran (1954). Hypergeometric functions of three Variables. *Ganita*, 5, No. 1.

ON THE INTERNAL BALLISTICS OF LEAKING GUNS

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1. INTRODUCTION

Corner's (1947 and 1950) theory of leaking guns, which is applicable to worn orthodox guns, smooth bore mortars and recoilless guns has, in addition to the usual assumptions in the theory of internal ballistics of orthodox guns, made the assumption that the setting up of the gas flow through convergent and divergent nozzle can be represented by the classical one-dimensional approach and the flow starts at a certain pressure called the nozzle-start pressure. The system of equations formulated on these assumptions can always be integrated numerically. Corner gave a simple solution for the case where the rate of burning is proportional to the pressure, having the same accuracy as Hunt-Hinds. A still simpler method consists in a reduction to an 'isothermal' model, which is closely related to Crow's method. Dr. Thiruvengkatachar and Mr. Venkatesan, on account of the values of Ψ involved being small, successfully attempted to solve the general equations for a leaking gun for tubular charge, i.e. $\theta = 0$, by a method of successive approximation based on an expansion in powers of Ψ . This method gives reasonable simple means of obtaining all the ballistic information of interest, i.e. the velocity, the maximum pressure, and the relation between shot-start and nozzle-start pressure.

In the present paper the author has extended these results (for tubular propellant) to the case of charges of any shape, i.e. for any value of θ .

2. THE FUNDAMENTAL EQUATIONS FOR A LEAKING GUN

In formulating the fundamental equations for a leaking gun we make the following assumptions:—

- (i) the rate of burning is proportional to the pressure,
- (ii) the initial resistance is represented by a shot-start pressure,
- (iii) the setting up of the nozzle-flow can be represented by the use of the equations for quasi-steady nozzle-flow beginning instantaneously at a certain pressure called the nozzle-start pressure,
- (iv) no unburnt propellant is lost through the nozzle.

During burning, when the nozzle is open and shot is in motion, we shall have the following equations:—

$$p\left(K_0 + Ax - \frac{C}{\delta}\right) = CNRT\left(1 + \frac{kCN}{6W}\right) \quad \dots \quad (1a)$$

$$W_1 \frac{d^2x}{dt^2} = Ap \quad \dots \quad (1b)$$

$$\text{where} \quad W_1 = W + \frac{1}{2}kCN \quad \dots \quad (1c)$$

$$D \frac{df}{dt} = -\beta p \quad \dots \dots \dots (1d)$$

$$Z = (1-f)(1+\theta f) \quad \dots \dots \dots (1e)$$

$$\frac{dN}{dt} = \frac{dZ}{dt} - \frac{\psi Sp}{C(RT)^{\frac{1}{2}}} \quad \dots \dots \dots (1f)$$

$$\frac{d(NT)}{dt} = -(\bar{\gamma}-1) \frac{Ap}{CR} \frac{dx}{dt} + T_0 \frac{dZ}{dt} - \frac{\gamma \psi Sp(RT)^{\frac{1}{2}}}{CR} \quad \dots \dots (1g^*)$$

These equations are different from those of corresponding conventional gun in three respects. Firstly, the altered Lagrange's correction factor occurs in (1a) and (1e). Secondly, an additional term appears on the right of the energy equation (1g) to account for the loss of energy by the gas-flow through the nozzle. And lastly (1f) is the nozzle-flow equation.

For the reduction of these equations to non-dimensional parameters the following transformations have been used:—

$$Al = K_0 - \frac{C}{\delta} \quad \dots \dots \dots (2a)$$

$$\xi = 1 + \frac{x}{l} \quad \dots \dots \dots (2b)$$

$$\tau = \left(\frac{\beta CRT_0}{ADl} \right) t \quad \dots \dots \dots (2c)$$

$$\zeta = \left(\frac{Al}{CRT_0} \right) p \quad \dots \dots \dots (2d)$$

$$\eta = \frac{d\xi}{d\tau} = \frac{AD}{C\beta RT_0} \frac{dx}{dt} \quad \dots \dots \dots (2e)$$

$$T' = \frac{T}{T_0} \quad \dots \dots \dots (2f)$$

$$M' = \frac{A^2 D^2}{\beta^2 CRT_0 W} \quad \dots \dots \dots (2g)$$

$$\Psi = \frac{\psi SD}{\beta C(RT_0)^{\frac{1}{2}}} \quad \dots \dots \dots (2h)$$

Thus equations (1) reduce to

$$\zeta \xi = NT' \left(1 + \frac{kCN}{6W} \right) \quad \dots \dots \dots (3a)$$

$$\eta \frac{d\eta}{d\xi} = \left(M' / 1 + \frac{kCN}{2W} \right) \zeta \quad \dots \dots \dots (3b)$$

$$\frac{df}{d\tau} = -\zeta \quad \dots \dots \dots (3c)$$

$$Z = (1-f)(1+\theta f) \quad \dots \dots \dots (3d)$$

* Heat loss to the barrel is taken into account by changing γ into $\bar{\gamma}$ in the first term.

$$\frac{dN}{d\tau} = \frac{dZ}{d\tau} - \Psi \zeta (T')^{-\frac{1}{2}} \quad \dots \quad (3e)$$

$$\frac{d(NT')}{d\tau} = -(\bar{\gamma} - 1) \zeta \frac{d\zeta}{d\tau} + \frac{dZ}{d\tau} - \gamma \Psi \zeta (T')^{\frac{1}{2}} \quad \dots \quad (3f)$$

Denoting the shot-start pressure and nozzle-start pressure by Z_0 and Z_N respectively, we have

$$\eta = 0 \text{ for } Z = Z_0 \quad \dots \quad (4a)$$

$$N = Z \text{ for } Z \leq Z_N \quad \dots \quad (4b)$$

Combining (3b), (3c) and (2e) we get

$$\eta \frac{d\eta}{d\zeta} = \frac{d\eta}{d\tau} = - \frac{M}{1 + \frac{kCN}{2W}} \frac{df}{d\tau} \quad \dots \quad (5)$$

Since the factor $1 + \frac{kCN}{2W}$ has a value close to unity throughout the period of burning, we may replace it by average value σ ($\sigma \sim 1$).

Thus

$$\frac{d\eta}{d\tau} = \frac{M}{\sigma} \frac{df}{d\tau}$$

Integrating the above differential equation and using the initial condition $f = f_0$ when $\eta = 0$, we have

$$\eta = \frac{M}{\sigma} (f_0 - f) \quad \dots \quad (6)$$

From equation (1e) we get

$$f = \frac{(\theta - 1) \pm \sqrt{(\theta + 1)^2 - 4\theta Z}}{2\theta} \quad \dots \quad (7a)$$

$$\therefore f_0 = \frac{(\theta - 1) \pm \sqrt{(\theta + 1)^2 - 4\theta Z_0}}{2\theta} \quad \dots \quad (7b)$$

\therefore Equation (6) becomes

$$\eta = \pm \frac{M}{2\theta\sigma} \left[\sqrt{(\theta + 1)^2 - 4\theta Z_0} - \sqrt{(\theta + 1)^2 - 4\theta Z} \right] \quad \dots \quad (8)$$

Since η is always positive we have to choose the positive sign in equation (8), whether θ is positive or negative.

Hence

$$\eta = \frac{M}{2\theta\sigma} \left[\sqrt{(\theta + 1)^2 - 4\theta Z_0} - \sqrt{(\theta + 1)^2 - 4\theta Z} \right] \quad \dots \quad (8a)$$

Again from (3c) and (3e) we get

$$\frac{dN}{d\tau} = \frac{dZ}{d\tau} + \Psi (T')^{-\frac{1}{2}} \frac{df}{d\tau}$$

Substituting for $\frac{df}{d\tau}$ from equation (3d) we have

$$\frac{dN}{d\tau} = \frac{dZ}{d\tau} + \Psi (T')^{\frac{1}{2}} \left[- \frac{1}{\sqrt{(\theta + 1)^2 - 4\theta Z}} \frac{dZ}{d\tau} \right] \quad \dots \quad (9)$$

Integrating the above equation,

$$N = Z - \Psi \int_{z_N}^Z \left[\frac{(T')^{-\frac{1}{2}}}{\sqrt{(\theta+1)^2 - 4\theta Z}} \right] dZ \quad \dots \quad (9a)$$

Using equations (3b), (3c), (3d) and (3f) we obtain

$$\frac{d}{d\tau}(NT') = -\frac{(\bar{\gamma}-1)\sigma}{M} \eta \frac{d\eta}{d\tau} + \frac{dZ}{d\tau} - \gamma \Psi (T')^{\frac{1}{2}} \frac{1}{\sqrt{(\theta+1)^2 - 4\theta Z}} \frac{dZ}{d\tau} \quad \dots \quad (10)$$

Assuming that the nozzle opens before the shot starts, the integration of equation (10) gives

$$NT' = -\frac{(\bar{\gamma}-1)\sigma}{2M} \eta^2 + Z - \gamma \Psi \int_{z_N}^Z \left[\frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2 - 4\theta Z}} \right] dZ \quad \dots \quad (10a)$$

Substituting for η and N from (8a) and (9a) we get

$$T' \left[Z - \Psi \int_{z_N}^Z \left\{ \frac{(T')^{-\frac{1}{2}}}{\sqrt{(\theta+1)^2 - 4\theta Z}} \right\} dZ \right] = -\frac{(\bar{\gamma}-1)\sigma}{2M} \frac{M^2}{4\theta^2\sigma^2} \left[a - \sqrt{(\theta+1)^2 - 4\theta Z} \right]^2 + Z \\ - \gamma \Psi \int_{z_N}^Z \left\{ \frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2 - 4\theta Z}} \right\} dZ \quad \dots \quad (10b)$$

$$\text{where } a = \sqrt{(\theta+1)^2 - 4\theta Z_0}.$$

Equation (10b) can be written as

$$ZT' - \Psi T' \int_{z_N}^Z \left\{ \frac{(T')^{-\frac{1}{2}}}{\sqrt{(\theta+1)^2 - 4\theta Z}} \right\} dZ + \gamma \Psi \int_{z_N}^Z \left\{ \frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2 - 4\theta Z}} \right\} dZ = A(Z) \quad (10c)$$

$$\text{where } A(Z) = Z - \frac{1}{2} \frac{M(\gamma-1)}{4\theta^2\sigma} [a - \sqrt{(\theta+1)^2 - 4\theta Z}]^2 \quad \dots \quad (11)$$

If the nozzle opens after the shot starts equation (10) is valid with $\Psi = 0$, up to the instant of nozzle opening. The solution of the system of equations (3) is thus reduced to the solution of the single equation (10c) for T' . When T' is obtained from (10c) as a function of Z , we find N from (9a) and NT from (10a). To obtain ξ we eliminate ζ between (3a) and (3b). Thus

$$\eta \frac{d\eta}{d\xi} = \frac{M}{\sigma} \frac{NT'}{\xi} \left(1 + \frac{kCN}{6W} \right) = \frac{(2+\sigma)M}{3\sigma} \frac{NT'}{\xi}.$$

Integrating we get

$$\log \xi = \frac{3\sigma}{(2+\sigma)M} \int_0^\eta \frac{\eta d\eta}{NT'}.$$

Using (8a) and (6) we obtain

$$\log \xi = \frac{3M}{2\theta\sigma(2+\sigma)} \int_{z_0}^Z \left\{ \frac{a}{\sqrt{(\theta+1)^2 - 4\theta Z}} - 1 \right\} \frac{dZ}{NT'} \quad \dots \quad (12)$$

The pressure ζ is given by (3a), as

$$\zeta = \frac{(2+\sigma)}{2} \frac{NT'}{\xi} \quad \dots \quad (13)$$

3. THE APPROXIMATE SOLUTION OF THE EQUATION

Introducing a new variable u defined by

$$u = \int_{z_N}^Z \left\{ \frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2 - 4\theta Z}} \right\} dZ \quad \dots \quad (14)$$

equation (10c) becomes

$$Z[(\theta+1)^2 - 4\theta Z] \left(\frac{du}{dZ} \right)^2 - A(Z) = \Psi \left[\{(\theta+1)^2 - 4\theta Z\} \left(\frac{du}{dZ} \right)^2 \int \frac{1}{\{(\theta+1)^2 - 4\theta Z\}} \left(\frac{dZ}{du} \right)^2 du - \gamma u \right] \quad \dots \quad (15)$$

For the solution of the above equation we substitute for u an expansion in powers of Ψ ,

$$u = [u^{(0)} + \Psi u^{(1)} + \Psi u^{(2)} + \dots] \quad \dots \quad (16)$$

and determine the coefficients of the successive terms in this expansion by comparison of coefficients of powers of Ψ on both sides of equation (15). Since Ψ is small in most of the practical cases, we neglect the terms involving Ψ^2 and higher powers of Ψ .

Thus

$$Z[(\theta+1)^2 - 4\theta Z] \left(\frac{du^{(0)}}{dZ} \right)^2 = A(Z) \quad \dots \quad (17)$$

so that

$$u^{(0)} = \int_{z_N}^Z \left[\frac{A(Z)}{Z\{(\theta+1)^2 - 4\theta Z\}} \right]^{\frac{1}{2}} dZ \quad \dots \quad (18)$$

$$Z\{(\theta+1)^2 - 4\theta Z\} \frac{du^{(1)}}{dZ} = \frac{1}{2} \left[\left[\frac{\{(\theta+1)^2 - 4\theta Z\} A(Z)}{Z} \right]^{\frac{1}{2}} \int_{z_N}^Z \left[\frac{Z}{A(Z)\{(\theta+1)^2 - 4\theta Z\}} \right]^{\frac{1}{2}} dZ - \gamma \left[\frac{Z\{(\theta+1)^2 - 4\theta Z\}}{A(Z)} \right]^{\frac{1}{2}} \int_{z_N}^Z \left[\frac{A(Z)}{Z\{(\theta+1)^2 - 4\theta Z\}} \right]^{\frac{1}{2}} dZ \right] \quad \dots \quad (19)$$

To the same order we have from (9a) and (10a)

$$\begin{aligned} N &= Z - \Psi \int_{z_N}^Z \left\{ \frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2 - 4\theta Z}} \right\} dZ \\ &= Z - \Psi \int_{z_N}^Z \left[\frac{Z}{A(Z)\{(\theta+1)^2 - 4\theta Z\}} \right]^{\frac{1}{2}} dZ \quad \dots \quad (20) \end{aligned}$$

$$\begin{aligned}
 NT' &= A(Z) - \gamma \Psi \int_{Z_N}^Z \left\{ \frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2 - 4\theta Z}} \right\} dZ \\
 &= A(Z) - \gamma \Psi u^{(0)} \\
 &= A(Z) - \gamma \Psi \int_{Z_N}^Z \left[\frac{A(Z)}{Z\{(\theta+1)^2 - 4\theta Z\}} \right]^{\frac{1}{2}} dZ \quad \dots \quad (21)
 \end{aligned}$$

Equations (20) and (21) give N and NT' as explicit functions of Z , from which the variation of T' with Z can be derived. Substituting the value of NT' from equation (21) in equation (12) we get ξ as a function of Z and thence ζ is given by (13). Thus all the ballistic parameters can be determined by evaluating a few integrals. These integrals can be easily evaluated by the use of any of the standard formulae for approximate quadrature, e.g. Simpson's Rule. Assuming Z_0 and Z_N to be small (which is the case), the integrands of these integrals can be expanded in powers of Z_0 and Z_N and we need retain only the terms up to the first order in Z_0 and Z_N , then the resulting integrals can be evaluated. Since this latter procedure is very laborious, it is convenient to evaluate the integrals by numerical quadrature.

4. MAXIMUM PRESSURE

The maximum pressure occurs when $\frac{d\zeta}{dZ} = 0$. Differentiating (13) with respect to Z we get

$$\frac{d\zeta}{dZ} = \frac{(2+\sigma)}{3} \left[\frac{1}{\xi} \frac{d}{dZ} (NT') - NT' \frac{1}{\xi^2} \frac{d\xi}{dZ} \right] = 0$$

or
$$\frac{d}{dZ} (NT') = NT' \frac{d}{dZ} (\log \xi) = NT' \frac{d}{dZ} (\log \xi).$$

Substituting for $\log \xi$ and NT' from equations (12) and (21) and denoting the value of Z for maximum pressure by Z_1 we get the following equation for the determination of Z_1

$$A'(Z_1) - \gamma \Psi \left[\frac{A(Z_1)}{Z_1\{(\theta+1)^2 - 4\theta Z_1\}} \right]^{\frac{1}{2}} = \frac{3M}{2\theta\sigma(2+\sigma)} \left[\frac{a}{\sqrt{(\theta+1)^2 - 4\theta Z_1}} - 1 \right] \quad (22)$$

The solution of (22) may be obtained by the method of successive approximations. Neglecting the term in Ψ and solving the equation we determine Z_1 to the first approximation. If we substitute the value of Z_1 in the coefficient of Ψ in (22) and again solve for Z_1 , we obtain Z_1 to the second approximation. In this way Z_1 can be calculated to any desired accuracy.

The peak pressure ζ_1 is then calculated from (21), (12) and (13),

$$\zeta_1 = \frac{(2+\sigma)}{3} \frac{(NT')_1}{\xi_1} \quad \dots \quad (23)$$

5. NOZZLE-START PRESSURE AND SHOT-START PRESSURE

Now we consider the relation between maximum pressure and nozzle-start pressure for a given value of shot-start pressure. If the shot starts after the nozzle opening, ($Z_N < Z_0$), it is clear that the instant of the shot-start will depend on

that of the nozzle-start for a given shot-start pressure. From (13) we obtain for $Z = Z_0$,

$$x = 0, \quad \xi = 1, \quad \zeta_0 = \frac{(2+\sigma)}{3} (NT')_0,$$

and hence from (21) and (11) we get

$$\zeta_0 = \frac{(2+\sigma)}{3} \left[Z_0 + \gamma \Psi \left\{ \sqrt{(\theta+1)^2 - 4\theta Z_0} - \sqrt{(\theta+1)^2 - 4\theta Z_N} \right\} \right] \quad \dots (24)$$

The variation of Z_N with Z_0 for a particular value of ζ_0 is expressed by the above equation. It may be noted that the relation is not linear, except when $\theta = 0$, i.e. for tubular propellants.

ABSTRACT

In this paper the author has extended the results obtained by Thiruvengkatachar and Venkatesan (1953) for the tubular propellants to the general case of a propellant of any shape, i.e. for any value of θ .

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REFERENCES

- Corner, J. (1947). The Internal Ballistics of Leaking Guns. *Proc. Roy. Soc.*, 188A.
 Corner, J. (1950). Theory of the Internal Ballistics of Guns. New York.
 H.M. Stationery Office, London (1951). Internal Ballistics.
 Thiruvengkatachar, V. R. and Venkatesan, N. S. (1953). On the Internal Ballistics of Leaking Guns. *Proc. Nat. Inst. Sci.*, 19, 829-837.

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THERMODYNAMICS OF IRREVERSIBLE PROCESSES APPLIED TO THERMAL TRANSPIRATION

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1. INTRODUCTION

A large number of non-equilibrium phenomena have been successfully treated by the thermodynamics of irreversible processes (Denbigh, 1951; de Groot, 1952). Recently the Soret-effect in certain binary mixtures has been interpreted in terms of non-equilibrium thermodynamic functions such as heats of transfer (Rutherford and Drickamer, 1954). In previous communications (Srivastava, Rastogi and Varma, 1954; Rastogi and Srivastava, 1955) expressions have been deduced for thermal effusion and thermo-molecular pressure effect for species involving isomeric reactions or the reactions of the type $X_n \rightleftharpoons nX$. Expressions for the chemical affinity and the deviation in concentration from the equilibrium values have been derived for each chamber in the steady state. In the present paper we propose to extend the treatment to the reaction of the type $xA \rightleftharpoons yB + zC$, which is more common.

2. ENTROPY PRODUCTION AND THE STATIONARY STATE OF THE FIRST ORDER

Our system consists of two chambers, I and II, communicating with each other through a narrow opening or a capillary. The two chambers are maintained at different temperatures, chamber II being maintained at a higher temperature. We suppose that the system contains a mixture of species A , B and C , amongst which the reaction $xA \rightleftharpoons yB + zC$ takes place, where x , y and z are the stoichiometric numbers. The superscripts I and II refer to the quantities in the corresponding chambers. Following the terminology as used in previous papers, we get the following expression for the entropy production:—

$$\sigma = \frac{d_i S}{dt} = - (\Delta T/T^2) J_u - \Delta(\mu_1/T) J_1 - \Delta(\mu_2/T) J_2 - \Delta(\mu_3/T) J_3 \\ + (A^I/T^I) J_I + (A^{II}/T^{II}) J_{II} \dots \dots \dots (1)$$

where Δ refers to the difference of the quantity in the two chambers, and the subscripts 1, 2 and 3 refer respectively to the chemical species A , B and C respectively. The respective flows are defined as follows:—

$$J_1 = - \sum_{h=1}^{\infty} L_{1h} \Delta(\mu_h/T) - L_{1u} (\Delta T/T^2) \\ J_2 = - \sum_{h=1}^3 L_{2h} \Delta(\mu_h/T) - L_{2u} (\Delta T/T^2)$$

$$\begin{aligned}
 J_s &= - \sum_{k=1}^3 L_{sk} \Delta(\mu_k/T) - L_{su} (\Delta T/T^2) \\
 J_u &= - \sum_{k=1}^3 L_{uk} \Delta(\mu_k/T) - L_{uu} (\Delta T/T^2) \\
 J_I &= L_{cc} (A^I/T^I) \\
 J_{II} &= L'_{cc} (A^{II}/T^{II}) \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)
 \end{aligned}$$

The affinity A is defined by the relation $A = x\mu_1 - y\mu_2 - z\mu_3$, where μ is the chemical potential. In the above expressions L_{ik} and L_{uk} represent the phenomenological coefficients. L_{cc} and L'_{cc} are the coefficients associated with the chemical reaction rate J .

Inserting equation (2) in equation (1) and making use of Onsager reciprocal relations we obtain

$$\begin{aligned}
 \sigma &= L_{11} \{ \Delta(\mu_1/T) \}^2 + L_{22} \{ \Delta(\mu_2/T) \}^2 + L_{33} \{ \Delta(\mu_3/T) \}^2 + L_{uu} \{ \Delta T/T^2 \}^2 \\
 &\quad + 2L_{12} \Delta(\mu_1/T) \cdot \Delta(\mu_2/T) + 2L_{13} \Delta(\mu_1/T) \cdot \Delta(\mu_3/T) + 2L_{23} \Delta(\mu_2/T) \cdot \Delta(\mu_3/T) \\
 &\quad + 2L_{1u} \Delta(\mu_1/T) \cdot (\Delta T/T^2) + 2L_{2u} \Delta(\mu_2/T) \cdot (\Delta T/T^2) + 2L_{3u} \Delta(\mu_3/T) \cdot (\Delta T/T^2) \\
 &\quad + L_{cc} (A^I/T^I)^2 + L'_{cc} (A^{II}/T^{II})^2 \quad \dots \quad \dots \quad \dots \quad \dots \quad (3)
 \end{aligned}$$

We shall consider the stationary state of the first order when ΔT is kept constant. Corresponding to each of the independent forces, $\Delta(\mu_1/T)$, $\Delta(\mu_2/T)$, $\Delta(\mu_3/T)$ and (A^I/T^I) we have the conditions

$$\frac{\delta \sigma}{\delta \Delta(\mu_1/T)} = 0; \quad \frac{\delta \sigma}{\delta \Delta(\mu_2/T)} = 0; \quad \frac{\delta \sigma}{\delta \Delta(\mu_3/T)} = 0; \quad \frac{\delta \sigma}{\delta (A^I/T^I)} = 0 \quad \dots \quad (4)$$

But these forces are related by the relation

$$A^{II}/T^{II} = A^I/T^I + x \cdot \Delta(\mu_1/T) - y \cdot \Delta(\mu_2/T) - z \cdot \Delta(\mu_3/T) \quad \dots \quad (5)$$

Using equations (4) and (5) we obtain the following set of relations from equation (3) :—

$$\left. \begin{aligned}
 L_{11} \Delta(\mu_1/T) + L_{12} \Delta(\mu_2/T) + L_{13} \Delta(\mu_3/T) + L_{1u} (\Delta T/T^2) - L_{cc} \cdot x \cdot \frac{A^I}{T^I} &= 0 \\
 L_{12} \Delta(\mu_1/T) + L_{22} \Delta(\mu_2/T) + L_{23} \Delta(\mu_3/T) + L_{2u} (\Delta T/T^2) + L_{cc} \cdot y \cdot \frac{A^I}{T^I} &= 0 \\
 L_{13} \Delta(\mu_1/T) + L_{23} \Delta(\mu_2/T) + L_{33} \Delta(\mu_3/T) + L_{3u} (\Delta T/T^2) + L_{cc} \cdot z \cdot \frac{A^I}{T^I} &= 0 \\
 L_{cc} \cdot A^I/T^I + L'_{cc} \cdot A^{II}/T^{II} &= 0
 \end{aligned} \right\} \quad (6)$$

For such a three-component system the chemical potential may be written in the form

$$\Delta(\mu_k/T) = -h_k (\Delta T/T^2) + v_k (\Delta P/T) + \sum_{i=1}^2 \left(\frac{\partial \mu_k}{\partial c_i} \right) \frac{\Delta c_i}{T} \quad (k=1, 2, 3) \quad \dots \quad (7)$$

where h_k and v_k are the partial specific enthalpy and partial specific volume of the component k , and c_i is the mass fraction of the species i .

On solving equations (6) we obtain

$$\frac{\Delta P}{\Delta T} = \frac{1}{T(v_1 d_1 + v_2 d_2 + v_3 d_3)} \cdot [(h_1 + L_1)d_1 + (h_2 + L_2)d_2 + (h_3 + L_3)d_3] \quad \dots \quad (8)$$

$$\frac{\Delta c_1}{\Delta T} = \frac{\left[(h_1 + L_1) \left(v_3 \frac{\partial \mu_2}{\partial c_2} - v_2 \frac{\partial \mu_3}{\partial c_2} \right) + (h_2 + L_2) \left(v_1 \frac{\partial \mu_3}{\partial c_2} - v_3 \frac{\partial \mu_2}{\partial c_2} \right) + (h_3 + L_3) \left(v_2 \frac{\partial \mu_1}{\partial c_2} - v_1 \frac{\partial \mu_2}{\partial c_2} \right) \right]}{T(v_1 d_1 + v_2 d_2 + v_3 d_3)} \quad \dots \quad (9)$$

$$\frac{\Delta c_2}{\Delta T} = \frac{\left[(h_1 + L_1) \left(v_2 \frac{\partial \mu_3}{\partial c_1} - v_3 \frac{\partial \mu_1}{\partial c_1} \right) + (h_2 + L_2) \left(v_3 \frac{\partial \mu_1}{\partial c_1} - v_1 \frac{\partial \mu_3}{\partial c_1} \right) + (h_3 + L_3) \left(v_1 \frac{\partial \mu_2}{\partial c_1} - v_2 \frac{\partial \mu_1}{\partial c_1} \right) \right]}{T(v_1 d_1 + v_2 d_2 + v_3 d_3)} \quad \dots \quad (10)$$

$$\frac{A^I}{T^I} \cdot \frac{1}{\Delta T} = \frac{L'_{cc}}{L_{cc} + L'_{cc}} \cdot \frac{1}{T^2} [zL_3 + yL_2 - xL_1]; \quad \frac{A^{II}}{T^{II}} \cdot \frac{1}{\Delta T} = \frac{-L_{cc}}{L_{cc} + L'_{cc}} \cdot \frac{1}{T^2} [zL_3 + yL_2 - xL_1] \quad \dots \quad (11)$$

where

$$L_1 = \frac{(a_1 L_{1u} + b_1 L_{2u} + c_1 L_{3u}) L_{cc} \cdot L'_{cc} + (a_2 L_{1u} + b_2 L_{2u} + c_2 L_{3u})(L_{cc} + L'_{cc})}{XL_{cc} \cdot L'_{cc} + Y(L_{cc} + L'_{cc})}$$

$$L_2 = \frac{(b_1 L_{1u} + e_1 L_{2u} + f_1 L_{3u}) L_{cc} \cdot L'_{cc} + (b_2 L_{1u} + e_2 L_{2u} + f_2 L_{3u})(L_{cc} + L'_{cc})}{XL_{cc} \cdot L'_{cc} + Y(L_{cc} + L'_{cc})}$$

$$L_3 = \frac{(c_1 L_{1u} + f_1 L_{2u} + g_1 L_{3u}) L_{cc} \cdot L'_{cc} + (c_2 L_{1u} + f_2 L_{2u} + g_2 L_{3u})(L_{cc} + L'_{cc})}{XL_{cc} \cdot L'_{cc} + Y(L_{cc} + L'_{cc})}$$

$$a_1 = z^2 L_{22} - 2yz L_{23} + y^2 L_{33}; \quad a_2 = L_{22} L_{33} - L_{23}^2$$

$$b_1 = -z^2 L_{12} + yz L_{13} - xz L_{23} + xy L_{33}; \quad b_2 = L_{13} L_{23} - L_{12} L_{33}$$

$$c_1 = yz L_{12} - y^2 L_{13} + xz L_{22} - xy L_{23}; \quad c_2 = L_{12} L_{23} - L_{13} L_{22}$$

$$e_1 = z^2 L_{11} + 2xz L_{13} + x^2 L_{33}; \quad e_2 = L_{11} L_{33} - L_{13}^2$$

$$f_1 = -yz L_{11} - xz L_{12} - xy L_{13} - x^2 L_{23}; \quad f_2 = L_{12} L_{13} - L_{11} L_{23}$$

$$g_1 = y^2 L_{11} + 2xy L_{12} + x^2 L_{22}; \quad g_2 = L_{11} L_{22} - L_{12}^2$$

$$X = -a_2 x^2 - e_2 y^2 - g_2 z^2 + 2b_2 xy + 2c_2 xz - 2f_2 yz$$

$$Y = L_{11} L_{23}^2 + L_{22} L_{13}^2 + L_{33} L_{12}^2 - L_{11} L_{22} L_{33} - L_{12} L_{13} L_{23}$$

$$d_1 = \left(\frac{\partial \mu_2}{\partial c_1} \cdot \frac{\partial \mu_3}{\partial c_2} - \frac{\partial \mu_2}{\partial c_2} \cdot \frac{\partial \mu_3}{\partial c_1} \right)$$

$$d_2 = \left(\frac{\partial \mu_3}{\partial c_1} \cdot \frac{\partial \mu_1}{\partial c_2} - \frac{\partial \mu_3}{\partial c_2} \cdot \frac{\partial \mu_1}{\partial c_1} \right)$$

$$d_3 = \left(\frac{\partial \mu_1}{\partial c_1} \cdot \frac{\partial \mu_2}{\partial c_2} - \frac{\partial \mu_1}{\partial c_2} \cdot \frac{\partial \mu_2}{\partial c_1} \right)$$

The equations (8), (9), (10) and (11) are fairly complicated and cannot be used in practice. As shown previously we can get an expression for affinity which does

not involve phenomenological coefficients. Thus, from equations (8), (9) and (10) we get

$$L_1 = T \left[\frac{\Delta P}{\Delta T} v_1 + \frac{\Delta c_1}{\Delta T} \cdot \frac{\partial \mu_1}{\partial c_1} + \frac{\Delta c_2}{\Delta T} \cdot \frac{\partial \mu_1}{\partial c_2} \right] - h_1 \quad \dots \quad (12)$$

$$L_2 = T \left[\frac{\Delta P}{\Delta T} v_2 + \frac{\Delta c_1}{\Delta T} \cdot \frac{\partial \mu_2}{\partial c_1} + \frac{\Delta c_2}{\Delta T} \cdot \frac{\partial \mu_2}{\partial c_2} \right] - h_2 \quad \dots \quad (13)$$

$$L_3 = T \left[\frac{\Delta P}{\Delta T} v_3 + \frac{\Delta c_1}{\Delta T} \cdot \frac{\partial \mu_3}{\partial c_1} + \frac{\Delta c_2}{\Delta T} \cdot \frac{\partial \mu_3}{\partial c_2} \right] - h_3 \quad \dots \quad (14)$$

Substituting the values of L_1 , L_2 and L_3 in equation (11) we obtain

$$\begin{aligned} \frac{A'}{T^2} \cdot \frac{1}{\Delta T} = \frac{L'_{cc}}{L_{cc} + L_{cc}} \cdot \frac{1}{T^2} \left[T \cdot \frac{\Delta P}{\Delta T} \cdot (zv_3 + yv_2 + xv_1) + T \cdot \frac{\Delta c_1}{\Delta T} \left(z \frac{\partial \mu_3}{\partial c_1} + y \frac{\partial \mu_2}{\partial c_1} - x \frac{\partial \mu_1}{\partial c_1} \right) \right. \\ \left. + T \cdot \frac{\Delta c_2}{\Delta T} \left(z \frac{\partial \mu_3}{\partial c_2} + y \frac{\partial \mu_2}{\partial c_2} - x \frac{\partial \mu_1}{\partial c_2} \right) - (zh_3 + yh_2 - xh_1) \right] \quad \dots \quad (15) \end{aligned}$$

If we consider a mixture of perfect gases, the above relation may be further simplified. For then,

$$v_k = \frac{RT}{M_k \cdot P}$$

and h_k can also be evaluated. For enthalpy we have earlier (Rastogi and Srivastava, 1954) used the expression

$$h_k = \frac{5}{2} \cdot \left(\frac{RT}{M_k} \right)$$

Strictly speaking this is true for the translational component of enthalpy. Even in a monatomic gas the atoms possess internal vibrations, but the vibrational energies are almost negligible and hence the above equation is correct for at least monatomic gases, particularly at not too high temperatures. For diatomic molecules there are three degrees of freedom of translation and two of rotation about the two axes perpendicular to the line joining the atoms. If we do not take into account the structure of atoms, we have

$$h_k = \frac{7}{2} \cdot \frac{RT}{M_k}$$

Similarly for a triatomic gas we have,

$$h_k = \frac{4RT}{M_k}$$

since a triatomic molecule has three translational and three rotational degrees of freedom. In this case we do not consider the vibrational degrees of freedom since the vibrations are usually not fully excited.

Thus for diatomic molecules equation (15) yields

$$\begin{aligned} \frac{A'}{T^2} \cdot \frac{1}{\Delta T} = \frac{L'_{cc}}{L_{cc} + L_{cc}} \cdot \frac{R}{T} \left[\frac{\Delta \log P}{\Delta \log T} \left(\frac{z}{M_3} + \frac{y}{M_2} - \frac{x}{M_1} \right) \right. \\ \left. + \frac{1}{\Delta \log T} (z \Delta \log c_3 + y \Delta \log c_2 - x \Delta \log c_1) - \frac{7}{2} \left(\frac{z}{M_3} + \frac{y}{M_2} - \frac{x}{M_1} \right) \right] \quad (16) \end{aligned}$$

From the definition of affinity it follows that

$$A = RT \cdot \delta c_1 \left[\frac{x}{c_1^0} + \frac{z}{c_3^0} \right] + RT \delta c_2 \left[\frac{z}{c_3^0} - \frac{y}{c_2^0} \right] \quad \dots (17)$$

for $\delta c_1 \ll c_1^0$ and $\delta c_2 \ll c_2^0$ where δc_k is the deviation in concentration in a particular chamber. Thus from equations (16) and (17) one can obtain a relation between the deviation in concentration of the respective species in a particular chamber and the temperature, pressure and the concentration prevailing in the two chambers. The above relation (16) can be easily checked provided suitable experiments are performed. In the next section equation (16) would be used to calculate the free energy change in either chamber for the thermal decomposition of deuterium iodide.

3. THERMAL DECOMPOSITION OF DEUTERIUM IODIDE

The kinetics of the reaction $2DI \rightleftharpoons D_2 + I_2$ has been recently studied (Taylor and Crist, 1941). The advantage in using this reaction is that L_{cc} and L'_{cc} can be readily calculated as the specific rate constants for both the forward and back reactions have been experimentally determined. In the following calculations experimental data for two temperatures, viz., 666.8°K. and 698.8°K. have been utilized for which the equilibrium concentration in moles/c.c. is given. We assume that chamber I is maintained at 666.8°K. and chamber II is maintained at 698.8°K. We further suppose that equilibrium concentrations and pressures are not affected significantly if the two chambers are connected by a narrow opening. For the case of thermal decomposition of deuterium iodide we shall have the following equation for affinity:—

$$\frac{A'}{T^I} \cdot \frac{1}{\Delta T} = \frac{L'_{cc}}{L_{cc} + L'_{cc}} \cdot \frac{R}{T} \left[\left(\frac{\Delta \log P}{\Delta \log T} - \frac{7}{2} \right) \left(\frac{1}{M_3} + \frac{1}{M_2} - \frac{2}{M_1} \right) \right. \\ \left. \Delta \log \bar{T} (\Delta \log c_3 + \Delta \log c_2 - 2 \Delta \log c_1) \right]$$

Here

$$\frac{L_{cc}}{L'_{cc}} = \frac{k[M_1^0]_{T^I}^2}{k'[M_1^0]_{T^{II}}^2},$$

$[M_1^0]$ represents the mass at equilibrium and k and k' are the specific rate constants for the decomposition at the two temperatures, T^I and T^{II} . The following table gives the values of affinity for different pressures and concentrations in each chamber:—

TABLE I

$T^I = 666.8^\circ\text{K.}; T^{II} = 698.8^\circ\text{K.}$

P^I (atm.)	P^{II} (atm.)	C_1^I	C_2^I	C_3^I	C_1^{II}	C_2^{II}	C_3^{II}	A (cal.)
1.099	1.333	0.857	0.00621	0.1202	0.8642	0.00771	0.1280	270.61
1.023	1.098	0.887	0.00991	0.1027	0.8816	0.00984	0.1084	63.06
1.064	1.125	0.872	0.00838	0.1200	0.8702	0.00858	0.1213	11.21
1.180	1.156	0.848	0.00630	0.1460	0.8563	0.00589	0.1362	-188.71

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ABSTRACT

The non-equilibrium thermodynamics of thermal transpiration of species involving the reaction of the type $xA \rightleftharpoons yB + zC$ has been developed by using Onsager reciprocal relations. Expressions for affinity which do not involve phenomenological coefficients are deduced for either chamber. The equilibrium data for the thermal decomposition of deuterium iodide has been used to calculate the value of affinity in order to have an estimate of the shift from thermodynamic equilibrium.

REFERENCES

- de Groot, S. R. (1952). *Thermodynamics of irreversible processes*, Amsterdam, North-Holland Publishing Company.
- Denbigh, K. G. (1951). *Thermodynamics of steady state*, London, Methuen.
- Rutherford, W. H. and Drickamer, H. G. (1954). Theory of thermal diffusion in liquids and the use of pressure to investigate the theory. *J. Chem. Phys.*, **22**, 1166.
- Rastogi, R. P. and Srivastava, R. C. (1955). Non-equilibrium thermodynamics of thermal transpiration of chemically reacting species. *Trans. Faraday Soc.*, **51**, 343.
- Srivastava, B. N., Rastogi, R. P. and Varma, A. S. (1954). Non-equilibrium thermodynamics with finite reaction velocity and liquid He II. *Physica*, **20**, 1271.
- Taylor, A. H. and Crist, R. H. (1941). Rate and Equilibrium studies on the thermal reaction of hydrogen and iodine. *J. Amer. Chem. Soc.*, **63**, 1376.

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INTERNAL BALLISTICS OF COMPOSITE CHARGES WITH NON-LINEAR LAW OF BURNING

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1. INTRODUCTION

The problem of using composite charges in a gun has been discussed in the 'Theory of Internal Ballistics of Guns' by Corner (1950) and in the 'Internal Ballistics' (1951) by reducing the problem to one of an equivalent single charge. In 'Internal Ballistics' (1951) only a particular case of two charges with different shapes and web sizes of the same propellant is considered and an approximate solution is given by making use of a modified form factor. Corner (1950) considers the more general problem of two charges of different shapes, sizes and compositions. The problem is reduced to that of a single equivalent charge with adjusted parameters.

Venkatesan and myself (1953) have, however, given a direct treatment of the general problem based on the Hunt-Hinds system under the assumptions that the co-volume correction is negligible for each propellant and that γ is the same for both the charges.

But in all these works, the linear law of burning has been assumed. In the present paper, I have discussed the problem of two composite charges (of different shapes, sizes and compositions) under the non-linear law of burning and for any values of the form-coefficients θ_1 and θ_2 . The treatment comes out similar to the Clemmow's system (1928 ; 1951) for a single charge.

2. ASSUMPTIONS

The following assumptions have been made in the discussion of this paper :—

- (i) That the co-volume of the gases equals the specific volume for each propellant, i.e., the co-volume correction is negligible.
- (ii) That $\gamma_1 = \gamma_2 = \gamma$. This assumption is justified by the fact that γ is practically the same for most of the propellants.
- (iii) That the pressure-index law is the same for both the propellants, i.e., $\alpha_1 = \alpha_2 = \alpha$.

Under the above assumptions, the theory is also applicable to a very large number of cases in which composite charges of different shapes and sizes and of the same propellant are used.

3. BASIC EQUATIONS

The classical basic equations of Internal Ballistics with a single charge have been derived in 'Internal Ballistics' (1951) and in our case they become,

$$\frac{F_1 C_1 z_1 + F_2 C_2 z_2}{A l} - p \left(1 + \frac{x}{l} \right) + \frac{\gamma - 1}{2 A l} w_1 v^2 \quad \therefore \quad \dots \quad (1)$$

where $w_1 = 1.06w + \frac{C_1}{3} + \frac{C_2}{3} \quad \dots \quad (1A)$

and $Al = K_0 - \left(\frac{C_1}{\delta_1} + \frac{C_2}{\delta_2} \right) \quad \dots \quad (1B)$

$$w_1 \frac{dv}{dt} = Ap \quad \dots \quad (2)$$

$$z_1 = (1-f_1)(1+\theta_1 f_1) \quad \dots \quad (3A)$$

$$z_2 = (1-f_2)(1+\theta_2 f_2) \quad \dots \quad (3B)$$

$$D_1 \frac{df_1}{dt} = -\beta_1 p^\alpha \quad \dots \quad (4A)$$

$$D_2 \frac{df_2}{dt} = -\beta_2 p^\alpha \quad \dots \quad (4B)$$

where $C_1, F_1, \beta_1, D_1, \theta_1, f_1, z_1$ refer to the first charge, and $C_2, F_2, \beta_2, D_2, \theta_2, f_2, z_2$ to the second charge, the letters having the usual meanings.

4. SOLUTION OF THE BASIC EQUATIONS

In order to make the quantities dimensionless, we make the following substitutions :—

$$\left(1 + \frac{x}{l} \right) = \xi \quad \dots \quad (5)$$

$$\frac{AD_1}{F_1 C_1 \beta_1} \left(\frac{F_1 C_1}{Al} \right)^{1-\alpha} v = \eta_1 \quad \dots \quad (6A)$$

$$\frac{AD_2}{F_2 C_2 \beta_2} \left(\frac{F_2 C_2}{Al} \right)^{1-\alpha} v = \eta_2 \quad \dots \quad (6B)$$

$$\frac{Al}{F_1 C_1} \cdot p = \zeta_1 \quad \dots \quad (6C)$$

$$\frac{Al}{F_2 C_2} \cdot p = \zeta_2 \quad \dots \quad (6D)$$

$$\frac{A^2 D_1^2}{F_1 C_1 \beta_1^2 w_1} \cdot \left(\frac{F_1 C_1}{Al} \right)^{2-2\alpha} = M_1 \quad \dots \quad (6E)$$

$$\frac{A^2 D_2^2}{F_2 C_2 \beta_2^2 w_1} \cdot \left(\frac{F_2 C_2}{Al} \right)^{2-2\alpha} = M_2 \quad \dots \quad (6F)$$

Then the equations (1), (2), (3) and (4) become

$$\left(z_2 + \frac{F_1 C_1}{F_2 C_2} z_1 \right) = \zeta_2 \xi + \frac{1}{2}(\gamma - 1) \frac{\eta_2^2}{M_2} \quad \dots \quad (7)$$

$$\eta_1 \frac{d\eta_1}{d\xi} = M_1 \zeta_1 \quad \dots \quad (8A)$$

$$\eta_2 \frac{d\eta_2}{d\xi} = M_2 \zeta_2 \quad \dots \quad (8B)$$

$$z_1 = (1-f_1)(1+\theta_1 f_1) \quad \dots \quad (9A)$$

$$z_2 = (1-f_2)(1+\theta_2 f_2) \quad \dots \quad (9B)$$

$$\eta_1 \frac{d\eta_1}{d\xi} = -\zeta_1^\alpha \quad \dots \quad (10A)$$

$$\eta_2 \frac{d\eta_2}{d\xi} = -\zeta_2^\alpha \quad \dots \quad (10B)$$

Differentiating (7) and using (8B), we get

$$\begin{aligned} d\left(z_2 + \frac{F_1 C_1}{F_2 C_2} z_1\right) &= \xi d\zeta_2 + \gamma \zeta_2 d\xi \\ &= \xi^{1-\gamma} d(\zeta_2 \xi^\gamma) \quad \dots \quad (11) \end{aligned}$$

Equations (8A) and (10A) give

$$\frac{d\eta_1}{df_1} = -M_1 \zeta_1^{1-\alpha} \quad \dots \quad (12A)$$

and equations (8B) and (10B) give

$$\frac{d\eta_2}{df_2} = -M_2 \zeta_2^{1-\alpha} \quad \dots \quad (12B)$$

\therefore from (12A) and (10A), we get

$$\frac{d}{df_1} \left(\zeta_1^\alpha \frac{d\xi}{df_1} \right) = M_1 \zeta_1^{1-\alpha} \quad \dots \quad (13A)$$

Similarly

$$\frac{d}{df_2} \left(\zeta_2^\alpha \frac{d\xi}{df_2} \right) = M_2 \zeta_2^{1-\alpha} \quad \dots \quad (13B)$$

Initially when the shot starts, we have

$$\left. \begin{aligned} x &= 0; v = 0; p = p_0; \xi = 1; f_1 = f_{10}; f_2 = f_{20}; \\ z_1 &= z_{10}; z_2 = z_{20}; \eta_1 = \eta_2 = 0. \end{aligned} \right\} \quad \dots \quad (14)$$

$$\text{Let} \quad \frac{\beta_1}{D_1} = \beta' \text{ and } \frac{\beta_2}{D_2} = \beta'' \quad \dots \quad (15)$$

Dividing (4A) by (4B) we get

$$\frac{df_1}{df_2} = \frac{\beta'}{\beta''} \quad \dots \quad (16)$$

which, on integrating and using initial conditions, give

$$\beta' f_2 - \beta'' f_1 = \beta' f_{20} - \beta'' f_{10} \quad \dots \quad (17)$$

Using (9A) and (9B), equation (17) becomes

$$\beta' \left(\frac{1+\theta_2}{4\theta_2} \right) \left[\sqrt{1-k_2 z_2} - \sqrt{1-k_2 z_{20}} \right] = \beta'' \left(\frac{1+\theta_1}{4\theta_1} \right) \left[\sqrt{1-k_1 z_1} - \sqrt{1-k_1 z_{10}} \right] \quad (18)$$

where

$$k_1 = \frac{4\theta_1}{(1+\theta_1)^2} \text{ and } k_2 = \frac{4\theta_2}{(1+\theta_2)^2} \quad \dots \quad (19)$$

Equations (9A) and (9B) give

$$df_1 = -\frac{1}{1+\theta_1} \cdot \frac{dz_1}{\sqrt{1-k_1 z_1}} \quad \dots \quad (20A)$$

and

$$df_2 = -\frac{1}{1+\theta_2} \cdot \frac{dz_2}{\sqrt{1-k_2 z_2}} \quad \dots \quad (20B)$$

From (18), we get

$$\sqrt{1-k_1 z_1} = \lambda \sqrt{1-k_2 z_2} + \mu \quad \dots \quad (21)$$

where

$$\lambda = \frac{\beta'}{\beta''} \cdot \frac{1+\theta_2}{1+\theta_1} \cdot \frac{\theta_1}{\theta_2} \quad \dots \quad (22A)$$

and

$$\mu = \sqrt{1-k_1 z_{10}} - \lambda \sqrt{1-k_2 z_{20}} \quad \dots \quad (22B)$$

Clearly when ignition takes place,

$$z_1 = z_2 = 0.$$

\therefore equation (21) gives

$$\mu = 1 - \lambda \quad \dots \quad (22C)$$

Hence

$$\sqrt{1-k_1 z_{10}} - \lambda \sqrt{1-k_2 z_{20}} = 1 - \lambda \quad \dots \quad (22D)$$

an expression giving the relation between z_{10} and z_{20} .

We can easily get with the help of (21)

$$z_2 + \frac{F_1 C_1}{F_2 C_2} z_1 = \mu' + \nu' z_2 - \frac{2\lambda\lambda'\mu}{k_2} \sqrt{1-k_2 z_2} \quad \dots \quad (23)$$

where

$$\left. \begin{aligned} \lambda' &= \frac{F_1 C_1}{F_2 C_2} \cdot \frac{k_2}{k_1} \\ \mu' &= \frac{1-\mu^2-\lambda^2}{k_2} \cdot \frac{F_1 C_1}{F_2 C_2} \cdot \frac{k_2}{k_1} \\ &= \frac{1-\mu^2-\lambda^2}{k_2} \lambda' \\ \nu' &= 1 + \lambda^2 \cdot \frac{F_1 C_1}{F_2 C_2} \cdot \frac{k_2}{k_1} \\ &= 1 + \lambda^2 \cdot \lambda' \end{aligned} \right\} \quad \dots \quad (24)$$

and

Now let

$$\begin{aligned} z_2 + \frac{F_1 C_1}{F_2 C_2} z_1 &= \mu' + \nu' z_2 - \frac{2\lambda\lambda'\mu}{k_2} \cdot \sqrt{1-k_2 z_2} \\ &= Z_2, \text{ say} \quad \dots \quad (25) \end{aligned}$$

and

$$\zeta_2 \zeta' = Y_2 \quad \dots \quad (26)$$

Then equation (11) becomes

$$d(Z_2) = \xi^{1-\gamma} dY_2 \quad \dots \quad (27)$$

From (13B), (20B) and (25), eliminating f_2 we get

$$(1+\theta_2)^2 \left[\lambda' \lambda \mu + \nu' \sqrt{1-k_2 z_2} \right] \cdot \frac{d}{dZ_2} \left[\xi_2^\alpha \{ \lambda' \lambda \mu + \nu' \sqrt{1-k_2 z_2} \} \frac{d\xi}{dZ_2} \right] \\ = M_2 \xi_2^{1-\alpha} \quad \dots \quad (28)$$

From (25), we have

$$\lambda' \lambda \mu + \nu' \sqrt{1-k_2 z_2} = a^{\frac{1}{2}} \sqrt{1-bZ_2} \quad \dots \quad (29)$$

where

$$a = \lambda'^2 \lambda^2 \mu^2 + \nu'^2 + k_2 \mu' \nu' \quad \dots \quad (30A)$$

and

$$b = \frac{k_2}{a} \nu' \quad \dots \quad (30B)$$

Let

$$bZ_2 = Z \quad \dots \quad (31A)$$

and

$$bY_2 = b\xi_2^\gamma Y = Y \quad \dots \quad (31B)$$

Then equation (27) gives

$$dZ = \xi^{1-\gamma} dY \quad \dots \quad (32)$$

and equations (28), (29) and (31) give

$$(1-Z) \left[\xi'' + \alpha \xi' \cdot \frac{\xi_2'}{\xi_2} \right] - \frac{1}{2} \xi' = \frac{M_2}{ab^2(1+\theta_2)^2} \cdot \xi_2^{1-2\alpha} \quad \dots \quad (33)$$

where the dashes ' and '' denote first and second derivatives with respect to Z .

From (32), we have

$$(\gamma-1) \frac{\xi'}{\xi} = \frac{Y''}{Y'} \quad \dots \quad (34A)$$

and

$$\frac{\xi''}{\xi'} = \frac{Y'''}{Y''} + \frac{2-\gamma}{\gamma-1} \cdot \frac{Y''}{Y'} \quad \dots \quad (34B)$$

From equation (31), we have

$$\frac{\xi_2'}{\xi_2} = \frac{Y'}{Y} - \frac{\gamma \xi'}{\xi} = \frac{Y'}{Y} - \frac{\gamma}{\gamma-1} \cdot \frac{Y''}{Y'} \quad \dots \quad (34C)$$

Eliminating ξ , ξ' , ξ_2' , ξ_1 and ξ_1' from equation (33) with the help of (31), (32), (34A), (34B) and (34C), we get

$$(1-Z) \left[\frac{Y'''}{Y''} + (n-2) \frac{Y''}{Y'} + \alpha \frac{Y'}{Y} \right] - \frac{1}{2} = Q \cdot \frac{(Y')^{2-2n}}{Y'' Y^{2\alpha-1}} \quad \dots \quad (35)$$

where

$$n = \frac{\gamma(1-\alpha)}{\gamma-1} \text{ and } Q = \frac{M_2(\gamma-1)}{(1+\theta_2)^2 ab^{3-2\alpha}} \quad \dots \quad (36)$$

—an equation similar to that obtained by Clemmow (1951; p. 123) for a single charge.

The equation (35) can be integrated numerically to give a series of values of Y , Y' , and Y'' in terms of Z for any specific given propellant (i.e. γ and α given) and a given value of Q , representing the loading conditions.

The initial conditions are :—

(i) when a finite shot-start pressure is assumed :

$$\begin{aligned} \text{then} \quad Z_0 &= bZ_{20} = b \left(z_{20} + \frac{F_1 C_1}{F_2 C_2} z_{10} \right) \\ Y_0 &= bY_{20} = b\zeta_{20} = b \cdot l \cdot p_0 \\ (Y')_0 &= 1 \text{ and } (Y'')_0 = 0 \end{aligned} \quad (i)$$

and (ii) when the shot-start pressure is taken to be zero :—

$$\begin{aligned} \text{then} \quad Z_0 &= 0; \quad Y_0 = 0 \\ (Y')_0 &= 1 \text{ and } (Y'')_0 = 0 \end{aligned} \quad (ii)$$

The shot-travel is obtained from (32) as

$$\xi^{\gamma-1} = \frac{dY}{dZ} \quad (37A)$$

$$\text{i.e.,} \quad x = l \left[(Y')^{\frac{1}{\gamma-1}} - 1 \right] \quad (37B)$$

and the pressure from (31B) as

$$\zeta_2 = \frac{\gamma}{b(Y')^{\frac{\gamma}{\gamma-1}}} \quad (38A)$$

$$\text{i.e.,} \quad p = \frac{F_2 C_2}{A b (Y')^{\frac{\gamma}{\gamma-1}}} \quad (38B)$$

Then equation (7) gives

$$\eta_2^2 = \frac{2M_2}{(\gamma-1)b} \left[Z - \frac{Y}{Y'} \right] \quad (39A)$$

$$\text{or} \quad v^2 = \frac{2F_2 C_2}{(\gamma-1)bw_1} \left[Z - \frac{Y}{Y'} \right] \quad (39B)$$

The equations (37A), (38A) and (39A), (or (37B), (38B) and (39B) respectively) giving the shot-travel, pressure and velocity in terms of Z , Y and the derivative Y' are valid so long as both the propellants are burning.

5. DIFFERENT CASES OF ALL-BURNT

Following Venkatesan and Patni (1953), we see from (17) that if

(i) $\beta'f_{20} < \beta''f_{10}$, then f_2 cannot become zero before f_1 . Hence charge C_1 must be burnt out first.

- (ii) $\beta'f_{20} < \beta''f_{10}$, then f_1 cannot become zero before f_2 . Hence charge C_2 must be burnt out first.
 (iii) $\beta'f_{20} = \beta''f_{10}$, then both the charges will have to be burnt out simultaneously.

Hence we see that two different cases arise, viz.—

I. The two propellants burn out at different times.

II. Both the propellants burn out simultaneously.

For the sake of definiteness, let us call that propellant which will burn first as C_1 .

Case I :—We have to consider this in two parts :—

- (i) When only C_2 is burning (and C_1 has been burnt out).
 (ii) When C_2 is also burnt out.

Case I(i) :—In this case, equation (7) becomes

$$\left(z_2 + \frac{F_1 C_1}{F_2 C_2}\right) = \zeta_2 \xi + \frac{1}{2}(\gamma - 1) \frac{\eta_2^2}{M_2} \quad \dots \quad (40)$$

which on differentiation and using (8B) gives

$$d\left(\frac{F_1 C_1}{F_2 C_2} + z_2\right) = \xi^{1-\gamma} d(\zeta_2 \xi^\gamma) \quad \dots \quad (41)$$

Let the suffix (2, 1) denote the position when the charge C_1 is just burnt out. Then the initial conditions for this case are :—

$$\left. \begin{aligned} x &= x_{2,1}; \quad v = v_{2,1}; \quad z_1 = 1; \quad z_2 = z_{2,1} \\ Z_2 &= Z_{2,1}; \quad \xi = \xi_{2,1}; \quad \eta_2 = \eta_{2,1} \text{ and } \zeta_2 = \zeta_{2,1} \end{aligned} \right\} \quad \dots \quad (42)$$

The equation (21) gives the value of z_2 at this position (2, 1) as

$$z_{2,1} = \frac{1}{k_2} \left[1 - \left\{ \frac{\sqrt{1-k_1-\mu}}{\lambda} \right\}^2 \right] \quad \dots \quad (43)$$

Also from (31) and (25), we get

$$Z_{2,1} = b \left(\frac{F_1 C_1}{F_2 C_2} + z_{2,1} \right) \quad \dots \quad (44)$$

Then $\xi_{2,1}$, $\zeta_{2,1}$ and $\eta_{2,1}$ are given by (37A) (38A), and (39A).

Let

$$\frac{F_1 C_1}{F_2 C_2} + z_2 = z_2 \quad \dots \quad (45)$$

and

$$\zeta_2 \xi^\gamma = Y_2, \text{ as before } \dots \quad (46)$$

Then (42) gives

$$d(z_2) = \xi^{1-\gamma} dY_2 \quad \dots \quad (47)$$

Proceeding, as for equations (32) and (33), we shall get here

$$dZ = \xi^{1-\gamma} dY \quad \dots \quad (48)$$

and

$$(1-Z) \left[\xi'' + \alpha \xi' \frac{\zeta_2'}{\zeta_2} \right] - \frac{1}{2} \xi' = \frac{M_2}{\alpha'(b')^2(1+\theta_2)^2} \cdot \zeta_2^{1-2\alpha} \quad \dots \quad (49)$$

where

$$a' = \frac{F_2 C_2 + F_1 C_1 k_2}{F_2 C_2} \quad \dots \quad (50A)$$

$$b' = \frac{k_2}{a'} \quad \dots \quad (50B)$$

$$Z = b' z_2 = b' \left(\frac{F_1 C_1}{F_2 C_2} + z_2 \right) \quad \dots \quad (50C)$$

and

$$Y = b' Y_2 = b' \zeta_2 \xi^{\gamma} \quad \dots \quad (50D)$$

Hence the corresponding equation (35) becomes in this case

$$(1-Z) \left[\frac{Y'''}{Y''} + (n-2) \frac{Y''}{Y'} + \alpha \frac{Y'}{Y} \right] - \frac{1}{2} = Q' \cdot \frac{(Y')^{2-2n}}{Y'' Y^{2\alpha-1}} \quad \dots \quad (51)$$

where

$$n = \frac{\gamma(1-\alpha)}{\gamma-1} \quad \text{and} \quad Q' = \frac{M_2(\gamma-1)}{a'(b')^{3-2\alpha}(1+\theta_2)^2} \quad \dots \quad (52)$$

with initial conditions

$$Z = Z_{2,1} = b' \left(\frac{F_1 C_1}{F_2 C_2} + z_{2,1} \right) \quad \dots \quad (53A)$$

$$Y = Y_{2,1} = b' \zeta_{2,1} \xi_{2,1}^{\gamma} \quad \dots \quad (53B)$$

$$Y' = (Y')_{2,1} = \xi_{2,1}^{\gamma-1} \quad \dots \quad (53C)$$

and

$$Y'' = (Y'')_{2,1} \quad \dots \quad (54D)$$

—the quantities $Y_{2,1}$, $(Y')_{2,1}$, $(Y'')_{2,1}$ being obtained from the numerical integration of equation (35) when $Z = Z_{2,1}$.

Then the shot-travel, pressure and velocity at any point between the position when C_1 is just burnt out and the position when C_2 is also burnt out, are given in terms of Z , Y and Y' by the equations

$$\xi^{\gamma-1} = Y' \quad \dots \quad (54A)$$

$$\text{i.e.} \quad x = \ell[(Y')^{\frac{1}{\gamma-1}} - 1] \quad \dots \quad (54B)$$

$$\zeta_2 = \frac{1}{b'} \cdot \frac{Y}{[Y']^{\frac{\gamma}{\gamma-1}}} \quad \dots \quad (55A)$$

$$\text{i.e.} \quad p = \frac{F_2 C_2}{A b'} \cdot \frac{Y}{[Y']^{\frac{\gamma}{\gamma-1}}} \quad \dots \quad (55B)$$

and

$$v_2^2 = \frac{2M_2}{b'(\gamma-1)} \left[Z - \frac{Y}{Y'} \right] \quad \dots \quad (56A)$$

$$\text{i.e.} \quad v^2 = \frac{2F_2 C_2}{(\gamma-1)w_1 b'} \left[Z - \frac{Y}{Y'} \right] \quad \dots \quad (56B)$$

The quantities x_2 , p_2 and v_2 corresponding to the all-burnt position are obtained by putting

$$Z = (Z)_2 = b' \left(\frac{F_1 C_1}{F_2 C_2} + 1 \right)$$

and the corresponding values $(Y)_2$ and $(Y')_2$ of Y and Y' as obtained from the numerical integration of equation (51) in equations (54B), (55B) and (56B).

Case I (ii):—In this case, the equation (7) becomes

$$1 + \frac{F_1 C_1}{F_2 C_2} = \zeta_2 \xi + \frac{1}{2}(\gamma - 1) \frac{\eta_2^2}{M_2} \quad \dots \quad (57)$$

Differentiating (57) and using (8B), we get

$$\xi^{1-\gamma} d(\zeta_2 \xi^\gamma) = 0$$

$$\therefore \zeta_2 \xi^\gamma = \text{constant} = \zeta_{2;2} \xi_2^\gamma$$

$$\text{i.e.} \quad \zeta_2 = \frac{\zeta_{2;2} \xi_2^\gamma}{\xi^\gamma} \quad \dots \quad (58A)$$

$$\text{or} \quad p = \frac{p_2 \xi_2^\gamma}{\xi^\gamma} \quad \dots \quad (58B)$$

where ξ_2 , $\zeta_{2;2}$ and $\eta_{2;2}$ are the values of ξ , ζ_2 and η_2 at all-burnt position, obtained from equations (54A), (55A) and (56A) as

$$\xi_2^{\gamma-1} = (Y')_2 \quad \dots \quad (59A)$$

$$\zeta_{2;2} = \frac{(Y)_2}{\frac{\gamma}{b'[(Y')_2]^{\gamma-1}}} \quad \dots \quad (59B)$$

and

$$\eta_{2;2}^2 = \frac{2M_2}{(\gamma-1)b'} \left[(Z)_2 - \frac{(Y)_2}{(Y')_2} \right] \quad \dots \quad (59C)$$

From equations (57) and (58A), we get

$$\eta_2^2 = \frac{2M_2}{\gamma-1} \left[\frac{F_2 C_2 + F_1 C_1}{F_2 C_2} - \frac{\zeta_{2;2} \xi_2^\gamma}{\xi^{\gamma-1}} \right] \quad \dots \quad (60A)$$

$$\text{or} \quad v^2 = \frac{2F_2 C_2}{(\gamma-1)w_1} \left[\frac{F_2 C_2 + F_1 C_1}{F_2 C_2} - \frac{\xi_2^\gamma \zeta_{2;2}}{\xi^{\gamma-1}} \right] \quad \dots \quad (60B)$$

The equations (58B) and (60B) give the values of pressure and velocity at any point after all-burnt position in terms of ξ and the various elements at the complete all-burnt position.

Case II:—Both the propellants are burnt out simultaneously,

$$\text{i.e.} \quad \beta' f_{20} = \beta'' f_{10}.$$

At complete 'all-burnt', we shall have from equations (31) and (25)

$$(Z)_2 = bZ_{2;2} = b \left(\frac{F_2 C_2 + F_1 C_1}{F_2 C_2} \right).$$

Then from equation (35), $(Y)_2$ and $(Y')_2$, the corresponding values of Y and Y' are calculated. Then equations (37), (38) and (39) give ξ_2 , $\zeta_{2;2}$ and $\eta_{2;2}$, the values of ξ , ζ_2 and η_2 at the complete 'all-burnt' as

$$\xi_2^{\gamma-1} = (Y')_2 \quad \dots \quad (61)$$

$$\zeta_{2;2} = \frac{1}{b} \cdot \frac{(Y)_2}{\frac{\gamma}{[(Y')_2]^{\gamma-1}}} \quad \dots \quad (62)$$

and
$$\eta_{2;2} = \frac{2M_2}{b(\gamma-1)} \left[(Z)_2 - \frac{(Y)_2}{(Y')_2} \right] \quad \dots \quad (63)$$

After all-burnt, the values of ζ_2 (giving pressure) and η_2 (giving velocity) at any point are given by equations (58A) and (60A).

6. MUZZLE VELOCITY

Case I.—(When the charges burn out separately):—

The muzzle velocity is given from equation (60B) as

$$v_3^2 = \frac{2Al}{(\gamma-1)w_1} \left[\frac{F_2 C_2 + F_1 C_1}{Al} - \frac{p_2 \xi_2^\gamma}{\xi_3^{\gamma-1}} \right] \quad \dots \quad (64)$$

where p_2 and ξ_2 are obtained from equations (59B) and (59A).

Case II.—(When the charges burn out simultaneously):—

In this case the muzzle velocity v_3 is given by the same equation (64) but p_2 and ξ_2 are obtained from equations (61) and (62).

7. MAXIMUM PRESSURE

The following cases may arise, viz., that the maximum pressure occurs when—

- (a) both the charges are burning;
- (b) C_1 is burnt out and C_2 is still burning; or
- (c) at the position of simultaneous 'all-burnt' of both the propellants.

Case (a).— From equation (38A),

$$\zeta_2 = \frac{Y}{\frac{\gamma}{b[Y']^{\gamma-1}}} \quad \dots \quad (38A)$$

$$\text{or } p = \frac{F_2 C_2}{Ab} \cdot \frac{Y}{\frac{\gamma}{[Y']^{\gamma-1}}} \quad \dots \quad (38B)$$

The pressure is maximum when

$$\frac{Y'}{Y} = \frac{\gamma}{\gamma-1} \cdot \frac{Y''}{Y'} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (65)$$

The equation (65) can be solved numerically for Z from the tabulated values of Z , Y , Y' and Y'' as obtained from (35) and then the value and the position of maximum pressure (i.e. p_1 and ξ_1) and the velocity v_1 at that instant can be determined from equations (38B), (37A) and (39B).

Conditions :—The conditions for the occurrence of maximum pressure in this position are

$$\left. \begin{array}{l} f_{1;1} > 0 \text{ and } f_{2;1} > 0 \\ \text{i.e. } z_{1;1} < 1 \text{ and } z_{2;1} < 1 \end{array} \right\} \quad (66)$$

where $z_{1;1}$ and $z_{2;1}$ are the values of z_1 and z_2 at the position of maximum pressure.

With the help of equations (29), (31) and (21), the conditions (66) reduce to

$$\left[a^* - \sqrt{1 - (Z)_1} \right] \left[b^* + \sqrt{1 - (Z)_1} \right] < \frac{\nu'^2 k_2}{a} \quad \dots \quad \dots \quad (67A)$$

$$\left[c^* - \sqrt{1 - (Z)_1} \right] \left[d^* + \sqrt{1 - (Z)_1} \right] < \frac{\nu'^2 k_1}{\lambda^2 a} \quad \dots \quad \dots \quad (67B)$$

where $(Z)_1$ = the value of Z obtained as the solution of equation (65), and

$$a^* = \frac{\nu' + \lambda \lambda' \mu}{\sqrt{a}} \quad \dots \quad \dots \quad \dots \quad (68A)$$

$$b^* = \frac{\nu' - \lambda \lambda' \mu}{\sqrt{a}} \quad \dots \quad \dots \quad \dots \quad (68B)$$

$$c^* = \frac{\nu' - \mu}{\lambda \sqrt{a}} \quad \dots \quad \dots \quad \dots \quad (68C)$$

$$d^* = \frac{\nu' + \mu}{\lambda \sqrt{a}} \quad \dots \quad \dots \quad \dots \quad (68D)$$

Case (b) :—Here

$$\beta' f_{20} > \beta'' f_{10}.$$

Equation (55B) gives

$$p = \frac{F_2 C_2}{A b'} \cdot \frac{Y}{\frac{\gamma}{[Y']^{\gamma-1}}} \quad \dots \quad \dots \quad \dots \quad (55B)$$

\therefore pressure is maximum, if

$$\frac{Y'}{Y} = \frac{\gamma}{\gamma-1} \frac{Y''}{Y'} \quad (69)$$

This equation can be solved numerically for Z from the tabulated values of Z , Y , Y' and Y'' obtained from equation (51). Then the value of maximum pressure can be found out from equation (55B).

Conditions :—The conditions for the occurrence of maximum pressure in this position are

$$\left. \begin{aligned} f_{1;1} &= 0 \text{ and } f_{2;1} > 0 \\ \text{i.e., } z_{1;1} &= 1 \text{ and } z_{2;1} < 1 \end{aligned} \right\} \quad \dots \quad (70)$$

But in view of the condition $\beta'f_{20} > \beta''f_{10}$ and equation (17), $f_{2;1}$ cannot be equal to zero.

Then with the help of (21) and (50C), the conditions reduce to

$$\beta'f_{20} > \beta''f_{10} \quad \dots \quad (71A)$$

$$\mu + \lambda\sqrt{1-k_2} < \sqrt{1-k_1} \quad \dots \quad (71B)$$

and

$$\frac{F_2C_2(Z)_1 - F_1C_1}{b'F_2C_2} < 1 \quad \dots \quad (71C)$$

where $(Z)_1$ is the value of Z obtained as the solution of equation (69).

Case (c) :—Here

$$\beta'f_{20} = \beta''f_{10}.$$

In this case the maximum pressure p_1 is given by the equations (65) and (38B).

Conditions :—The conditions for the occurrence of maximum pressure in this position are

$$\left. \begin{aligned} \beta'f_{20} &= \beta''f_{10} \\ f_{1;1} &= f_{2;1} = 0 \\ z_{1;1} &= z_{2;1} = 1 \end{aligned} \right\} \quad \dots \quad (72)$$

or

These conditions are reduced to

$$\left[a^* - \sqrt{1-(Z)_1} \right] \left[b^* + \sqrt{1-(Z)_1} \right] = \frac{\nu'^2 k_2}{a} \quad \dots \quad (73A)$$

$$\left[c^* - \sqrt{1-(Z)_1} \right] \left[d^* + \sqrt{1-(Z)_1} \right] = \frac{\nu'^2 k_1}{\lambda^2 a} \quad \dots \quad (73B)$$

where a^* , b^* , c^* and d^* are given by (68A), (68B), (68C) and (68D) and $(Z)_1$ has the same value as in (67A) and (67B).

8. A SPECIAL CASE. ($\theta_1 = \theta_2 = 0$)

When both of θ_1 and θ_2 or any one of them be zero, some of the equations derived above present some difficulty as they may take the undetermined form. Accordingly those equations are to be modified. For a long tubular propellant, we generally take $\theta = 0$; but as is theoretically determined, this $\theta = \frac{R_0 - r_0}{H_0}$; so even if H_0 (i.e. the length of the grain) be 200–500 times the annulus ($R_0 - r_0$), θ is greater than .002 and we may not neglect it. With this device, the theory developed in the previous sections is equally applicable to tubular propellants also. However, we give here principal equations which differ from those given above when θ_1 and θ_2 both are taken exactly zero.

Since

$$\theta_1 = \theta_2 = 0, \quad k_1 = k_2 = 0.$$

The equations (18), (21), (25), (32) and (33) become here

$$\beta''z_1 - \beta'z_2 = \beta''z_{10} - \beta'z_{20} \quad \dots \quad (74)$$

$$z_1 = \frac{\beta'}{\beta''}z_2 + \frac{\beta''z_{10} - \beta'z_{20}}{\beta''} \quad \dots \quad (75)$$

$$z_2 + \frac{F_1C_1}{F_2C_2}z_1 = \lambda^*z_2 + \mu^* = Z \quad \dots \quad (76)$$

$$dZ = \xi^{1-\gamma}dY \quad \dots \quad (77)$$

$$\xi'' + \alpha\xi' \frac{\zeta_2'}{\zeta_2} = \frac{M_2}{\lambda^2} \zeta_2^{1-2\alpha} \quad \dots \quad (78)$$

where

$$\lambda^* = 1 + \frac{F_1C_1}{F_2C_2} \frac{\beta'}{\beta''} \quad \dots \quad (79)$$

$$\mu^* = \frac{F_1C_1}{F_2C_2} \left[z_{10} - \frac{\beta'}{\beta''}z_{20} \right] \quad \dots \quad (80)$$

and

$$Y = \zeta_2 \xi^\gamma \quad \dots \quad (81)$$

Accordingly the fundamental equation (35) becomes here

$$\frac{Y'''}{Y''} + (n-2) \frac{Y''}{Y'} + \alpha \frac{Y'}{Y} = Q^* \cdot \frac{(Y')^{2-2n}}{Y'' Y^{2\alpha-1}} \quad \dots \quad (82)$$

where

$$n = \frac{\gamma(1-\alpha)}{\gamma-1} \text{ and } Q^* = \frac{M_2(\gamma-1)}{(\lambda^*)^2} \quad \dots \quad (83)$$

and the independent variable is $Z = \lambda^*z_2 + \mu^*$.

The equation (82) does not contain the independent variable Z explicitly; it can, therefore, be solved numerically by treating Y' as a function of Y .

With zero shot-start pressure, the series solution of (82) takes the form as given by Clemmow (1951, p. 124),

$$Y = Z \left[1 + \frac{Z^*}{4-2\alpha} + \frac{\{3-2\alpha-n(2-\alpha)\}(Z^*)^2}{(5-3\alpha)(4-2\alpha)} + \dots \right] \quad \dots \quad (84)$$

where

$$Z^* = \frac{Q^*Z^{3-2\alpha}}{(3-2\alpha)(2-\alpha)} \quad \dots \quad (85)$$

The quantities ξ , ζ_2 and η_2 giving the shot-travel (x), pressure (p) and velocity (v) when both the charges are burning, are given by

$$\xi^{\gamma-1} = \frac{dY}{dZ} \quad \dots \quad (86)$$

$$\zeta_2 = \frac{Y}{[Y']^{\frac{\gamma}{\gamma-1}}} \quad \dots \quad (87)$$

and

$$\eta_2^2 = \frac{2M_2}{\gamma-1} \cdot \left[Z - \frac{Y}{Y'} \right] \quad \dots \quad (88)$$

where Z , Y and Y' have the values obtained from (82). Clearly the equations (86), (87) and (88) correspond to equations (37A), (38A) and (39A) of the general case.

9. DIFFERENT CASES OF ALL BURNT

Case I (i).—(When C_1 is burnt out and C_2 is still burning.)

The initial conditions for this case are

$$z_{2;2,1} = \frac{\beta''(1-z_{10}) + \beta'z_{20}}{\beta'} \quad \dots \quad (89)$$

$$Z_{2,1} = \lambda^* z_{2;2,1} + \lambda^* \quad \dots \quad (90)$$

and $\xi_{2,1}$, $\zeta_{2;2,1}$ and $\eta_{2;2,1}$ are given by (86), (87) and (88) for the value

$$Z = Z_{2,1}.$$

Now let

$$\left. \begin{aligned} z_2 + \frac{F_1 C_1}{F_2 C_2} &= Z \\ \zeta_2 \xi^\gamma &= Y \end{aligned} \right\} \quad \dots \quad (91)$$

and

Then here we shall have

$$dZ = \xi^{1-\gamma} dY \quad \dots \quad (92)$$

$$\xi'' + \alpha \xi' \frac{\zeta_2'}{\zeta_2} = M_2 \gamma_2^{1-2\alpha} \quad \dots \quad (93)$$

and the equation corresponding to (51) is

$$\frac{Y'''}{Y''} + (n-2) \frac{Y''}{Y'} + \alpha \frac{Y'}{Y} = Q'^* \cdot \frac{(Y')^{2-2n}}{Y'' Y^{2\alpha-1}} \quad \dots \quad (94)$$

where

$$n = \frac{\gamma(1-\alpha)}{\gamma-1} \quad \text{and} \quad Q'^* = M_2(\gamma-1) \quad \dots \quad (95)$$

The equation (94) is similar to (82) with Q'^* for Q^* and with initial conditions

$$\left. \begin{aligned} Z &= Z_{2,1} \\ Y &= \zeta_{2;2,1} \xi_{2,1}^\gamma \\ Y' &= \xi_{2,1}^{\gamma-1} \\ Y'' &= (Y'')_{2,1} \end{aligned} \right\} \quad \dots \quad (96)$$

and

= the value of Y'' corresponding to $Z = Z_{2,1}$.

The quantities ξ , ζ_2 and η_2 are given by (86), (87) and (88) but now the quantities Z , Y and Y' are given by (91) and (94).

Case I (ii).—(When C_1 and C_2 both are burnt out.)

In this case ζ_2 and η_2 (giving pressure p and velocity v respectively) at any point after all-burnt are given in terms of ξ by equations (58A) and (60A) where ξ_2 , $\zeta_{2;2}$ and $\eta_{2;2}$ (the values of ξ , ζ_2 and η_2 respectively at the complete all-burnt position) are given by equations (86), (87) and (88) for

$$Z = (Z)_2 = \frac{F_1 C_1 + F_2 C_2}{F_2 C_2},$$

and for Y and Y' obtained from equation (94) with the above value of Z .

Case II.—Both the propellants are burnt out simultaneously, i.e.

$$\beta' f_{20} = \beta'' f_{10},$$

$$\text{or} \quad \beta' z_{10} - \beta' z_{20} = \beta'' - \beta'.$$

At 'all-burnt', we have from equations (86), (87) and (88)

$$\left. \begin{aligned} \xi_2^{\gamma-1} &= (Y')_2 \\ \zeta_{2;2} &= \frac{(Y)_2}{[(Y')_2]^{\gamma/\gamma-1}} \\ \eta_{2;2} &= \frac{2M_2}{\gamma-1} \left[(Z)_2 - \frac{(Y)_2}{(Y')_2} \right] \end{aligned} \right\} \quad \dots \quad (97)$$

and

where

$$(Z)_2 = \frac{F_1 C_1 + F_2 C_2}{F_2 C_2} \quad \text{and } (Y)_2 \text{ and } (Y')_2 \text{ are}$$

obtained from equation (82) for this value of $Z = (Z)_2$.

After 'all-burnt' position, ζ_2 and η_2 (giving pressure p and velocity v) at any point are given in terms of ξ by (58A) and (60A) with ξ_2 , $\zeta_{2;2}$ and $\eta_{2;2}$ as given by (97).

10. MAXIMUM PRESSURE

Case (a).—When both the charges are burning, the maximum pressure p_1 is given from equation (87) as

$$p_1 = \frac{F_2 C_2}{Al} \cdot \frac{Y}{\frac{\gamma}{(Y')^{\gamma-1}}} \quad \dots \quad (98)$$

subject to the condition

$$\frac{Y'}{Y} = \frac{\gamma}{\gamma-1} \cdot \frac{Y''}{Y'} \quad \dots \quad (99)$$

which should be solved numerically from the tabulated values of Z , Y , Y' and Y'' as obtained from equation (82).

Conditions.—The conditions of equation (66) for the occurrence of maximum pressure in this position are simplified with the help of equations (75) and (76) as follows:—

$$\left. \begin{aligned} (Z)_1 &< \lambda^* + \mu^* \\ (Z)_1 &< \frac{\beta'' \lambda^*}{\beta'} - \frac{\mu^*}{\lambda^* - 1} \end{aligned} \right\} \quad \dots \quad (100)$$

i.e. $(Z)_1$, the value of Z obtained as the solution of equation (99), should be less than the smaller of

$$\lambda^* + \mu^*$$

and

$$\left(\frac{\beta''\lambda^*}{\beta'} - \frac{\mu^*}{\lambda^*-1} \right).$$

*Case (b) :—*Here the maximum pressure p_1 is given by the same equations (98) and (99) but the latter equation should be solved from the tabulated values of Z , Y , Y' and Y'' as obtained from equation (94).

*Conditions :—*The conditions of equation (70) for the occurrence of maximum pressure in this position are simplified to

$$\left. \begin{aligned} \beta'f_{20} &> \beta''f_{10} \\ (Z)_1 &< 1 + \frac{F_1C_1}{F_2C_2} \end{aligned} \right\} \dots \dots \dots (101)$$

*Case (c) :—*Here both the charges are burnt out simultaneously, i.e. $\beta'f_{20} = \beta''f_{10}$ and the maximum pressure occurs at 'all-burnt' position.

In this case the maximum pressure is given by the same equations (98) and (99).

*Conditions :—*The conditions of equation (72) for the occurrence of maximum pressure in this position are simplified in this case to

$$\left. \begin{aligned} \beta'f_{20} &= \beta''f_{10} \\ \text{and } (Z)_1 &= \lambda^* + \mu^* \\ &= \frac{\beta''\lambda^*}{\beta'} - \frac{\mu^*}{\lambda^*-1} \end{aligned} \right\} \dots \dots \dots (102)$$

where λ^* and μ^* are given by equations (79) and (80) and $(Z)_1$ has the same value as found from equation (99).

11. SUMMARY

In this paper the problem of using composite charges in conventional guns has been discussed. A solution of the basic equations of Internal Ballistics when non-linear power law for rate of burning is assumed, is derived for any values of the form-coefficients under certain conditions, viz., the value of γ —the ratio of specific heats—is the same for both the charges and the power-index law also is the same for both the propellants.

12. ACKNOWLEDGEMENT

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13. REFERENCES

- Clemmow, C. A. (1928). Theory of Internal Ballistics based on a Pressure-index Law of Burning for Propellants. *Phil. Trans. Roy. Soc., A*, 227, 345.
 ——— (1951). Internal Ballistics. H.M.S.O. publication. Chapters X and XI.
 Corner, J. (1950). Theory of Internal Ballistics of Guns. John Wiley & Sons, New York: 328-333.
 Venkatesan, N. S. and Patni, G. C. (1953). Influence of Composite charges on max. pressure and muzzle velocity. *Defence Science Journal*, 3-1, January, 1953.

PRODUCTION OF MESONS IN NUCLEON-NUCLEON COLLISIONS

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INTRODUCTION

The theory of π -meson production has been worked out by various authors under different assumptions in order to explain the various results of observations. Heisenberg, and later on, Lewis, Oppenheimer and Wouthysen (1948) put forward the theory of 'multiple production' of mesons. Heitler and Janossy (1949, 1950), on the other hand, proposed the 'plural theory' of meson production on some plausible assumptions. The results of cloud chamber observations as well as the nature of stars recorded in photographic plates are in many cases in qualitative agreement with the plural theory. Recent experiments of Schultz (1954), however, lend more support to the theory of multiple production of meson showers. A quantitative comparison between the theoretical and observed results, however, is not feasible without making considerable improvements in the theoretical as well as observational results.

It is now evident from various results of observation that the development of the nucleons and mesons should follow a cascade process. The most uncertain factor in the theoretical development is the form of the cross-section for the energy loss of the nucleon and that for the production of mesons in the nucleon-nucleon collisions, which are usually deduced from the theory of meson production. Hamilton, Heitler and Peng (1943) calculated the cross-sections, which were, however, modified later by Heitler and Walsh (1945). Peng (1944) used the results of Hamilton, Heitler and Peng and estimated the number of nucleons and mesons through the cascade process, but ignored the effect of β -decay. Chakravarty (1947) calculated the average number of nucleons and mesons generated by a primary nucleon of given energy passing through a given depth of material. The development of the showers followed a cascade process and the form of the cross-section given by Heitler and Walsh was used. The effect of β -decay in the scheme of the cascade process was also taken into account. The results obtained by Chakravarty differ considerably from those of Peng and is due to the difference in the form of the cross-sections assumed and also partly due to the neglect of decay of mesons by Peng. To avoid the uncertainties that are inherent in the theory of meson production, Heitler and Janossy (1949) have derived the cross-sections for the energy loss in a nucleon-nucleon collision in a phenomenological way, which compares well, except in details, with those deduced previously. Heitler and Janossy have used this cross-section in estimating the absorption of nucleons in matter. Messel (1951) has used a different cross-section also suggested by Heitler and Janossy and has taken into account the production of recoil nucleons. This cross-section, however, gives an average energy loss of the order of $5/24$ ths of the primary into the meson component. As such, the results obtained by Messel in the estimation of the size of the nucleon cascade is much larger than that obtained by Heitler and Janossy. In a later paper, Messel has, however, used a different cross-section for the production of mesons and recoil nucleons.

Detailed theoretical results on the various theories of meson production are essential for a proper comparison with the observed results and thus one should also know by how much these results vary with the choice of the form of cross-section. In the present paper an attempt has been made to estimate the number of nucleons as well as mesons produced by a primary nucleon of given energy E_0 in passing through nuclear matter. We have also taken into consideration, following Heitler and Walsh and Chakravarty, the effect of the decay of mesons. In a later section, we have compared our results with those of Heitler and Janossy and also of Messel, which will show, on the one hand, the effect of the form of cross-section on the size of a nucleon and meson shower and, on the other hand, will indicate the possible nature of meson production when compared with the different results of observation.

Following Heitler and Janossy (1949) we have assumed that a nucleon of energy E may lose an energy between ϵ and $\epsilon + d\epsilon$ in a single collision with another nucleon and the probability for such an energy loss,

$$\phi(E, \epsilon) d\epsilon = \alpha(1 - \epsilon/E)^\alpha d\epsilon/E \quad \dots \quad (1)$$

We have also assumed that the average cross-section for the production of a recoil nucleon of energy between ϵ' and $\epsilon' + d\epsilon'$ by a nucleon of energy E , losing an amount of energy between ϵ and $\epsilon + d\epsilon$ is given by $\psi(E, \epsilon, \epsilon') d\epsilon d\epsilon'$ when the meson produced takes the energy $\epsilon - \epsilon'$ where

$$\psi(E, \epsilon, \epsilon') d\epsilon d\epsilon' = \alpha(1 - \epsilon/E)^{\beta_1} (1 - \epsilon'/E)^{\beta_2} \frac{d\epsilon d\epsilon'}{E^2} \quad \dots \quad (2)$$

and

$$\alpha = 15, \quad \beta_1 = 2, \quad \beta_2 = 1$$

Consider a layer of the substance on the surface of which a primary nucleon with energy E_0 impinges normally. We propose to find out the energy distribution of the mesons and nucleons at any depth below the surface of the layer. The mesons, together with the recoil nucleons, are produced by the fast nucleons and they are absorbed through ionisation and β -decay. The unit in which depth or thickness is measured is the average distance between two consecutive collisions and has been defined as collision units by Heitler and Janossy. Let $P(E, t) dE$ be the total number of nucleons at depth t having energies between E and $E + dE$ and $M(E, t) dE$ the total number of mesons having energies between E and $E + dE$. Then the diffusion equations may be written as follows:—

$$\begin{aligned} \frac{\partial}{\partial t} P(E, t) - \beta \frac{\partial}{\partial E} P(E, t) &= \int_E^\infty P(E', t) \phi(E', E' - E) dE' - P(E, t) \int_0^E \phi(E, E') dE' + \\ &+ \int_E^\infty P(E', t) \Psi_1(E/E') \frac{dE'}{E'} \quad \dots \quad (3) \end{aligned}$$

and

$$\frac{\partial}{\partial t} M(E, t) - \gamma \frac{\partial}{\partial E} M(E, t) = S(E, t) - \frac{b}{Et} M(E, t) \quad \dots \quad (4)$$

where

$$\frac{1}{E'} \Psi_1(E/E') = \int_E^{E'} \phi(E', \epsilon, E) d\epsilon \quad \dots \quad (5)$$

$$S(E, t) = \int_E^\infty P(E', t) \chi(E/E') dE'/E' \quad \dots \quad (6)$$

so that

$$\frac{1}{E'} \chi(E/E') = \int_E^{E'} \psi(E', \epsilon, \epsilon - E) d\epsilon \quad \dots \quad (7)$$

β and γ are the rates of ionisation loss of the nucleons and mesons and b is a quantity which depends on the proper lifetime of the mesons at rest and for the atmosphere $b \approx 9.5$ in natural meson units.

In order to solve equation (3) we introduce a function defined by

$$p(s, t) = \int_0^\infty E^{s-1} P(E, t) dE \quad \dots \quad (8)$$

By Mellin's transform then we have

$$P(E, t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} E^{-s} p(s, t) ds \quad \dots \quad (9)$$

where σ is such that when $R(s) > \sigma$, $p(s, t)$ is analytic. Multiplying equation (3) by E^{s-1} and integrating with respect to E from 0 to ∞ we have

$$\frac{\partial}{\partial t} p(s, t) + \beta(s-1) p(s-1, t) - (M+N) p(s, t) + a p(s, t) = 0 \quad \dots \quad (10)$$

where

$$\left. \begin{aligned} M(s) &= \frac{\alpha}{s+\beta_1} \\ N(s) &= \frac{\alpha}{1+\beta_1} \cdot \frac{\Gamma(s)\Gamma(\beta_1+\beta_2+2)}{\Gamma(s+\beta_1+\beta_2+2)} \\ a &= \frac{\alpha}{1+\beta_1} \end{aligned} \right\} \quad \dots \quad (11)$$

In the present paper we shall neglect the ionisation loss and estimate its effect in a later paper. We then have

$$\frac{\partial}{\partial t} p(s, t) + K_s p(s, t) = 0 \quad \dots \quad (12)$$

where

$$-K_s = M(s) + N(s) - a \quad \dots \quad (13)$$

so that

$$p(s, t) = C e^{-K_s t} \quad \dots \quad (14)$$

For an incident primary nucleon of energy E_0 , we have $P(E, 0) = \delta(E_0 - E)$ and hence $p(s, 0) = E_0^{s-1}$, so that

$$p(s, t) = E_0^{s-1} e^{-K_s t} \quad \dots \quad (15)$$

and

$$P(E, t) = \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^s e^{-K_s t} ds \quad \dots \quad (16)$$

The energy spectrum of nucleons at a depth θ gm./cm.² of the absorber is then given by $P(E, \theta)$ where

$$P(E, \theta) = \int_0^\infty P(E, t) F(\theta, t) dt \quad \dots \quad (17)$$

$F(\theta, t)$ being the probability for t to lie in the interval dt when the nucleon traverses θ gm./cm.² of material (Heitler and Janossy, 1949).

Also

$$F(\theta, t) = \frac{1}{2\pi i} \int_{\lambda' - i\infty}^{\lambda' + i\infty} \exp \{ \lambda t - \bar{p} f(\lambda) \} d\lambda \quad \dots \quad (18)$$

where

$$f(\lambda) = 1 - 2 \frac{1 - (1 + \lambda)^{-\lambda}}{\lambda^2} \quad \dots \quad (19)$$

and \bar{p} is the average number of collisions when passing through a thickness θ and is equal to $\theta n \phi_A$ (20)

ϕ_A being the geometrical cross-section and n the number of nuclei per gram.

Substituting in (17), we have

$$P(E, \theta) = \int_0^\infty dt \frac{1}{2\pi i} \int_{\lambda' - i\infty}^{\lambda' + i\infty} \exp \{ \lambda t - \bar{p} f(\lambda) \} d\lambda \frac{1}{2\pi i E_0} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E} \right)^s e^{-K_s t} ds \quad \dots \quad (21)$$

$$= \frac{1}{2\pi i E_0} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E} \right)^s e^{-\bar{p} f(K_s)} ds \quad \dots \quad (22)$$

We assume a power law spectrum for the primary nucleons so that

$$P(E_0, 0) = \begin{cases} \frac{\gamma E_c^{\gamma+1}}{E_0^{\gamma+1}}, & E_0 > E_c \\ 0, & E_0 < E_c \end{cases} \quad \dots \quad (23)$$

where E_c is the latitude cut-off energy.

Then the number of the nucleons in the energy range $(E, E + dE)$ at any depth θ is $P_1(E, \theta) dE$ given by

$$\begin{aligned} P_1(E, \theta) &= \int_{E_c}^\infty P(E, \theta) dE_0 \cdot P(E_0, 0) \\ &= \frac{\gamma}{2\pi i E_c} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_c}{E} \right)^s e^{-\bar{p} f(K_s)} \frac{ds}{\gamma - s + 1} \quad \dots \quad (24) \end{aligned}$$

If $Q(E, \theta)$ be the total number of nucleons above a certain energy E , at a depth θ , then

$$\begin{aligned} Q(E, \theta) &= \int_E^{\infty} P_1(E, \theta) dE \\ &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_c}{E}\right)^{s-1} \frac{\gamma}{(s-1)(\gamma-s+1)} e^{-\bar{W}(K_s)} ds \quad \dots (25) \\ &= \frac{\gamma}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{B(s, \bar{p})} ds \end{aligned}$$

$$\text{where} \quad B(s, \bar{p}) = (s-1) \log_e \frac{E_c}{E} - \bar{p}f(K_s) - \log_e (s-1) - \log_e (\gamma-s+1) \quad \dots (26)$$

Heitler and Janossy have taken α in the cross-section to be of the order 10.5 and so for the purpose of comparison we have also taken $\alpha = 10.5$ for calculating $Q(E, \theta)$ and the results obtained are given in Table 1, together with those of $T(E, \theta)$ obtained by Heitler and Janossy (1949).

TABLE 1

Total number of nucleons $Q(E, \theta)$ at a depth θ . $T(E, \theta)$ represents the same expression obtained by Heitler and Janossy.

$\frac{\log E/E_c}{\bar{p}}$	log $Q(E, \theta)$			log $T(E, \theta)$		
	-2	-1	0	-2	-1	0
10	-0.4	-1.15	-2.0	-0.5	-1.2	-2.25
20	-1.8	-2.6	-3.5	-2.05	-3.1	-4.45
30	-3.1	-4.2	-5.9	-3.9	-5.15	-6.6

The table shows clearly, as is expected, the contribution of the recoil nucleons at the different depths.

If now $N(E, t)$ be the total number of nucleons at a depth t , with energy greater than E , produced by a primary of energy E_0 , then

$$\begin{aligned} N(E, t) &= \int_E^{\infty} P(E, t) dE \\ &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^{s-1} \cdot \frac{1}{s-1} e^{-Kt} ds \quad \dots \dots (27) \end{aligned}$$

To compare the results of the present paper with those of Messel and of Chakravarty, we have calculated the values of $N(E, t)$ for three different values of $\log_e (E_0/E)$, viz. 2, 5, 8 and the results have been shown in Fig. 1. As expected, Messel's analysis gives a larger multiplication and a slow-rate of absorption. The

values derived by Chakravarty are less than those of the present paper and this is possibly due to the difference of the cross-section assumed.

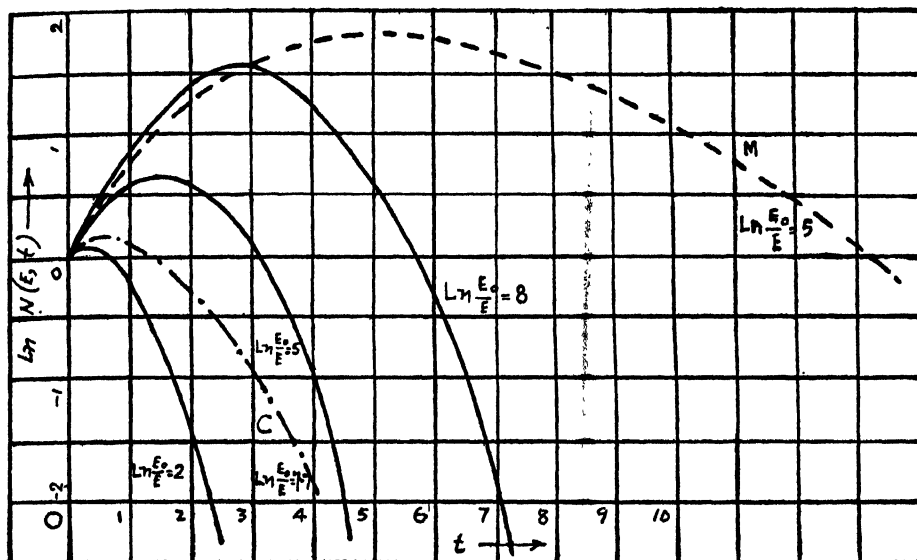


FIG. 1. Total number of nucleons at different depths (— Present Paper; --- derived from the analysis of Messel; - · - deduced by Chakravarty).

Substituting the value of $P(E, t)$ in (4) it can be shown that

$$M(E, t) = \left(\frac{E}{\gamma t}\right)^{\frac{b}{E+\gamma t}} \int_0^t s(E+\gamma t-\gamma t'; t') (\gamma t')^{\frac{b}{E+\gamma t}} \frac{dt'}{(E+\gamma t-\gamma t')^{b/(E+\gamma t)}} \\ = \left(\frac{E}{t}\right)^{\frac{b}{\eta}} \int_0^t s(\eta-\gamma t'; t') \left(\frac{t'}{\eta-\gamma t'}\right)^{\frac{b}{\eta}} dt' \quad \dots \quad (28)$$

where

$$\eta = E + \gamma t \quad \dots \quad (29)$$

and γ is the rate of ionisation loss of the mesons.

Using (6) and (16) and after making some simplifications we get

$$M(E, t) = \left(\frac{E}{t}\right)^{\frac{b}{\eta}} \int_0^t \left(\frac{t'}{\eta-\gamma t'}\right)^{\frac{b}{\eta}} dt' \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{\eta-\gamma t'}\right)^s e^{-Kt'} F_s ds \quad \dots \quad (30)$$

where

$$F(s) = \int_{\eta-\gamma t'}^{\infty} \left(\frac{\eta-\gamma t'}{E'}\right)^s \chi \left(\frac{\eta-\gamma t'}{E'}\right) \frac{dE'}{E'} \\ = \frac{5}{24} \left[\frac{3}{s} - \frac{8}{s+1} + \frac{6}{s+2} - \frac{1}{s+4} \right] \quad \dots \quad (31)$$

Equation (30) can be written in the form

$$M(E, t) = \left(\frac{E}{t}\right)^{\frac{b}{\eta}} \frac{1}{E_0} \int_0^t \left(\frac{t'}{\eta-\gamma t'}\right)^{\frac{b}{\eta}} W(E_0, \eta, t') dt' \quad \dots \quad (32)$$

where

$$W(E_0, \eta, t') = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{\eta - \gamma t'} \right)^s e^{-K s'} F_s ds \quad \dots \quad (33)$$

$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{s y_0 - K s' + \log_s F_s} ds \dots \quad (34)$$

y_0 being equal to $\log_s \left(\frac{E_0}{\eta - \gamma t'} \right)$

The integral in (34) is evaluated by the saddle point method for some integral values of t , viz., $t = 0, 1, 2, 3, 4, 5, 6, 7, 8$, and 10 and for different y_0 's.

The equation (32) can be simplified without introducing much error and we can take

$$M(E, t) \approx \frac{1}{E_0} \int_0^t \left(\frac{t'}{t} \right)^{\frac{b}{\eta}} W(E_0, \eta, t') dt' \dots \quad (35)$$

The error in putting $[E/(\eta - \gamma t')]^{\frac{b}{\eta}} \approx 1$ is only to the extent of 10% even when η is small. We have checked by calculating the values for two cases where $E_0 = 10^5$ ($E = 5$ and $E = 50$). The values are found to be 52.5 and 10.2 according to formula (32) as against 58 and 11, found by formula (35). The integral in (35) is then solved by applying a process of numerical integration.

If $\bar{M}(E_1, t)$ be the total number of mesons above a certain energy E_1 , at a depth t , then

$$\bar{M}(E_1, t) \approx \int_{E_1}^{\infty} dE \int_0^t \left(\frac{t'}{t} \right)^{\frac{b}{E + \gamma t}} dt' \cdot \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E + \gamma t - \gamma t'} \right)^s e^{-K s'} F_s ds \dots \quad (36)$$

$$\approx \frac{1}{E_0} \int_0^t dt' \cdot \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} E_0^s e^{-K s'} F_s ds \int_{E_1}^{\infty} \left(\frac{t'}{t} \right)^{\frac{b}{E + \gamma t}} \frac{dE}{(E + \gamma t - \gamma t')^s} \dots \quad (37)$$

Again,

$$\begin{aligned} & \int_{E_1}^{\infty} \left(\frac{t'}{t} \right)^{\frac{b}{E + \gamma t}} \frac{dE}{(E + \gamma t - \gamma t')^s} \\ &= \int_{E_1 + \gamma t}^{\infty} \left(\frac{t'}{t} \right)^{\frac{b}{\eta}} \frac{d\eta}{(\eta - \gamma t')^s} \quad \text{if } \eta = E + \gamma t \\ &= \int_{E_1 + \gamma t}^{\infty} \left\{ 1 + \frac{b \log_s t'/t + \gamma s t'}{\eta} \right\} \frac{d\eta}{\eta^s} \dots \quad (38) \end{aligned}$$

(expanding in powers of $\frac{1}{\eta}$ and neglecting higher powers of $\frac{1}{\eta}$).

The integrand in (38)

$$-\frac{1}{s-1} \frac{1}{(E_1 + \gamma t)^{s-1}} + \left[\frac{b}{s} \log t'/t + \gamma t' \right] \frac{1}{(E_1 + \gamma t)^s}$$

Substituting in (37) we have

$$\begin{aligned} \overline{M}(E_1, t) = & \int_0^t dt' \left[\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E_1 + \gamma t} \right)^{s-1} \frac{e^{-Ks'}}{s-1} F_s ds \right. \\ & \left. + \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E_1 + \gamma t} \right)^s \left\{ \frac{b}{s} \log \frac{t'}{t} + \gamma t' \right\} e^{-Ks'} F_s ds \right] \quad (39) \end{aligned}$$

We have evaluated (39) by the method indicated before and the results for $M(E, t)$ and $\overline{M}(E_1, t)$ have been given in Tables 2 and 3.

TABLE 2
Values of $M(E, t) \times 10^3$.

$E_0/\mu c^2$	$\begin{matrix} E/\mu c^2 \\ t \end{matrix}$	5	10	20	30	50
10^2	2	43	21	9.8	6.0	3.1
	3	34	17	8.2	4.8	2.8
	4	26	13	6.7	4.4	2.7
	5	17	11	5.8	4.1	2.6
	6	14	9	5.1	3.9	2.5
10^3	2	73	44	25	17	10.3
	3	106	54	28	21	10.6
	4	93	51	26	20	10.5
	5	73	43	23	17	9.8
	6	59	36	20	15	9.1
10^4	2	84	45	26	19	13
	3	161	91	48	33	20
	4	236	124	63	39	22
	5	230	117	59	38	22
	6	197	106	53	36	20
10^5	2	58	31	22	16	11
	3	233	127	61	41	26
	4	351	230	105	69	40
	5	564	278	128	84	46
	6	459	274	135	58	44

TABLE 3
Values of $\bar{M}(E_1, t)$.

$E_1/\mu c^2$	$E_0/\mu c^2$	10^2	10^3	10^4	10^5
	t				
5	2	·497	1·252	1·457	1·182
	3	·421	1·438	2·503	3·435
	4	·369	1·368	3·145	5·538
	5	·311	1·175	2·973	6·952
	6	·280	1·035	2·712	6·629

The results for $M(E, t)$ show as is expected, that with higher energies the maximum intensity occurs at greater depths. The values of $M(E, t)$ obtained here are larger than those given by Chakravarty (1947) but there is a general tendency of agreement between the two sets of results. These results can be used for the estimation of the meson intensity at different depths of the atmosphere for a proper distribution of the primary nucleons incident on the top of the atmosphere. A comparison with the results of observation on the size frequency distribution of meson showers will then indicate how far the cascade production process can explain the observed results and also whether any other generation process should be taken into consideration.

ABSTRACT

Nucleon-nucleon collisions are responsible for the production of mesons and the energy loss in the process is partly taken over by the recoil nucleons. The average cross-sections for the energy losses through meson production and through production of recoil nucleons have been obtained by Heitler and Janossy. In the present paper an attempt has been made to study the effect of different cross-sections for the energy loss as well as meson production on the size of the shower. The diffusion equations for the average number of nucleons and mesons have been solved. The integral and differential meson spectra have been evaluated for an incident primary nucleon.

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REFERENCES

- Chakravarty, S. K. (1947). Generation of mesons and its dependence on meson spin. *Bull. Cal. Math. Soc.*, **39**, 186.
 Hamilton, J., Heitler, W., and Peng, H. W. (1943). Theory of Cosmic ray mesons. *Phys. Rev.*, **64**, 78.
 Heitler, W., and Janossy, L. (1949). On the size frequency distribution of penetrating showers. *Proc. Phys. Soc. Lond.*, **A 62**, 669.
 ——— (1949). On the absorption of meson producing nucleons. *Ibid.*, **A 62**, 374.
 ——— (1950). Further investigations on the plural production of meson showers. *Helv. Phys. Acta*, **23**, 417.
 Heitler, W., and Walsh, P. (1945). Theory of Cosmic ray mesons. *Rev. Mod. Phys.*, **17**, 251.
 Lewis, H. W., Oppenheimer, J. R., and Wouthysen, S. A. (1948). Multiple Production of mesons. *Phys. Rev.*, **73**, 127.
 Messel, H. (1951). On the nucleon Cascade with ionisation loss. *Ibid.*, **83**, 21.
 Messel, H., Potts, R. B., and McCusker, C. B. A. (1952). Theoretical Results on the high energy nuclear collisions in light and heavy elements. *Phil. Mag.*, **43**, 889.
 Peng, H. W. (1944). On the Cascade production of meson. *Proc. R. Irish Ac.*, **49 A**, 14, 245.
 Schultz, H. (1954). On the production of penetrating showers at different altitudes and in different materials. *Z. Naturforschg.*, **9 a**, 419.

A NOTE ON THE FRAGMENTATION OF CONICAL 'LINERS' AND ITS RELATION TO THE THEORY OF 'SHAPED-CHARGE'—II

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1. INTRODUCTION

In a previous note (Kothari and Singh, 1953, to be referred to as I), a fragmentation-pattern on a mild steel 'witness-plate' was obtained by base detonating a 'shaped-charge' having an 'inverted' mild steel conical liner of angle 45° . The fragmentation-pattern consists of 'radial streaks' confined within a rather sharply defined annular region (see Fig. 2 of I). From the dimensions of the pattern, the values of velocities of collapse have been calculated, which agree (to order of magnitude) with those of Eichelberger and Pugh (1952).

In this note results for the fragmentation of 45° mild steel conical liners of different calibres and of 80° mild steel conical liner are presented. The formation of 'radial streaks' in the fragmentation-pattern is treated in some detail.

2. EXPERIMENTAL WORK

A. Conical 'liners' of angle 45° and of different base diameters (referred to as calibre and denoted as D , thickness of liner $0.032D$) were machined from rods of mild steel. The 'inverted' conical liner was soldered at one end of a gas pipe. The liner was snug fitted (bearing-fit) in a recess machined in a massive mild steel block (Fig. 1). High explosive (70 TNT, 30 Tetryl) was cast in the equipment. On base

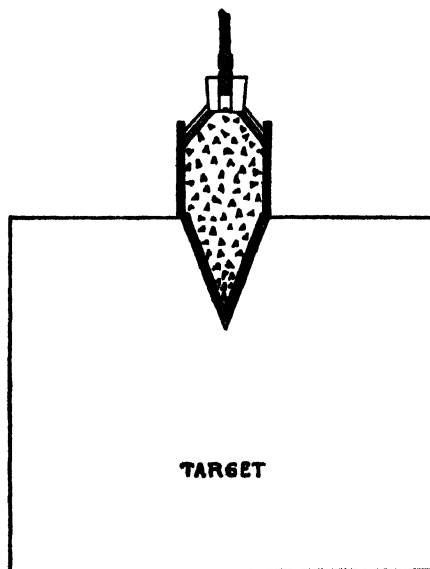


FIG. 1. Lay-out of the equipment with an inverted 45° mild steel conical liner, which is bearing-fit in a recess machined in a mild steel block.

detonating the equipment, the recess was found to have enlarged and the liner was also recovered. The recovered liners are photographed (Fig. 2). The cracks on the surface of the recovered liners are nearly parallel to the axis of propagation of detonation and the liner splits up into wedge-shaped strips.

B. The experimental set-up was the same as described in Part I, except that the conical liner was made of two metals (half of the conical liner towards the base made of copper and the other half towards the apex of mild steel or vice versa). On base detonating the equipment, the fragmentation-pattern (Fig. 3) was obtained which consisted of two annular rings. The outer ring is formed by fragments of the metal towards the base and the inner ring by fragments of the metal towards the apex of the liner.

In another experiment, annular rings of metal were removed at predetermined places in the mild steel 'witness-plate'. On base detonating the equipment, a fragmentation-pattern (Fig. 4) was obtained.

C. The equipment having an 'inverted' conical liner (material—mild steel, calibre—2.0 in., angle—45° and thickness—0.032D) was detonated on a cardboard 'witness-plate' (placed one above the other and clamped together). The fragments penetrated deep into the cardboard target and the co-ordinates of the holes were as expected by the mechanism of collapse of the liner (refer Part I). In some holes, one fragment was found and its surface was dull (slight bluish tinge); in some others, many fragments were recovered from each hole and often it was possible to fit these together to form a single fragment (the broken pieces had a clean surface appearance); and in the remaining holes, many tiny (passing 14 mesh, B.S.S.) fragments were found. Some of the fragments at the periphery of the pattern penetrated as much as 3½ in. of cardboard. The fragments were separated into different sizes and photographed (Fig. 5). The percentage recovery of fragments, i.e. total weight of fragments/weight of original conical liner was about 66%.

D. The equipments having 'inverted' conical liners (material—mild steel, angle—45° and thickness—0.032D) of different calibres were detonated on mild steel 'witness-plates'. The dimensions of the fragmentation-patterns indicate that the angles δ_0 and δ_1 (δ represents the angle between the collapsing liner element of the liner and normal to the original liner surface, δ_0 and δ_1 denote the value of δ for the liner element near the base and the apex of the liner respectively) are 9.4° and 4.4° respectively and are independent of the calibre within the experimental error.

E. The equipment having an 'inverted' conical liner (material—mild steel, calibre—3.0 in., angle—80° and thickness—0.096 in.) was detonated on a mild steel 'witness-plate' (distance of the base of the liner from the witness-plate = 4.5 in.). A typical fragmentation-pattern, which consists of a ring is shown in Fig. 6. The metal in the ring is scooped out in a radial direction and each radial streak has got a scaly appearance. The calculated values of δ_0 and δ_1 are 9.7° and 8.4°.

3. DISCUSSIONS

A. In the experiments described in section (A) above, on base detonating the equipment, there takes place a plastic expansion of the cone material but the expansion of the cone is retarded by the mild steel block. In the recovered conical liners, there are cracks which are nearly parallel to the axis of propagation of detonation. We assume, that in case of those experiments, where we get a fragmentation-pattern on a mild steel 'witness-plate', the material of the cone undergoes plastic expansion and the cracks appear parallel to the axis of propagation of the detonation wave and thus the liner splits up into many wedge-shaped strips.* On further

* Dr. S. Paterson (1954) also suggested in a personal communication to Professor D. S. Kothari that the radial streaks are cut by wedge-shaped strips of the conical liner.

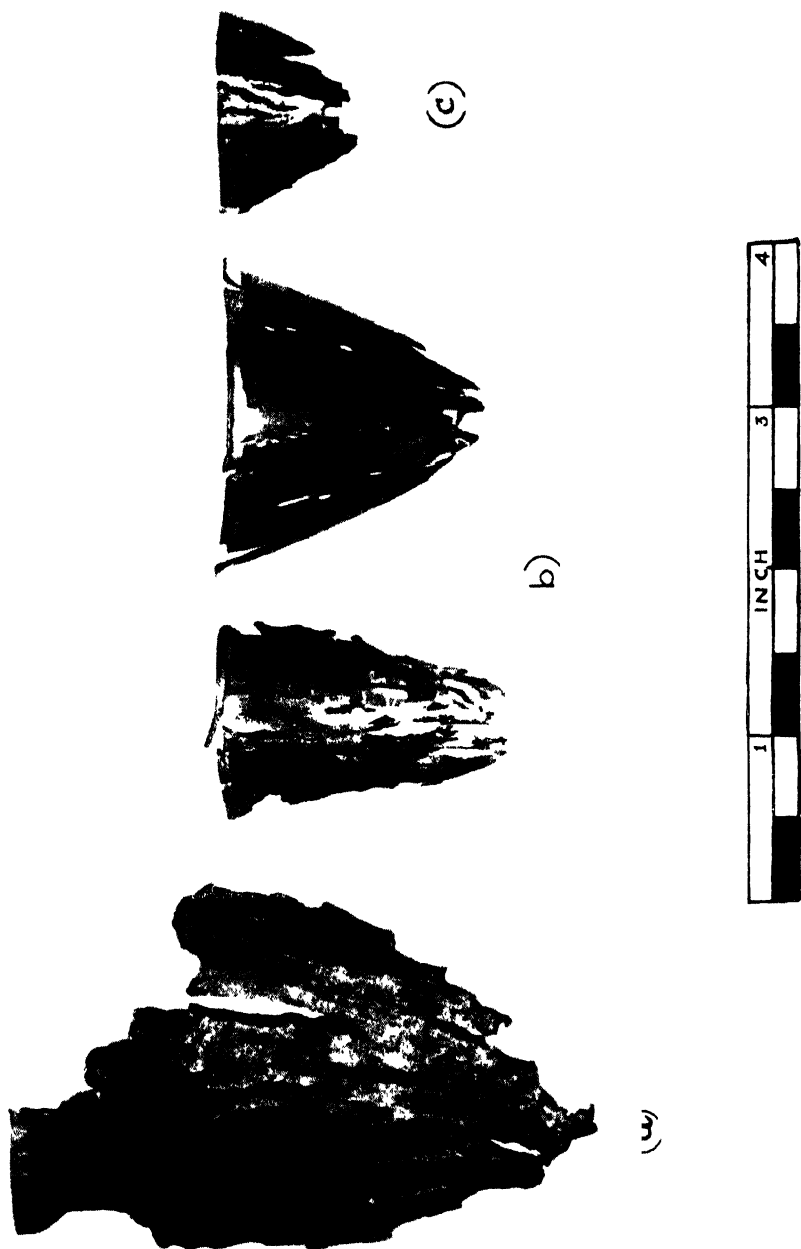


FIG. 2. Photograph of the recessed conical heads of firings showing the cracks nearly parallel to the axis of propagation of detonation. Outside base diameter of the conical heads before firings are (a) 3 in., (b) 1½ in., and (c) 1 in.

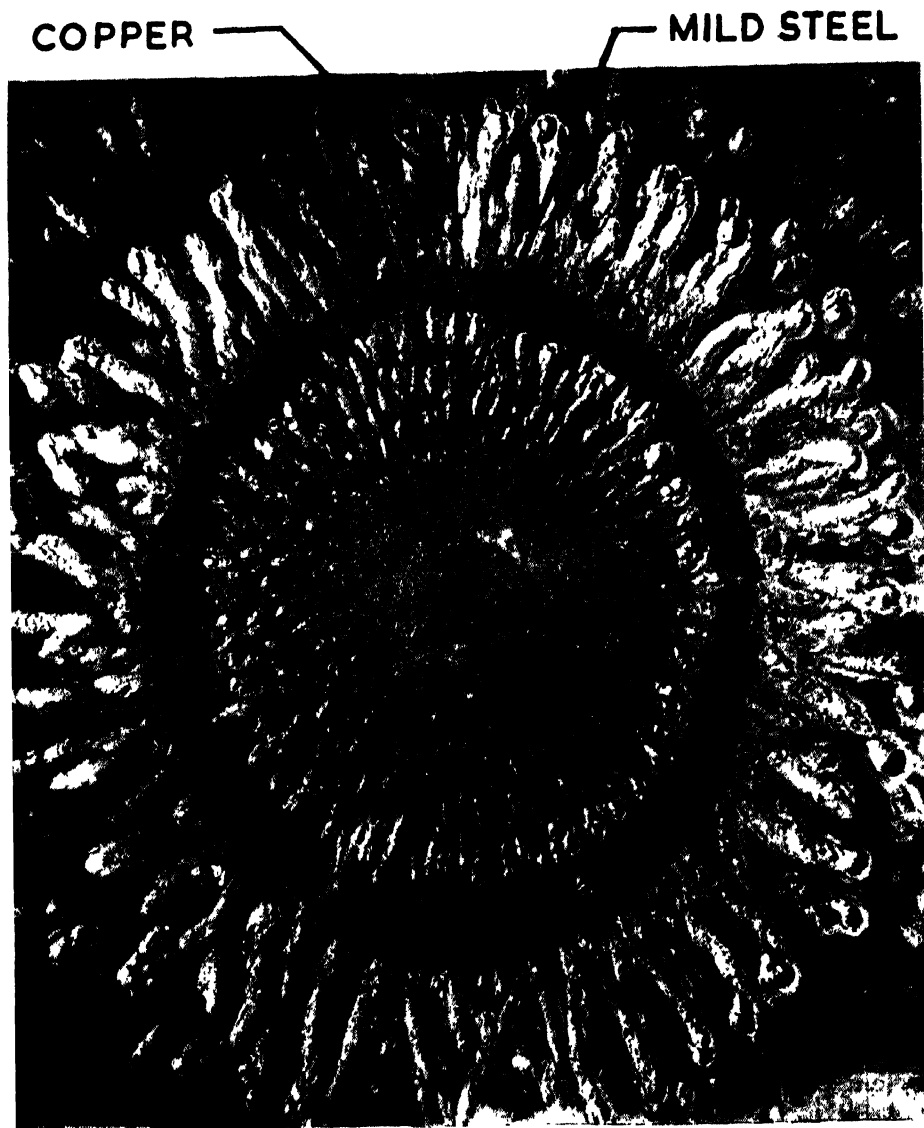


FIG. 3. Typical fragmentation-pattern on a mild steel 'witness-plate' by detonating a 'shaped-charge' having an inverted bi-metallic conical liner (half of the conical liner towards the base made of copper and the other half towards the apex made of mild steel).



FIG. 4. Typical fragmentation-pattern on a mild steel 'witness-plate' (four annular quarter rings of metal were removed from the witness-plate before the firing).

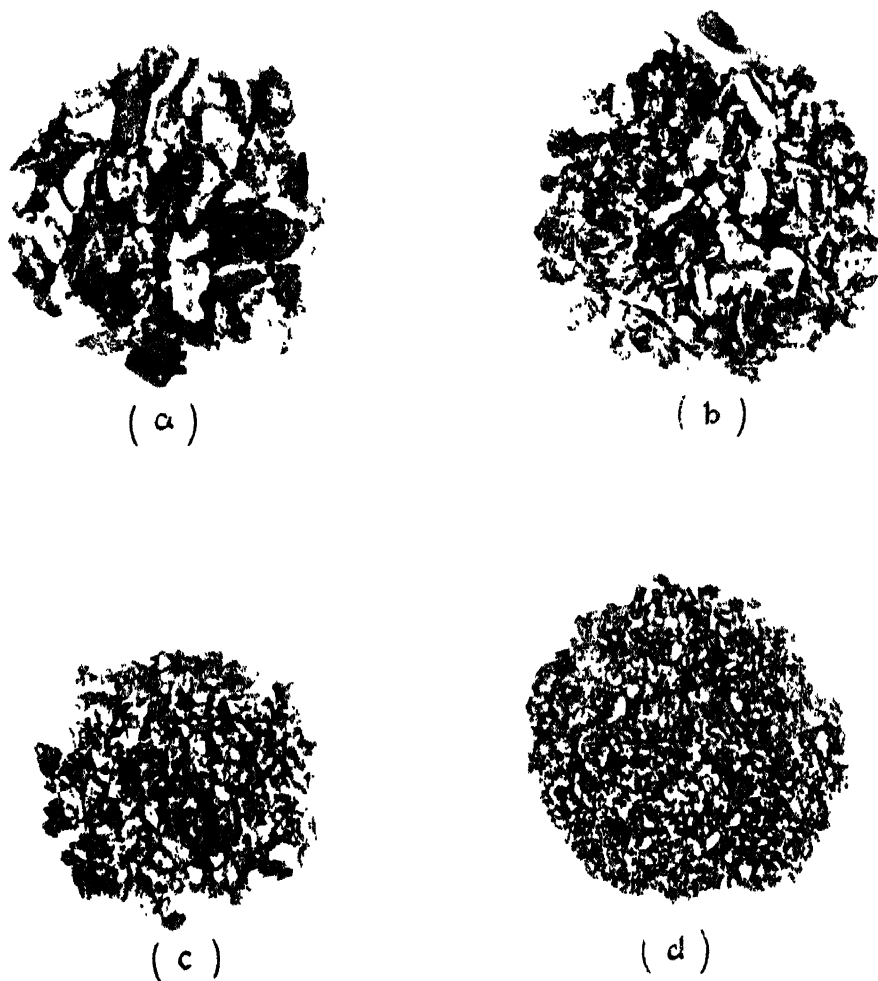


FIG. 5. Fragments collected on a 'witness plate' of card-board.

(a) 11.3% (mean wt. of a fragment = 0.12 gm.)

(b) 28.8% (mean wt. of a fragment = 0.02 gm.)

(c) 6.3% (retained on 14 mesh, B.S.S.)

(d) 19.9% (passing 14 mesh, B.S.S.)

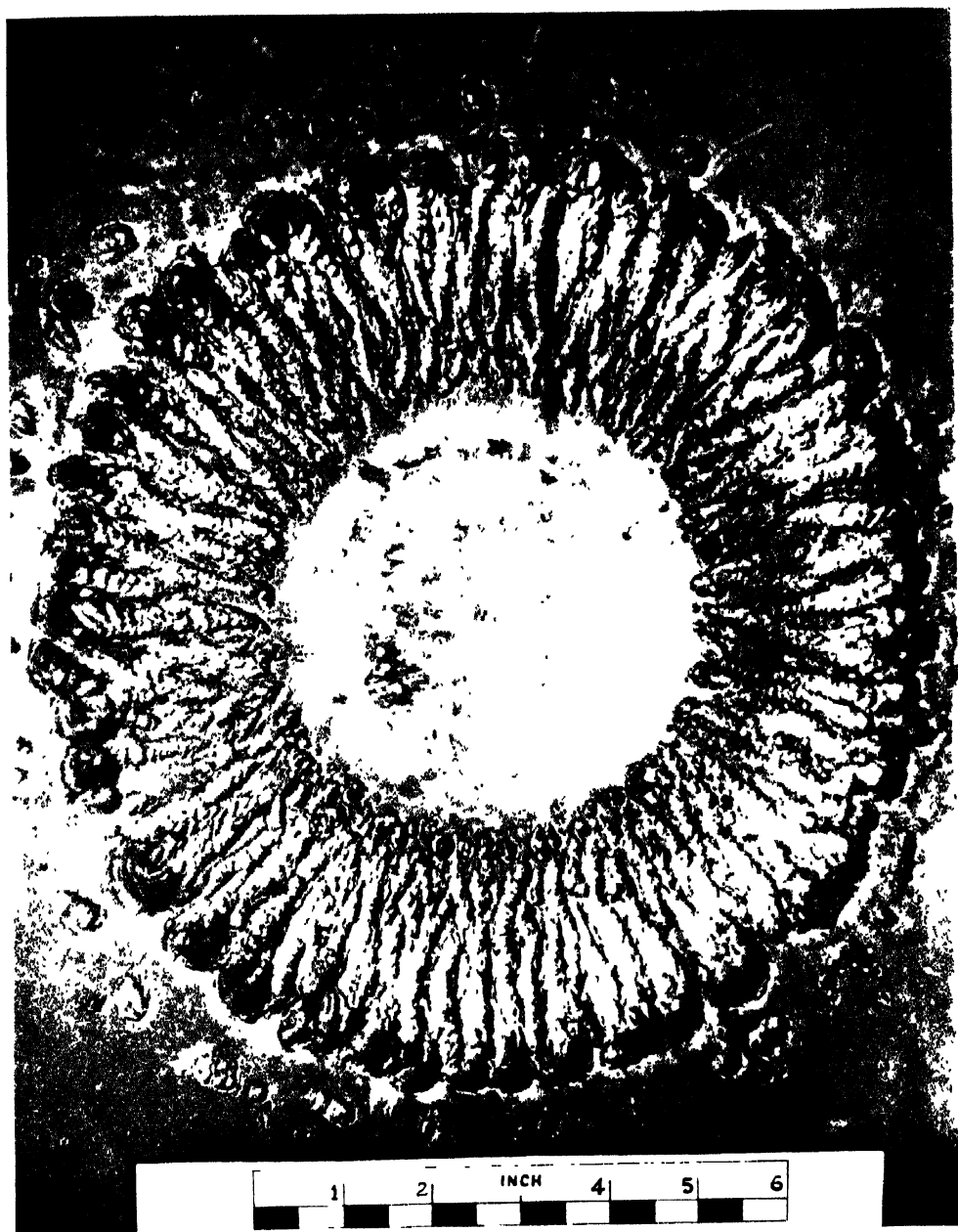


FIG. 6. Typical fragmentation-pattern on a mild steel 'witness-plate' by firing a 80 - mild steel conical liner.

expansion, each strip fragments and as such on striking a mild steel 'witness-plate' forms a fragmentation-pattern having the scooping of the metal in a radial direction. As the detonation wave takes a finite time to travel from the base of the liner to the apex, so the fragments arising from the base of the liner start first as compared to that from the apex.

B. The experiments indicate that the outer and the inner circumferences of the ring are due to the impact of fragments from the metal near the base and the apex of the liner respectively. The fragmentation-pattern (Fig. 4) indicates that by cutting a slot in the centre of the ring, the travel of radial streaks cannot be stopped and that adjacent fragments do not affect each other appreciably so far as their impact on witness plate is taken into consideration. This also indicates that the velocity with which a fragment travels is also independent of the velocities of the neighbouring elements.

C. The experiments on the collection of fragments on a cardboard 'witness-plate' indicate that there are two types of fragments. In such cases, where only one fragment is found at the bottom of a hole and the surface is dull (slight bluish tinge) the fragment is denoted as a 'primary fragment'. Obviously the primary fragments arise by the breaking of the liner when the explosive detonates. In those cases where many fragments are recovered from one hole and their clean surfaces can be joined to form a single fragment, the fragments are denoted as 'secondary fragments'. Obviously a single fragment after striking a cardboard target and/or while penetrating the target breaks into many fragments. It is these tiny fragments, which strike the mild steel 'witness-plate' at a tremendous velocity (of the order of 2 to 3 km./sec.) that produce a fragmentation-pattern.

Pearson and Rinehart (1952) recovered the fragments of low-carbon steel cylinders, which were internally loaded with explosive charges. The authors have not published the experimental details of the method of recovery of fragments and also have not discussed the two types of fragments, i.e. 'primary' and 'secondary'.

D. The experiments on firings of 45° mild steel 'inverted' conical liners indicate that the velocities of collapse of the base and the apex of the liner are 2.5 and 1.2 km./sec. respectively and are independent of calibre.

E. The experiments on firings of 80° mild steel 'inverted' conical liners indicate that the velocities of collapse of the base and the apex of the liner are 3.1 and 2.7 km./sec. respectively.

F. The possible bearing of the results on the theory of 'shaped-charge' is that the velocities of collapse are independent of the calibre. Let β represent the half angle of collapse, i.e. the angle which the collapsing liner makes with the axis of the equipment. Let β_0 denote the value of β for the liner element near the apex of the liner (in 'shaped-charges', subscript 0 and I refer to the apex and the base of liner respectively). We assume that

$$\beta_0 = \alpha + 2\delta_0.$$

This indicates that β_0 is also independent of the calibre. The author (1953) has given an explicit expression for the length of jet. It has been shown that the minimum length of jet depends on calibre, δ_0 , δ_1 and β_0 . As the angles δ_0 , δ_1 and β_0 are independent of calibre, therefore the minimum length of jet when fully formed is directly proportional to the calibre of the equipment. The depth of penetration by a jet into a given target depends upon its length and density. The interesting conclusion is that, to a first approximation, in 'shaped-charges' lined with 45° conical liners, the penetration and the calibre are linearly related. The theoretical calculations indicate that in case of 45° and 80° mild steel conical liners, the minimum timings for the complete formation of jets (that is, the time for the detonation wave to travel from the apex to the base of the liner plus the time for the element at the base of the liner to reach the axis) are $6.5D$ and $3.7D$ μ sec. (where D is in cm.); and the minimum lengths of the jets are $3D$ and $1D$, respectively. This also

indicates that the penetration by a jet from 45° mild steel conical liner will be greater than that of 80° conical liner. The author (1954) has shown that in 'shaped-charges', the penetration and calibre are linearly related and 45° conical liners give greater penetration than that of 80° liner.

4. SUMMARY

The results presented above can be partly summarised as follows:

(i) The metal of the conical liner first fragments into wedge-shaped strips, which further fragment and give rise to 'radial streaks'. The outer and the inner circumferences of the ring are due to the impact of fragments from the metal near the base and the apex of the liner respectively.

(ii) The collection of fragments on a cardboard 'witness-plate' indicate that there are two types of fragments—'primary' and 'secondary'. 'Primary fragments' arise by the breaking of the liner when the explosive detonates. A 'primary fragment' after striking a cardboard target and/or while penetrating the target breaks up into many fragments, which are denoted as 'secondary fragments'.

(iii) The possible bearing of the results on 'shaped-charges' is that the length of jet is proportional to the calibre and the minimum length of jet by 45° conical liner is more than that of 80° conical liner.

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ABSTRACT

Recently Kothari and Singh (1953) obtained a fragmentation-pattern by base detonating a 'shaped-charge' having an 'inverted' conical liner. The fragmentation of 45° conical liners of different calibres and of 80° conical liner is described, and the possible bearing of the results on the theory of 'shaped-charge' is presented. The formation of 'radial streaks' in the fragmentation-pattern is also described.

REFERENCES

- Eichelberger, R. J., and Pugh, E. M. (1952). Experimental Verification of the Theory of Jet Formation by Charges with Lined Conical Cavities. *J. Appl. Phys.*, **23**, 537-542.
 Kothari, D. S., and Singh, S. (1953). A note on the Fragmentation of Conical 'Liners' and its relation to the theory of 'Shaped-Charge'. *Proc. Nat. Inst. Sci. India*, **19**, 507-510.
 Pearson, J., and Rinehart, J. S. (1952). Deformation and Fracturing of Thick-Walled Steel Cylinders under Explosive Attack. *J. Appl. Phys.*, **23**, 434-441.
 Singh, S. (1953). Note on the Length of High-Velocity 'Munroe' Jets. *Proc. Nat. Inst. Sci. India*, **19**, 357-360.
 ——— (1954). Target Damages by Explosives with lined Conical Cavities. *Ibid.*, **20**, 274-279.

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TEMPERATURE DEPENDENCE OF THE DIELECTRIC CONSTANT OF IONIC CRYSTALS

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1. INTRODUCTION

Although the temperature coefficients of the dielectric constant of several ionic crystals have been measured over a wide range of temperature (Eucken and Büchner, 1935), very little work has been done on the temperature dependence of the coefficients themselves. Recently, Yamashita (1951), (1952) has developed a quantum mechanical theory of the dielectric constant (ϵ) of an ionic crystal and has extended his theory for the calculation of the temperature coefficient $\left(\frac{1}{\epsilon} \frac{d\epsilon}{dT}\right)$ of the dielectric constant of lithium fluoride. He has shown theoretically that the value of the coefficient for LiF is not independent of temperature. In the present investigation the variation of the quantity $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ with temperature has been experimentally determined in the case of three cubic crystals, namely, lithium fluoride, sodium chloride and magnesium oxide. The results obtained thereby have been compared with those to be expected on Yamashita's theory.

The quantum mechanical theory of Yamashita has been extended to the case of MgO for the calculation of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ and its variation with T . For this purpose a knowledge of the values of the overlap and exchange terms occurring in the theory is essential. A simple method of evaluating these quantities has been outlined here and the details are presented below. The agreement between theory and experiment is good in the case of MgO. The reasons for the failure of the theory in the case of the highly polar crystal, LiF, have also been discussed here. Details of the theoretical calculations and experimental technique will be now presented. Atomic units have been used throughout the present calculations.

2. THEORETICAL

According to Yamashita (1951) the value of $\frac{d\epsilon}{dT}$ for a crystal can be calculated from a knowledge of the change in its free energy (ΔA) since

$$\frac{d(\Delta A)}{dT} = -\frac{1}{2} \frac{d\chi}{dT} F^2; \quad \frac{d\epsilon}{dT} = 4\pi \frac{d\chi}{dT} \quad \dots \quad (1)$$

(χ = susceptibility). The expression for $\frac{d(\Delta A)}{dT}$ in terms of the quantities a (inter-ionic distance), ρ (compressibility term) and α_M (Madelung constant) is as follows (Yamashita, 1951).

$$\begin{aligned}
\frac{d(\Delta A)}{dT} = \frac{1}{2a^3} \left\{ 3 \left[2xF + \frac{1}{2} \frac{4\pi}{3} \frac{(2x)^2}{2a^3} + \frac{4\pi}{3} \frac{(4\tilde{r}^2\lambda)(2x)}{2a^3} - \frac{2\alpha_M}{3a^2} \left(\frac{1}{\rho} - \frac{2}{a} \right) x^2 + \right. \right. \\
+ 4\tilde{r}^2 F\lambda + \frac{1}{2} \frac{4\pi}{3} \frac{(4\tilde{r}^2)^2 \lambda^2}{2a^3} - 3\lambda^2 - A\lambda^2 - B\lambda x \left. \right] \frac{\delta a}{a} + \\
\left[\frac{1}{2} \frac{4\pi}{3} \frac{(2x)^2}{2a^3} + \frac{4\pi}{3} \frac{(4\tilde{r}^2\lambda)(2x)}{2a^3} + \frac{1}{2} \frac{4\pi}{3} \frac{(4\tilde{r}^2)^2 \lambda^2}{2a^3} \right] \frac{\delta a}{a} - \\
- \left[\frac{2\alpha_M}{3a^2} \left(\frac{a}{\rho^2} - \frac{2}{\rho} - \frac{2}{a} \right) x^2 \right] \frac{\delta a}{a} - \left[\frac{Aa\lambda^2}{\rho} + \frac{Ba}{\rho} \lambda x \right] \frac{\delta a}{a} + \\
+ \left[\frac{h\omega}{kT^2} \cdot \frac{e^{\frac{h\omega}{kT}}}{\left(e^{\frac{h\omega}{kT}} - 1 \right)^2} \cdot h(\Delta\omega_x + 2\Delta\omega_y) \cdot \delta T \right] \left. \right\} \dots \dots (2)
\end{aligned}$$

The quantities x and λ are the perturbation parameters while the quantities A and B are the overlap and exchange terms. The last term of equation (2) is the entropy term obtained by use of the Einstein model. (ω = Eigenfrequency). \tilde{r}^2 is given by the equation

$$\tilde{r}^2 = \int_0^a |P(r)|^2 r^2 dr \quad \dots \quad \dots \quad (3)$$

where $P(r)$ is the probability term in the wave function. If ψ_0 is the unperturbed wave function and ψ the perturbed wave function we have according to Kirkwood (1932)

$$\psi = \psi_0(1 + \lambda r \cos \theta) \quad \dots \quad \dots \quad (4)$$

(λ = perturbation parameter). For actual calculation of $\frac{d\epsilon}{dT}$ we need the values of \tilde{r}^2 , x and λ as well as those of A and B . The compressibility data on the crystal under consideration are also required for the evaluation of ρ (Yamashita, 1952). Yamashita has calculated the values of \tilde{r}^2 , A and B by directly evaluating the corresponding integrals. The values thus arrived at are, in general, not self-consistent since the calculated and observed values of the dielectric constant do not agree. One of the main reasons for this discrepancy lies with the difficulty in the choice of an accurate wave function for an ion in a crystal. Owing to this fact in the present paper we have used the observed value of the dielectric constant in order to arrive at the values of the parameters.

3. VALUES OF THE PARAMETERS AND THE CALCULATION OF $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$

For the evaluation of \tilde{r}^2 we proceed as follows. The polarisability α_f of an ion in the free state is related to \tilde{r}^2 by the equation

$$\alpha_f = \frac{(4\tilde{r}^2)^2}{6} \quad \dots \quad \dots \quad (5)$$

The values of α_f have been already given by Pauling (1927) who has calculated these values on a quantum mechanical basis. Using the value of \tilde{r}^2 thus obtained

we next proceed to calculate the value of the overlap term A . Now, the change in the total energy (ΔE_0) of an ionic crystal in a high frequency field is given by the equation (Yamashita, 1952)

$$\Delta E_0 = -4\tilde{r}^2 F\lambda - \frac{1}{2} \frac{4\pi}{3} \frac{(4\tilde{r}^2)^2}{2a^3} \lambda^2 + 3\lambda^2 + A\lambda^2 \quad \dots \quad (6)$$

We have also

$$\Delta E_0 = -\frac{1}{2} \chi_0 F^2; \epsilon_0 = 1 + \frac{4\pi\chi_0}{2a^3} \quad \dots \quad (7)$$

(ϵ_0 = dielectric constant in the high frequency field). Equation (6) may be written in the form

$$\Delta E_0 = -4\tilde{r}^2 F\lambda + N\lambda^2 \quad \dots \quad (8)$$

By the variational principle we have $\frac{\partial \Delta E_0}{\partial \lambda} = 0$

Therefore,

$$N\lambda = 2\tilde{r}^2 F \quad \dots \quad (9)$$

Inserting this value of $N\lambda$ in equation (8) we can calculate the value of N since the value of ΔE_0 can be obtained from equation (7) using the observed value of ϵ_0 . Needless to say, the value of A is easily obtained once the value of N is known.

From the known values of \tilde{r}^2 and A we can evaluate the exchange term B as follows. We make use of the expression given by Yamashita (1952) for the change (ΔE) in the total energy of a crystal in a static field. As stated earlier the value of ρ occurring in the equation can be obtained from the compressibility data. Substituting the values of \tilde{r}^2 , A and ρ in Yamashita's equation we calculate the value of ΔE by giving B different values (ranging usually from -1 to $+1$). The values of ΔE thus calculated are plotted against the respective B values when a smooth curve is obtained. From the observed static dielectric constant of the crystal the true value of ΔE can be obtained from an equation similar to (7). Thus the value of B corresponding to the true value of ΔE can be found from the graph. Although the above methods of calculating the values of \tilde{r}^2 , A and B are phenomenological they are quite accurate on account of their self-consistency and are also simple since the values of these parameters are obtained without tedious evaluation of the integrals.

The value of A for LiF obtained by the present method is 1.76 ($\tilde{r}^2 = 1.90$). This value of A is nearly twice that ($A = 0.8$) given by Yamashita. The present value of A is more consistent with experimental data and since this parameter may be viewed upon as a correction term this new value of A is to be preferred. Similarly in the case of MgO the value of A obtained here is 3.51 as compared to Yamashita's value of 1.9 only. It may be pointed out here that the value of the dielectric constant for high frequency field calculated by the use of Yamashita's values is very much higher than the observed values. [LiF: $\epsilon_0 = 2.33$ (calculated) 1.92 (observed); MgO: $\epsilon_0 = 4.56$ (calcd.) 2.95 (observed).] However, good agreement is obtained with regard to the value of \tilde{r}^2 . The value of \tilde{r}^2 for MgO obtained by the present method is 3.14 as compared to Yamashita's value of 3.19. In the case of LiF, the value of $B = 0.52$ obtained by the present method agrees very well with that ($B = 0.50$) obtained by Yamashita by direct evaluation of the integrals. As pointed out earlier, since the values of A and B obtained by the use of the experimental data on dielectric constants are self-consistent, these values are to be preferred in the calculations of the temperature coefficients of dielectric constant.

Table I and Fig. 1 show the method of calculation of B for LiF. The observed value for the static dielectric constant being 9.27 (Mott and Gurney, 1940) the value of ΔE corresponding to this is $-35.5 F^2$. From Fig. 1 it will be seen that the value

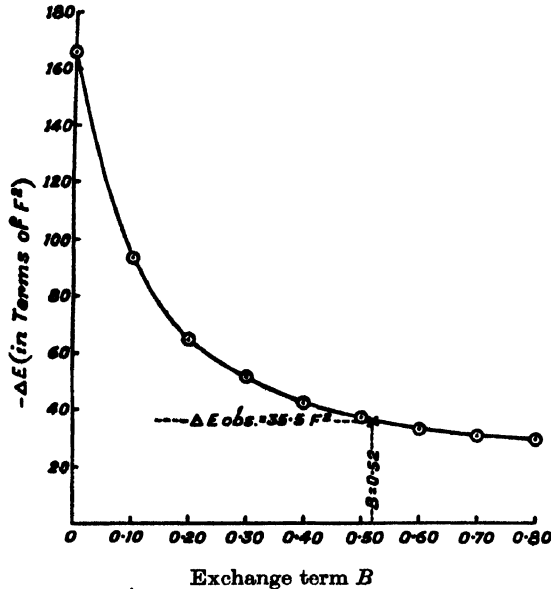


FIG. 1. Variation of ΔE with B
Lithium fluoride

of B for this value of ΔE is 0.52. Table II and Fig. 2 show similarly the ΔE and B values for MgO . The observed static dielectric constant of MgO being 9.8 the

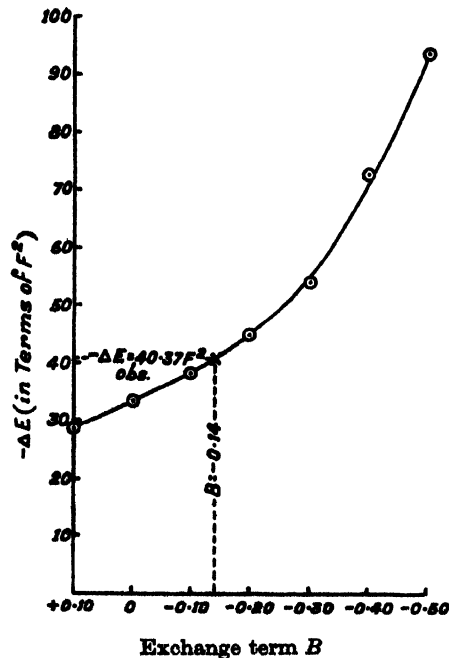


FIG. 2. Variation of ΔE with B
Magnesium oxide

value of $\Delta E = -40.37 F^2$. From Fig. 2 we find that the value of B corresponding to this ΔE value is -0.14 . The values of λ and x given in Tables I and II are obtained by the use of the variational condition $\frac{\partial \Delta E}{\partial \lambda} = 0$; $\frac{\partial \Delta E}{\partial x} = 0$.

TABLE I
B and ΔE values for LiF.

<i>B</i>	λ	<i>x</i>	ΔE
0	11.00 <i>F</i>	124.40 <i>F</i>	-166.15 <i>F</i> ²
0.10	5.71 <i>F</i>	70.31 <i>F</i>	-92.00 <i>F</i> ²
0.20	3.70 <i>F</i>	50.47 <i>F</i>	-64.56 <i>F</i> ²
0.30	2.62 <i>F</i>	40.45 <i>F</i>	-50.45 <i>F</i> ²
0.40	1.92 <i>F</i>	34.69 <i>F</i>	-42.00 <i>F</i> ²
0.50	1.41 <i>F</i>	31.22 <i>F</i>	-36.57 <i>F</i> ²
0.60	0.985 <i>F</i>	29.22 <i>F</i>	-32.97 <i>F</i> ²
0.70	0.599 <i>F</i>	28.42 <i>F</i>	-30.70 <i>F</i> ²
0.80	0.200 <i>F</i>	28.89 <i>F</i>	-29.48 <i>F</i> ²

TABLE II
B and ΔE values for MgO

<i>B</i>	λ	<i>x</i>	<i>E</i>
0.10	2.72 <i>F</i>	12.24 <i>F</i>	-29.33 <i>F</i> ²
0	3.05 <i>F</i>	13.92 <i>F</i>	-33.10 <i>F</i> ²
-0.10	3.49 <i>F</i>	16.09 <i>F</i>	-37.97 <i>F</i> ²
-0.20	4.07 <i>F</i>	19.01 <i>F</i>	-44.55 <i>F</i> ²
-0.30	4.90 <i>F</i>	23.15 <i>F</i>	-53.93 <i>F</i> ²
-0.40	6.18 <i>F</i>	29.49 <i>F</i>	-72.15 <i>F</i> ²
-0.50	8.40 <i>F</i>	40.46 <i>F</i>	-93.20 <i>F</i> ²

Table III gives the values of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ for LiF calculated by the use of equations (1) and (2) with and without the entropy term. In Table IV the values of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ of

TABLE III
 $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ of LiF (*Theoretical*)

Temp. Range °C.	$\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ calcd. (ΔS term neglected)	$\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ calcd. (ΔS term included)
27-127	49.0 $\times 10^{-5}$	35.0 $\times 10^{-5}$
127-227	56.5 $\times 10^{-5}$	40.5 $\times 10^{-5}$
227-327	64.0 $\times 10^{-5}$	46.0 $\times 10^{-5}$
327-527	75.5 $\times 10^{-5}$	56.0 $\times 10^{-5}$
527-727	91.0 $\times 10^{-5}$	70.5 $\times 10^{-5}$

MgO are given and these have been calculated without the entropy term which is rather uncertain. The thermal expansion data of Eucken and Dannöhl (1934) on LiF and of Austin (1931) on MgO have been used in the calculations.

TABLE IV

$\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ of MgO (*Theoretical*)

Temp. Range °C.	$\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ calcd. (ΔS term neglected)
25- 50 ..	13.5×10^{-5}
50-100 ..	16.0×10^{-5}
100-150 ..	19.6×10^{-5}
150-200 ..	21.5×10^{-5}
200-250 ..	22.5×10^{-5}

It will be seen from Tables III and IV that in the case of LiF and MgO a variation in the value of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ with T is to be expected. We shall now give the details of the experimental method for the determination of the temperature coefficient of the dielectric constant of crystals at various temperatures and then we shall see how far the predictions of the theory are in accord with experimental results.

4. EXPERIMENTAL

The capacity measurements were made with a heterodyne beat apparatus which has been already described elsewhere (Narasimhan, 1953). The A.C. power supply for the apparatus was obtained through a constant voltage transformer. The beat note was extremely steady after a warm-up period of 3 to 4 hours. The changes in the capacity brought about by changes in temperature were measured by means of a stepped-rod type condenser (Watson, Rao and Ramaswamy, 1939). This condenser was placed in parallel with the crystal condenser which consisted of a brass container (3" diameter; $2\frac{1}{2}$ " height; $\frac{5}{8}$ " thick) with a removable lid screwed on top. This assembly was first silvered and then given a thick coating of gold in the interior. In the inside bottom of the cell was mounted a circular piece of ceramic insulator on which was screwed a gold plated brass disc ($1\frac{1}{2}$ " diameter; $\frac{1}{8}$ " thick) carrying a terminal for connecting the r.f. lead. The crystal under investigation was placed on the disc after aluminising the broad parallel surfaces of the crystal plate. The contact for the earth electrode of the crystal plate was obtained by a metal probe connected to the brass cell. The lead from the terminal of the brass disc was taken out through a hole ($\frac{1}{8}$ " diameter) made at the side of the outer brass vessel. Covering this hole was a ceramic insulator carrying a terminal to which was connected the above-mentioned lead from the insulated disc. The crystal cell could be heated by means of a wire resistance heater strip placed at the outside bottom of the cell. The cell was connected to the heterodyne beat apparatus by means of a shielded cable with the outer brass vessel forming the earthed electrode.

Temperatures were measured by means of a thermocouple. Since the difference in temperature between the inner brass disc and the outer brass vessel

was found to be less than 0.5°C . over the range employed it was found convenient to place the thermocouple on the outer wall of the cell for purposes of temperature measurement. Regulation at desired temperatures was carried out by means of an electronic relay in conjunction with a 'contact head' which was operated by means of the expansion of the brass cell itself. The regulation thus obtained was within 1°C . In the present investigation we have worked up to 250°C . only partly on account of the fact that the capacity changes up to this range only could be followed by the Watson condenser in one continuous experiment and partly on account of the fact that the aluminised surfaces of the crystal tended to 'crack-up' into flakes at higher temperatures. The experimental procedure comprised of firstly the determination of the capacity change of the empty cell over the desired temperature range and then the determination of the capacity change over the same temperature range with the crystal plate inside the cell. The difference in the readings gives the capacity change (ΔC) due to the crystal plate in arbitrary condenser units and thus the value of $\frac{\Delta C}{\Delta T}$ could be obtained. The ΔC values were checked up by taking the readings both on heating and cooling when reproducible results could be obtained. In the present work the value of the capacity (C) of the crystal plate in arbitrary condenser units was obtained at room temperature by direct measurement and subsequent capacity (C) values at different temperatures have been obtained by the addition of the corresponding ΔC values. According to Eucken and Büchner (1935) $\frac{1}{\epsilon} \frac{d\epsilon}{dT} = \frac{1}{C} \frac{\Delta C}{\Delta T} - \alpha$ where α is the coefficient of linear expansion of the crystal.

Thus using the known values of α we have obtained the value of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ from the capacity data. The error in the values of $\frac{1}{C} \frac{\Delta C}{\Delta T}$ reported here is estimated to be $\pm 0.2 \times 10^{-5}$ corresponding to an error of 2-3 divisions in the measuring condenser settings.

LiF and MgO crystals used in the present work were clear crystals grown from the melt while the NaCl crystal used was a flawless natural one.

5. RESULTS

The experimental results on the dependence of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ on T for LiF, NaCl and MgO are given in Tables V, VI and VII. The capacity readings are in arbitrary condenser units.

TABLE V

$\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ of LiF in the temperature range $25^{\circ}\text{C}.$ – $250^{\circ}\text{C}.$

Temp. Range $^{\circ}\text{C}.$	Cap. change due to cell ΔC (cell)	Cap. change with LiF	ΔC	Initial cap. C	$\frac{1}{C} \frac{\Delta C}{\Delta T}$ $\times 10^5$	$\alpha \times 10^5$	$\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ $\times 10^5$
25–50 ..	364	1,338	974	105,726	35.4	3.6	31.8
50–100 ..	971	3,398	2,427	106,700	45.5	3.9	41.6
100–150 ..	1,122	3,925	2,803	109,127	51.3	4.2	47.1
150–200 ..	2,657	6,072	3,415	111,930	61.0	4.5	56.5
200–250 ..	5,836	9,909	4,073	115,345	70.6	4.8	65.8

TABLE VI

 $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ of NaCl in the temperature range 25°C.–250°C.

Temp. Range °C.	Cap. change due to cell ΔC (cell)	Cap. change with NaCl	ΔC	Initial cap. C	$\frac{1}{C} \frac{\Delta C}{\Delta T}$ $\times 10^5$	$\alpha \times 10^5$	$\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ $\times 10^5$
25– 50 ..	406	740	334	33,768	39.6	4.04	35.6
50–100 ..	902	1,674	772	34,102	45.3	4.05	41.3
100–150 ..	2,235	3,213	978	34,874	56.1	4.06	52.0
150–200 ..	5,088	6,285	1,197	35,852	66.8	4.08	62.7
200–250 ..	8,456	9,806	1,350	37,049	72.9	4.09	68.8

TABLE VII

 $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ of MgO in the temperature range 25°C.–250°C.

Temp. Range °C.	Cap. change due to cell ΔC (cell)	Cap. change with MgO	ΔC	Initial cap. C	$\frac{1}{C} \frac{\Delta C}{\Delta T}$ $\times 10^5$	$\alpha \times 10^5$	$\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ $\times 10^5$
25– 50 ..	406	581	175	46,364	15.1	0.67	14.4
50–100 ..	902	1,290	388	46,539	16.7	0.79	15.9
100–150 ..	2,235	2,690	455	46,927	19.4	0.97	18.4
150–200 ..	5,088	5,637	549	47,382	23.2	1.06	22.1
200–250 ..	8,456	9,036	574	47,931	24.0	1.11	22.9

The ΔC (cell) values for NaCl and MgO are different from those for LiF owing to a few alterations made in the original cell for purposes of accommodating the larger NaCl and MgO crystals.

6. DISCUSSION

The variations of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ with T for LiF, NaCl and MgO have been shown graphically in Figs. 3–5. It will be seen that the experimentally observed variation of

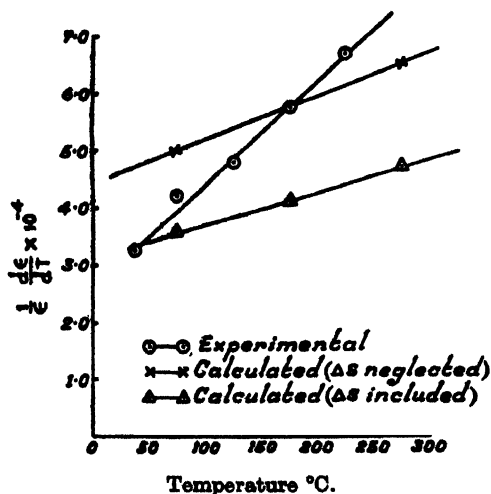


FIG. 3. Variation of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ of LiF with temperature

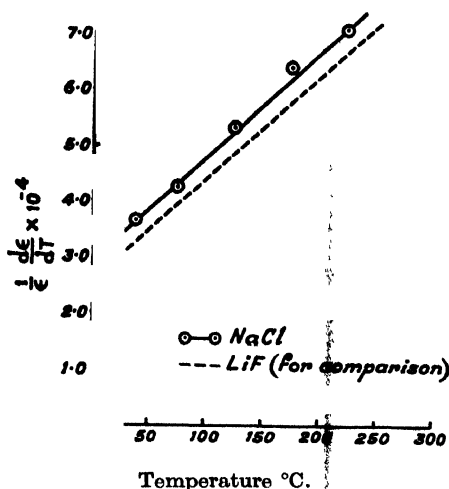


FIG. 4. Variation of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ of NaCl with temperature

$\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ of LiF is much larger than that theoretically predicted (both with and without ΔS term). The temperature dependence of the coefficient in the case of NaCl follows very closely that observed for LiF (see Fig. 4). The coefficient in the

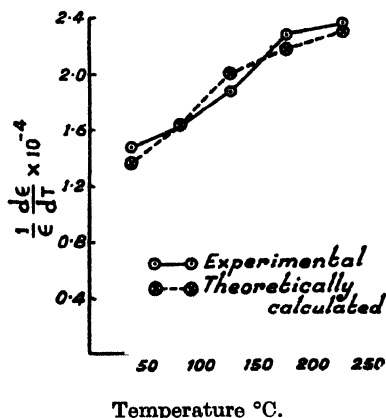


FIG. 5. Variation of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ of MgO with temperature

case of NaCl is displaced towards larger values. This shift may be qualitatively accounted for on the basis of the polar nature of the crystals as has been shown by Eucken and Büchner. In the case of MgO, however, the agreement between theory and experiment is quite satisfactory considering the neglect of the ΔS term.

Of the reasons for the discrepancy between theory and experiment in the case of the ionic crystals like LiF we may mention here the neglect of long range forces in the theory. In other words, only the nearest neighbours of an ion are considered. Further, the calculation of the repulsive energy term from compressibility data

may not be accurate. It must be also stated that the use of the Einstein model for the calculation of the ΔS term makes the calculations approximate. Above all the variations in the $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ values are theoretically obtained because of variations in the thermal expansion coefficient. In this respect the variations in the thermal expansion coefficient seem to govern ultimately the $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ values since it will be seen from equation (2) that the value of $\frac{\delta a}{a}$ is used for multiplication. The experimental results do not seem to favour this relationship at least in the case of LiF. It is interesting to note here that while the α values of NaCl do not vary much the $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ values are profoundly influenced by temperature change (refer Table VI).

From the observed variation of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ with T in the three crystals under consideration the following empirical expression may be deduced

$$\frac{1}{\epsilon} \frac{d\epsilon}{dT} \cdot \frac{1}{T^{\frac{1}{2}}} = \text{constant. } T = ^\circ\text{K.} \quad \dots \quad (10)$$

As will be seen from Table VIII equation (10) represents the variation of the $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ values to a fair degree of accuracy. The value of the constant is of course characteristic of the crystal.

TABLE VIII

For LiF, NaCl and MgO

Temp. Range °C.	$\frac{1}{\epsilon} \frac{d\epsilon}{dT} \cdot \frac{1}{T^{\frac{1}{2}}}$		
	LiF	NaCl	MgO
25-50 ..	0.058	0.065	0.026
50-100 ..	0.064	0.064	0.025
100-150 ..	0.059	0.066	0.023
150-200 ..	0.060	0.066	0.023
200-250 ..	0.059	0.062	0.021

It must be emphasized here that the basis of equation (10) is purely empirical. But considering the constancy of the values obtained in Table VIII the equation seems to represent a true relationship at least for LiF and NaCl. The values obtained for MgO are progressively lower with increasing temperature but further work at higher temperatures is needed in order to throw more light on this aspect of the problem.

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ABSTRACT

The temperature coefficient $\left(\frac{1}{\epsilon} \frac{d\epsilon}{dT}\right)$ of the dielectric constant of LiF, NaCl and MgO has been measured over the temperature range 25°C.-250°C. It is shown that the value of the coefficient itself is dependent on temperature. The quantum mechanical theory of Yamashita has been used for the theoretical calculation of the values of $\frac{1}{\epsilon} \frac{d\epsilon}{dT}$ at different temperatures. From the present experimental data the above theory is shown to fail in the case of LiF while in the case of MgO good agreement is obtained. The drawbacks of the theory have also been discussed in the light of the above data. An empirical relationship between the value of the temperature coefficient of ionic crystals and the temperature has been pointed out.

REFERENCES

- Austin, J. B. (1931). The thermal expansion of some refractory oxides. *J. Amer. Ceram. Soc.*, **14**, 795-810.
- Eucken, A. and Büchner, A. (1935). Die Dielektrizitätskonstante schwach polare Kristalle und ihre Temperaturabhängigkeit. *Z. physik. Chem.*, **127**, 321-349.
- Eucken, A. and Dannöhl, W. (1934). Die thermische Ausdehnung einiger Alkalihalogenide und Metalle bei hohen Temperaturen. *Z. Elektrochem.*, **40**, 814-821.
- Mott, N. F. and Gurney, R. W. (1940). Electronic processes in ionic crystals. Oxford.
- Narasimhan, P. T. (1953). Dipole moments of camphor compounds—Part I. *Proc. Ind. Acad. Sci.*, **A37**, 551-556.
- Pauling, L. (1927). The sizes of ions and the structure of ionic crystals. *J. Amer. Chem. Soc.*, **49**, 765-790.
- Watson, H. E., Gundu Rao, G. and Ramaswamy, K. L. (1934). The dielectric coefficients of gases. Part II. *Proc. Roy. Soc. (Lond.)*, **A143**, 558-588.
- Yamashita, J. (1951). Temperature dependence of the dielectric constant of ionic crystals. *Busseiron Kenkyu.*, No. 35, 33-39; *Chem. Abstr.* (1952), **46**, 6884 h.
- (1952). The theory of the dielectric constant of ionic crystals—Part I. *Prog. theor. Phys.*, **8**, 280-290.

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SOLID-LIQUID EQUILIBRIA IN BINARY AND MULTI-COMPONENT MIXTURES

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1. INTRODUCTION

Solid solutions are an important class of mixtures on account of their applicability to metallic solutions and alloys. Such solutions are amenable to theoretical treatment as most of statistical theories assume crystalline or quasi-crystalline model which is certainly true even for the liquid mixtures near the freezing-point. Significant progress has been made in recent years in interpreting the concept of superlattices and the order-disorder phenomenon in alloys (Guggenheim, 1952). Thermodynamic interpretation of solidus and liquidus curves has been previously attempted (Seltz, 1934; Scatchard and Hamer, 1935) but not to a satisfactory degree. In the present paper it is intended to discuss solid-liquid equilibria in greater detail in the light of newer thermodynamic theories.

It is generally believed that for two electrolytes to form a solid solution, the following conditions must be satisfied :—

- (1) The formulae of the two components must be similar.
- (2) There must be similar unit cells containing exactly the same number of positive and negative ions in geometrically similar positions.
- (3) The size of the ions should be approximately the same.
- (4) The polarization of the ions must not be different.

Similarly for non-electrolytes, the conditions for the formation of solid solutions are :—

- (1) The formulae and structure of the components must be similar.
- (2) The crystal structure of the components should be similar.
- (3) The molar volumes should not differ appreciably.

In this paper we shall consider only solid solutions of non-electrolytes. Such solutions may be ideal, regular or non-ideal. The conditions for ideal or regular solutions have been predicted from statistical mechanics (Rushbrooke, 1949) according to which an ideal solution must satisfy the following microscopic conditions :—

- (i) The systems of the assembly can be regarded as independent systems, the partition function of any one of them being entirely unaffected by the composition of the solution.
- (ii) The systems in the condensed phase are of approximately equal size and shape and pack in the same way and have the same number of nearest neighbours.
- (iii) There are no preferential interactions between different systems within the condensed phase so that all the atoms move in the potential field which is not sensibly affected by the composition of the solution. Further the potential energy can be regarded as the sum of contributions from pairs of nearest neighbours.
- (iv) The vapour phase behaves as a mixture of perfect gases.
- (v) The volume of the condensed phase is negligible compared to that of vapour.

Non-ideal but regular solutions are those in which all the above conditions are satisfied except condition (iii). The gaseous mixtures at low pressures may be expected to behave ideally as the molecules are far apart. The liquid mixtures, in general, may or may not be ideal. The solid solutions in most of the cases cannot be expected to behave ideally since the molecules are usually very close and the interactions cannot be neglected. These solutions may approximate to regular solutions in a few cases.

2. EQUILIBRIUM CONDITIONS IN A MULTI-COMPONENT SYSTEM

We now proceed to deduce the conditions of solid-liquid equilibrium in a multi-component non-ideal system consisting of r components capable of existing in liquid as well as solid phase. Following the procedure adopted previously (Srivastava and Rastogi, 1953) it can be shown that

$$\sum_{i=1}^r N_i(\bar{S}'_i - \bar{S}_i) = \left\{ N_1 \sum_{i=1}^{r-1} \left(\frac{\partial \mu'_1}{\partial N'_i} \right)_{T, P, N'_j} dN'_i + N_2 \sum_{i=1}^{r-1} \left(\frac{\partial \mu'_2}{\partial N'_i} \right)_{T, P, N'_j} dN'_i + \dots \text{up to } r \text{ terms} \right\} \quad (1)$$

where \bar{S}'_i and \bar{S}_i are the partial molar entropy of the component i in the liquid and solid phase respectively. N_1, N_2, \dots are the mole-fractions in the solid phase while N'_1, N'_2, \dots are the mole-fractions in the liquid phase. μ_i represents the chemical potential of the component i . For condensed systems the pressure changes have insignificant effect and hence the term due to pressure changes does not occur in the above equations. Now, it can be easily seen

$$N_1(\bar{S}'_1 - \bar{S}_1) + N_2(\bar{S}'_2 - \bar{S}_2) + \dots \text{up to } r \text{ terms} = \sum_{i=1}^r \frac{N_i \lambda_i}{T} \cdot dT \quad (2)$$

where $\sum N_i \lambda_i$ is the heat of melting of one mole containing N_1, N_2, \dots mole-fractions of the respective components in the solid phase without sensibly altering the composition of the liquid phase. λ_i is the heat of melting of the component i and includes the partial molar heat function for the component i . Using equation (2), equation (1) yields

$$\sum_{i=1}^r \frac{N_i \lambda_i}{T} \cdot dT = \left\{ N_1 \sum_{i=1}^{r-1} \left(\frac{\partial \mu'_1}{\partial N'_i} \right)_{T, P, N'_j} dN'_i + N_2 \sum_{i=1}^{r-1} \left(\frac{\partial \mu'_2}{\partial N'_i} \right)_{T, P, N'_j} dN'_i + \dots \text{up to } r \text{ terms} \right\} \quad (3)$$

The above equation holds for any non-ideal multi-component system when the solid phase is in equilibrium with the liquid phase.

Similarly it is easy to show that

$$-\sum_{i=1}^r \frac{N'_i \lambda_i}{T} \cdot dT = \left\{ N'_1 \sum_{i=1}^{r-1} \left(\frac{\partial \mu_1}{\partial N_i} \right)_{T, P, N_j} dN_i + N'_2 \sum_{i=1}^{r-1} \left(\frac{\partial \mu_2}{\partial N_i} \right)_{T, P, N_j} dN_i + \dots \text{up to } r \text{ terms} \right\} \quad (4)$$

3. IDEAL SOLUTIONS

For ideal solutions the chemical potential is given by

$$\mu_i = \mu_i^0 + RT \log N_i$$

where μ_i^0 depends on temperature and pressure. Accordingly we have

$$\left(\frac{\partial \mu_i}{\partial N_i}\right)_{T,P} = \frac{RT}{N_i}$$

Three cases may arise. Firstly the liquid phase may be alone ideal, the solid phase remaining non-ideal. Secondly the solid phase may be alone ideal. Thirdly both the phases may be ideal. Taking the first case we have from equation (3)

$$\sum_{i=1}^r \frac{N_i \lambda_i}{T} \cdot dT = RT \left[\frac{N_1}{N'_1} dN'_1 + \frac{N_2}{N'_2} dN'_2 + \dots \frac{N_r}{N'_r} dN'_r \right]$$

which for a bicomponent system yields

$$\frac{N_1 \lambda_1^0 + N_2 \lambda_2^0 - H_M}{T} \cdot dT = RT \left(\frac{N_1}{N'_1} - \frac{N_2}{N'_2} \right) dN'_1 \quad \dots \quad (5)$$

where λ_1^0 and λ_2^0 are the heat of fusion of pure components and H_M is the heat of mixing in the solid phase. The heat of mixing will be zero if the solid phase is also ideal. If we put $N_1/N'_1 = \alpha$; $N_2/N'_2 = \beta$ where α and β are the separation ratios of the respective components, the maxima or minima in the melting-point composition curve will occur when $\alpha = \beta$.

From equation (5) a rule similar to that of Konowaloff for liquid-vapour system can be deduced for the present case. Thus, if dN'_1/dT is positive the first component will get enriched in the solid phase as the temperature is increased showing thereby that fractional crystallization is possible. Again if the solid phase exclusively consists of the solvent, i.e., component 1, we obtain

$$\frac{dN'_1}{dT} = \frac{N_1 \lambda_1^0}{RT^2} \quad \dots \quad (6)$$

which on integration yields the van't Hoff's law for the depression of freezing-point. On the other hand, if the solid phase is constituted by only the solute, equation (5) readily yields an expression relating solubility with temperature

Taking the second case into consideration (when solid phase is alone ideal), we have from equation (4)

$$- \sum_{i=1}^{i=r} \frac{N'_i \lambda_i}{T} \cdot dT = RT \left\{ \frac{N'_1}{N_1} dN_1 + \frac{N'_2}{N_2} dN_2 + \dots \frac{N'_r}{N_r} dN_r \right\} \quad \dots \quad (7)$$

which for a two-component system yields

$$\frac{dN_1}{dT} = \frac{-(N'_1 \lambda_1^0 + N'_2 \lambda_2^0 - H'_M)}{RT^2} \left/ \frac{N'_1}{N_1} - \frac{N'_2}{N_2} \right. \quad \dots \quad (8)$$

where H'_M is the heat of mixing for the liquid phase. Again a similar condition for stationary melting point can be deduced from equation (8). It also follows from equation (8) that if dN_1/dT is positive $\alpha > \beta$. The heat of mixing would be zero if the liquid phase is ideal.

From equations (5) and (8) it follows that if both the phases are ideal

$$\frac{dN_1}{dN'_1} = \frac{N_1 \lambda_1^0 + N'_2 \lambda_2^0}{N_1 \lambda_1^0 + N'_2 \lambda_2^0} \cdot \frac{N_1 N_2}{N'_1 N'_2} \quad \dots \quad (9)$$

On integrating the above equation we get the following relation between the composition of the two phases at any instant

$$\lambda_2^0 \log \frac{N_1}{N_1'} = \lambda_1^0 \log \frac{N_2}{N_2'} + \text{constant} \quad \dots \quad (10)$$

It follows that if $\log \frac{N_1}{N_1'}$ is plotted against $\log \frac{N_2}{N_2'}$, straight line would be obtained.

This can be a convenient test for testing the ideality of a binary solid solution.

Applying the above considerations to a simple binary eutectic system, the liquidus curve of the component 1 would be represented by equation (5) when $N_2 = 0$ while the liquidus curve for the component 2 will again be given by equation (5) when $N_1 = 0$. Thus the equation for the liquidus curve of component 1 is given by

$$\log N_1' = \frac{-\lambda_1^0}{RT} + \text{constant} \quad \dots \quad (11)$$

Similarly for the component 2 we have

$$\log N_2' = \frac{-\lambda_2^0}{RT} + \text{constant} \quad \dots \quad (12)$$

We suppose that T_1^0 and T_2^0 are respectively the freezing-points of the respective components. At the eutectic point the two curves meet. If N_1^e and T_e be the mole-fraction of the component 1 and the eutectic temperature, we have from equations (11) and (12)

$$\log N_1^e = \frac{-\lambda_1^0}{R} \left[\frac{1}{T_e} - \frac{1}{T_1^0} \right]; \quad \log (1-N_1^e) = \frac{-\lambda_2^0}{R} \left[\frac{1}{T_e} - \frac{1}{T_2^0} \right]$$

Eliminating T_e between the two equations, we have

$$\frac{\log N_1^e}{\lambda_1^0} - \frac{\log (1-N_1^e)}{\lambda_2^0} = \frac{1}{R} \left[\frac{1}{T_1^0} - \frac{1}{T_2^0} \right] \quad \dots \quad (13)$$

The above equation should hold at the eutectic point provided the mixture behaves ideally. On eliminating N_1^e we get the relation which gives the value of the eutectic temperature.

Let us examine the above relations for the system naphthalene-phenanthrene which appears to be ideal from considerations of solubility parameters as defined by Hildebrand (1950). The data for this system can be taken from International Critical Tables (1929). Taking the value of heat of fusion of naphthalene and phenanthrene as 19.07 and 18.1 kilojoules/gm. mol., the mole-fraction of naphthalene at the eutectic point is found to be 0.66 whereas the observed value is 0.62. The agreement is fairly satisfactory.

4. REGULAR MIXTURES

The ideal solutions are the simplest to deal with from the quantitative point of view, but they are only exceptions. After them regular solutions are the simplest to be amenable to theoretical analysis. The statistical theory predicts that any mixture of two kinds of molecules of similar size and shape should obey certain laws called the laws of strictly regular solution to which formula for regular solution

are a useful approximation. Hildebrand has experimentally demonstrated that a considerable number of binary mixtures follow exactly or approximately the laws of regular solutions. For deducing the laws of strictly regular solutions various procedures are adopted. The simplest of them assumes a completely random distribution of two kinds of molecules in spite of non-zero energy of mixing. This is called the zeroth approximation. The quasi-chemical treatment of 'strictly regular solution' (Guggenheim, 1935; Rushbrooke, 1938) is reasonably rigorous. The model used involves a quasi-lattice of co-ordination number z in which each molecule occupies a single lattice site. This situation is more nearly appropriate to solid solutions than to liquid solutions. The model would still be a reality for liquid solutions at the freezing point of the components. Various improvements have been made on this approximation. Bethe's method (Bethe, 1935; Rushbrooke, 1938) has been shown to be equivalent to quasi-chemical treatment. The essential basis of these approximations is 'the hypothesis of non-interference of pairs' which cannot hold in a closed-packed lattice and one must take into account triangular triplets or tetrahedral quadruplets instead of pairs. Expressions for partial pressures and chemical potentials have been deduced even for such a case but cannot be used in practice (Guggenheim, 1952). For practical purposes the zeroth approximation is much more helpful than the more refined approximations. Using zeroth approximation the chemical potential of the component i is given by

$$\begin{aligned}\mu_1 &= \mu_1^0 + RT \log N_1 + N_2^2 w \\ \mu_2 &= \mu_2^0 + RT \log N_2 + N_1^2 w \quad \dots \quad \dots \quad \dots \quad (14)\end{aligned}$$

where w , the interchange energy is defined as the decrease in potential energy on exchange of a single pair of molecules between the pure components. In terms of pair energies ϵ_{11} , ϵ_{12} and ϵ_{22}

$$w = \frac{1}{2}z(\epsilon_{12} - \epsilon_{11} - \epsilon_{22}) \quad \dots \quad \dots \quad \dots \quad (15)$$

It can be shown that if solid phase is regular

$$\left(\frac{\partial \mu_i}{\partial N_i}\right)_{T,P} = \frac{RT}{N_i} - 2(1 - N_i)w \quad (i = 1, 2) \quad \text{and} \quad H_M = N_1 N_2 w,$$

on the other hand, if liquid phase is regular

$$\left(\frac{\partial \mu'_i}{\partial N'_i}\right)_{T,P} = \frac{RT}{N'_i} - 2(1 - N'_i)w' \quad (i = 1, 2) \quad \text{and} \quad H'_M = N'_1 N'_2 w',$$

where w and w' are the interchange energies in the solid and liquid phases respectively.

We shall consider the following cases:

A. Liquid phase regular but solid phase ideal.

Substituting relevant values in equations (3) and (4) we have

$$\frac{dN'_1}{dT} = \frac{N_1 \lambda_1^0 + N_2 \lambda_2^0}{T} \bigg/ RT \left(\frac{N_1}{N'_1} - \frac{N_2}{N'_2} \right) + 2w' (N'_1 N_2 - N'_2 N_1) \quad \dots \quad (16)$$

and

$$- \frac{dN_1}{dT} = \frac{N'_1 \lambda_1^0 + N'_2 \lambda_2^0 - N'_1 N'_2 w'}{RT^2} \bigg/ \left(\frac{N'_1}{N_1} - \frac{N'_2}{N_2} \right) \quad \dots \quad (17)$$

B. *Solid phase regular but liquid phase ideal.*

For this case equations (3) and (4) yield

$$\frac{dN'_1}{dT} = \frac{N_1\lambda_1^0 + N_2\lambda_2^0 - N_1N_2w}{RT^2} \left/ \left(\frac{N_1}{N'_1} - \frac{N_2}{N'_2} \right) \right. \quad \dots \quad (18)$$

and

$$-\frac{dN_1}{dT} = \frac{N'_1\lambda_1^0 + N'_2\lambda_2^0}{RT^2} \left/ \left(\frac{N'_1}{N_1} - \frac{N'_2}{N_2} \right) \right. \quad \dots \quad (19)$$

C. *Both phases regular.*

For this case we have the following equation for the liquidus curve

$$\frac{dN'_1}{dT} = \frac{N_1\lambda_1^0 + N_2\lambda_2^0 - N_1N_2w}{T} \left/ RT \left(\frac{N_1}{N'_1} - \frac{N_2}{N'_2} \right) + 2w'(N'_1N_2 - N'_2N_1) \right. \quad \dots \quad (20)$$

The corresponding equation for solidus curve is given by

$$-\frac{dN_1}{dT} = \frac{N'_1\lambda_1^0 + N'_2\lambda_2^0 - N'_1N'_2w'}{T} \left/ RT \left(\frac{N'_1}{N_1} - \frac{N'_2}{N_2} \right) - 2w(N_2N'_1 - N_1N'_2) \right. \quad (21)$$

For cases A and B, the necessary and sufficient condition of stationary melting point is

$$N_1/N_2 = N'_1/N'_2 \quad \dots \quad (22)$$

which is identical with the condition found for ideal mixtures. But for C, the maxima or minima in the solidus or liquidus curve will occur when

$$\begin{aligned} N_1/N_2 &= N'_1/N'_2 \\ 2w \cdot N_1N_2 &= RT \\ \text{or } 2w'N'_1N'_2 &= RT \end{aligned}$$

The condition $N_1/N_2 = N'_1/N'_2$ simply means that the solidus and liquidus should touch at the maxima or minima. However, if they do not touch and still have a maxima or minima, the other condition has to be satisfied. This means that in an ideal mixture if maxima or minima occurs the two curves must touch each other. The above equation also tells us that for maxima or minima the minimum value of interchange energy that is permissible is $4T$. If for any system the value of interchange energy is less than this amount, only the first condition will determine the existence of the stationary melting point.

Considering an eutectic system, the liquidus or the freezing point curves for the two components are given by the following equations:—

Case A.

$$\frac{dN'_1}{dT} = \frac{N'_1\lambda_1^0}{RT^2 - 2wTN'_1N'_2}; \quad \frac{dN'_2}{dT} = \frac{N'_2\lambda_2^0}{RT^2 - 2w'TN'_1N'_2} \quad \dots \quad (23)$$

Case B.

$$\frac{dN'_1}{dT} = \frac{N'_1\lambda_1^0}{RT^2}; \quad \frac{dN'_2}{dT} = \frac{N'_2\lambda_2^0}{RT^2} \quad \dots \quad (24)$$

The relations are identical with those obtained for ideal mixtures,

For the Case C we obtain

$$\frac{dN'_1}{dT} = \frac{N'_1 \lambda_1^0}{RT^2 - 2w'TN'_1N'_2}; \quad \frac{dN'_2}{dT} = \frac{N'_2 \lambda_2^0}{RT^2 - 2w'TN'_1N'_2} \quad \dots \quad (25)$$

5. ATHERMAL MIXTURES—SOLUTIONS OF MACROMOLECULES

It has been suggested that the departures from the laws of ideal solutions due to wide differences in size between the two species of molecules can be at least semi-quantitatively described by means of relatively simple formulae in which the only parameter is the ratio of the molar volumes (Guggenheim, 1952; Hildebrand, 1950).

The Gibbs function of a mixture containing n_1 moles of component 1 and n_2 moles of component 2 can be written as

$$G = n_1 \mu_1^0 + n_2 \mu_2^0 + RT \left\{ n_1 \log \frac{n_1}{n_1 + rn_2} + n_2 \log \frac{rn_2}{n_1 + rn_2} \right\} \quad \dots \quad (26)$$

where r denotes the ratio of the volumes of a molecule of type 2 to that of a molecule of type 1. From the above equation we obtain,

$$\begin{aligned} \mu_1 &= \mu_1^0 + RT \left\{ \log \frac{N_1}{1 + (r-1)N_2} + \frac{(r-1)N_2}{1 + (r-1)N_2} \right\} \\ \mu_2 &= \mu_2^0 + RT \left\{ \log \frac{rN_2}{1 + (r-1)N_2} - \frac{(r-1)N_1}{1 + (r-1)N_2} \right\} \quad \dots \quad (27) \end{aligned}$$

These equations yield

$$\begin{aligned} \left(\frac{\partial \mu_1}{\partial N_1} \right)_{T, P} &= RT \left\{ \frac{1}{N_1} - \frac{(r-1)^2 N_1}{1 + (r-1)N_2} \right\} \\ \text{and} \quad \left(\frac{\partial \mu_2}{\partial N_2} \right)_{T, P} &= RT \left\{ \frac{1}{N_2} + \frac{(r-1)^2 N_1}{1 + (r-1)N_2} \right\} \quad \dots \quad (28) \end{aligned}$$

and similarly for the liquid phase.

Evidently the form of equations (26) and (27) implies that heat of mixing is zero. Using equations (3) and (4) and (28) we obtain

$$\frac{N_1 \lambda_1^0 + N_2 \lambda_2^0}{T} \cdot dT = RT \left\{ \left(\frac{N_1}{N'_1} - \frac{N_2}{N'_2} \right) - \frac{(r'-1)^2 (N_1 N'_2 + N'_1 N_2)}{1 + (r'-1)N'_2} \right\} dN'_1 \quad (29)$$

$$\frac{N'_1 \lambda_1^0 + N'_2 \lambda_2^0}{T} \cdot dT = RT \left\{ \left(\frac{N'_1}{N_1} - \frac{N'_2}{N_2} \right) - \frac{(r-1)^2 (N'_1 N_2 + N'_2 N_1)}{1 + (r-1)N_2} \right\} dN_1 \quad (30)$$

These formulae will hold if the heat of mixing is zero and non-ideality mainly results due to differences in size. These conditions are favourable to solutions of macromolecules.

The strictly regular solution model has some serious limitations since, in order to fit in the regular lattice, molecules of the different species must be of nearly the same size, although the method has been extended to include molecules occupying a series of lattice sites. Thus for athermal mixtures of high polymers

$$\mu_1 = \mu_1^0 + RT \left[\log \frac{N_1}{1 + (r-1)N_2} + \frac{1}{2} z \cdot \frac{r - (r-1)N_1}{q - (q-1)N_1} \right]$$

The number q for a single chain or branched chain r -mer is related to r by the following relation

$$\frac{1}{2}z(r-q) = r-1$$

This follows because we consider a particular r -mer occupying a group of r -sites, each of these sites having z neighbouring sites some occupied by other elements of the same r -mer. Then zq is the number of pairs of neighbouring sites of which one is a member of the group occupied by the given r -mer and the other is not. Flory's approximation is usually obtained by making $z \rightarrow \infty$ (Flory, 1941). These and other more rigorous formulae can be used for yielding equations of solidus and liquidus for a solution of high polymer.

6. SOLID-LIQUID EQUILIBRIUM DATA FOR CHLORINE-BROMINE SYSTEM

In this section it is intended to apply the ideas developed in previous sections to interpret the available equilibrium data (International Critical Tables, 1929) for the system chlorine-bromine and test equations (5), (8), (16), (17), (18), (19), (20) and (21). For an ideal or regular system it is necessary to choose the components in such a way that the size and shape of the molecules are approximately the same. Moreover, the solutions have to conform to other restrictive conditions which are realized in practice with great difficulty. The system chosen for investigation is relatively simple. The crystals of chlorine and bromine are weakly bound crystals where the attraction is mainly due to Van der Waals' forces. The Van der Waals' interaction between two non-dipole molecules does not affect in first approximation their interaction with a third molecule and consequently there is no tendency towards the formation of bigger groups containing several molecules. For the sake of comparison the molar volumes of chlorine and bromine in solid as well as liquid phases are given below.

		Molar volume in the solid phase	Molar volume in the liquid phase
Chlorine	37.37 ml.	45.3 ml.
Bromine	47.04 ml.	51.3 ml.

In the crystal of chlorine there are 16 chlorine atoms in a tetragonal unit at -185°C . The molecules of chlorine are packed in such a way that each chlorine has its molecular partner 1.82 \AA away and neighbours in adjacent molecules as near as 2.52 \AA (Keesom and Taconis, 1936). Crystalline bromine has also been shown to be possessing definitely a molecular structure, the Br_2 molecule being easily recognised in the crystalline structure from the observed interatomic distances. In crystalline bromine each atom has one nearest neighbour at a distance 2.27 \AA , the other member of bromine molecule (Vonnegut and Warren, 1936). This agrees with the band spectrum value of 2.28 \AA and the electron diffraction value of 2.28 \AA . Each atom has three other close neighbours in the same reflection plane (plane of paper) at distances $3.30, 3.30, 3.75 \text{ \AA}$. Eight more neighbours in the planes above and below are at slightly larger distances $4.00, 4.00, 3.98$ and 4.10 . Thus it can be seen that the structure of orthorhombic bromine is apparently different from the crystal structure of chlorine. It may also be noted that the intramolecular distance for chlorine is 1.82 \AA while that for bromine is 2.27 \AA . Nevertheless complete series of solid solutions are formed between chlorine and bromine. From considerations of molar volume and crystal structure, the mixture of chlorine and bromine cannot be expected to behave rigorously as an ideal or regular mixture. The interchange of molecules in a lattice can only cause strain. The problem of packing may not be difficult in the liquid phase and hence we may reasonably expect the liquid phase to behave approximately as regular mixture owing to the

fact that the molar volumes of the two species differ by only 12% in the liquid phase.

The liquidus and solidus curves for the system have been drawn by using the experimental data given in I.C.T. and are shown in Fig. 1. The temperatures are correct to $\pm 2^\circ\text{C}$. and the mole-fractions are correct to $\pm 4\%$.

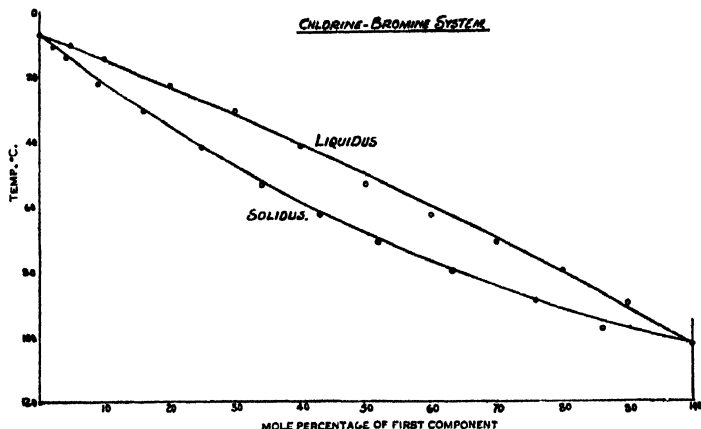


FIG. 1.

The quantities required for testing eqs. (5), (8), (16), (17), (18), (19), (20) and (21) can be easily found out. At any temperature a line drawn parallel to abscissa gives the composition of liquid and solid phase at equilibrium at the points of intersection of the liquidus and solidus curves, respectively. The latent heat of fusion are known at the freezing-point. The heat of fusion can be known at any other

temperature by using the formula $\frac{d\lambda_i}{dT} = C_L - C_S + \frac{\lambda_i}{T}$ where C_S and C_L are the specific heat of solid and liquid species respectively. Inserting these values dN_1/dT and dN'_1/dT can be calculated from equations (5) and (8) and compared with the experimental slope determined from Fig. 1. The theoretical and experimental values are shown in Table I. The values show wide divergence and hence the mixture cannot be ideal.

The equations for regular solutions involve one or two adjustable parameters, i.e., w and w' , the interchange energy in the solid and liquid phase respectively. For the present purpose we may assume them to be temperature independent. Now, in order to test the equations of regular solutions w and w' may be calculated at a series of temperatures by employing the values of experimental slopes of liquids and solidus curves. If the equations hold for the system under consideration the values of w and w' should come out to be the same. The actual calculations for the present system reveal that this is not the case and consequently the two phases cannot be simultaneously regular nor any phase can behave as an ideal one, since in that case the interchange energy would have been zero. These conclusions agree with those previously deduced on the basis of considerations of molar volume, crystal structure and intra-molecular distances. However, we may assume the liquid phase to behave ideally, the solid phase remaining non-ideal.

The heat of mixing for non-ideal solutions is empirically represented as follows (Scatchard *et al.*, 1952; Barker, Brown and Smith, 1953)

$$H_M = N_1 N_2 [A + B(N_1 - N_2) + C(N_1 - N_2)^2 + D(N_1 - N_2)^3 + \dots] \quad (31)$$

where A , B , C , D are constants which are determined empirically. We suppose that they are practically independent of temperature. Assuming the liquid phase

to be regular and heat of mixing to be given by equation (31), we have the following equation for the liquidus curve

$$\frac{dN'_1}{dT} = \frac{N_1\lambda_1^0 + N_2\lambda_2^0 - N_1N_2[A + B(N_1 - N_2) + C(N_1 - N_2)^2 + \dots]}{RT} \left/ \left(\frac{N_1}{N'_1} - \frac{N_2}{N'_2} \right) + \frac{2w'}{RT} (N'_1N_2 - N'_2N_1) \right. \quad (32)$$

Accordingly we use equation (32) to fit the data. The observed experimental values of dN'_1/dT and those calculated from equation (32) are compared in the following table which also gives the values calculated from equation (5) which holds for ideal solutions. The observed values are in satisfactory agreement with those calculated by equation (32).

TABLE I

$$w' = -1080 \text{ cal./gm. mol.}, H_M = N_1N_2[1236 + 13250(N_1 - N_2)^2]$$

N_1 (chlorine)	N'_1	$T^\circ \text{ K}$	λ_1^0 cal./gm. mol.	λ_2^0 cal./gm. mol.	$\left(\frac{dN'_1}{dT}\right)_{\text{obs.}}$	$\left(\frac{dN'_1}{dT}\right)_{\text{cal.}}$ from eq. (32)	$\left(\frac{dN'_1}{dT}\right)_{\text{cal.}}$ from eq. (5)
0.20	0.335	237	1,817	2,407	0.01000	0.00984	0.03363
0.30	0.46	225	1,782	2,355	0.01015	0.01015	0.03327
0.40	0.59	213	1,747	2,264	0.01054	0.01087	0.03047
0.50	0.70	203	1,718	2,204	0.01020	0.01000	0.02498
0.60	0.78	195	1,695	2,161	0.01020	0.01027	0.02357
0.76	0.90	183	1,660	2,085	0.00880	0.00620	0.01692

It is interesting to note that the heat of mixing in the solid phase is positive while that in the liquid phase is negative. The heat of mixing in the liquid phase can only be negative when $\epsilon_{12} > \epsilon_{11} + \epsilon_{22}$.

We take this opportunity to discuss the possibility of existence of bromine chloride in the condensed phase. Measurements of the transmission of chlorine-bromine light filters, containing an excess of liquid bromine demonstrate the existence of bromine chloride in mixtures of gaseous bromine and chlorine (Gray and Style, 1930). The freezing-point diagram of the system chlorine-bromine does not indicate the formation of stable bromine chloride molecule. However, it has been argued that it is 'compatible with the existence of a compound, if it is partly dissociated and in the solid forms a continuous series of solid solutions with both of its components'. (Sidgwick, 1933). It is obvious that if a stable type of molecule is formed in the condensed phase it would be a ternary system and the phase diagram would have been definitely complicated. Thus it seems that if at all bromine chloride is formed in the condensed phase it would be unstable as the complexes formed in a solution of alcohol and water are. Moreover, the formation of bromine chloride in the gaseous phase does not necessarily mean the formation in the condensed phase. The negative heat of mixing in the liquid phase does point to the fact that the interaction between unlike molecules is more prominent than between like molecules.

From the considerations of crystal structure, it can be conceived that since the size of chlorine molecules differ with bromine molecules by about 20% in the solid phase, the substitution of one molecule for the other would cause a considerable amount of strain or distortion in the crystal. This fact may be responsible for making the solid phase non-ideal. The amount of strain developed in the solid

solution of chlorine and bromine is probably insufficient to cause separation in two phases. Employing fairly arbitrary assumptions the strain energy in a binary solid solution of atoms of different sizes has been deduced (Lawson, 1947; Hildebrand and Scott, 1950). Using Scott's expression, i.e.,

$$\Delta E^D = \frac{2\delta_1\delta_2(V_1V_2)^{\frac{1}{2}}(V_1 - V_2)^2}{(N_1V_1^2 + N_2V_2^2)} N_1N_2 \dots \dots \dots (33)$$

where δ_1 and δ_2 are the solubility parameters of the two species and V_1 and V_2 are the molar volumes, ΔE^D , the strain energy for a equi-molar mixture of chlorine and bromine is found to be equal to 137 cal. which does not seem to be unreasonable.

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SUMMARY

Thermodynamics of solid-liquid equilibria has been investigated. Equations of solidus and liquidus curves have been deduced for ideal, regular, athermal and non-ideal mixtures including solutions of high polymers. The conditions for stationary melting-point have also been deduced. Binary eutectic system has been discussed. The eutectic composition for naphthalene-phenanthrene system predicted from theory is in satisfactory accord with the observed value. The solid-liquid equilibrium data of chlorine-bromine system has been examined. It is found that the system is neither ideal nor regular but the equations assuming the liquid phase to be regular and solid phase to be non-ideal fit the data. Heat of mixing for the liquid phase is found to be negative while it is positive for the solid phase. The formation of bromine chloride molecules in the condensed phase is discussed. The interaction between chlorine and bromine molecules is more prominent than the interaction between the like molecules in the liquid phase, but it seems that the stable compound is not formed. The strain energy in the solid solution has been calculated and appears to be of the correct order of magnitude.

REFERENCES

- Barker, J. A., Brown, I., and Smith, F. (1953). Thermodynamics of alcohol solutions, the system ethanol and carbon tetrachloride. *Faraday Society Discussion on Equilibrium properties of solutions*, 142.
- Bethe, H. A. (1935). Statistical theory of superlattices. *Proc. Roy. Soc.*, **150**, 552.
- Flory, P. J. (1941). Thermodynamics of high polymer solutions. *J. Chem. Phys.*, **9**, 660.
- Gray, L. T. M., and Style, D. W. G. (1930). Absorption of light by chlorine, bromine and their gaseous mixtures. *Proc. Roy. Soc.*, **126**, 603.
- Guggenheim, E. A. (1935). The Statistical Mechanics of regular solutions. *Ibid.*, **148**, 304.
- (1952). *Mixtures*, Clarendon Press, Oxford.
- Hildebrand, J. H., and Scott, R. L. (1950). *Solubility of non-electrolytes*, Reinhold Publishing Corporation, New York.
- Keesom, W. H., and Taconis, K. W. (1936). Crystal Structure of chlorine. *Physica*, **3**, 237.
- (1936). *Proc. K. Acad. Wetensch. Amsterdam*, **39**, 314.
- Lawson, A. W. (1947). Binary solid solutions. *J. Chem. Phys.*, **15**, 831.
- Rushbrooke, G. S. (1949). *Statistical Mechanics*, Oxford University Press.
- (1938). A note on Guggenheim's theory of strictly regular binary liquid mixtures. *Proc. Roy. Soc.*, **166**, 296.
- Scatchard, G., and Hamer, W. J. (1935). The application of equations for the chemical potential to equilibrium between solid solution and liquid solution. *J. Amer. Chem. Soc.*, **57**, 1809.
- Scatchard, G., Ticknor, L. B., Groats, J. R., and McCartney, E. R. (1952). Heats of mixing in some non-electrolyte solutions. *J. Amer. Chem. Soc.*, **74**, 3721.
- Seltz, H. (1934). Thermodynamics of solid solutions. I. Perfect Solutions. *J. Amer. Chem. Soc.*, **56**, 307.
- Sidgwick, N. V. (1933). Interhalogen Compounds. *Chem. Soc. Ann. Reports*.
- Srivastava, B. N., and Rastogi, R. P. (1953). Thermodynamics of systems of any number of components. *Proc. Nat. Inst. Sci. India*, **19**, 653.
- Vonnegut, B., and Warren, B. E. (1936). Structure of crystalline bromine. *J. Amer. Chem. Soc.*, **58**, 2461.

FLOW OF A COMPRESSIBLE FLUID AROUND A CORNER

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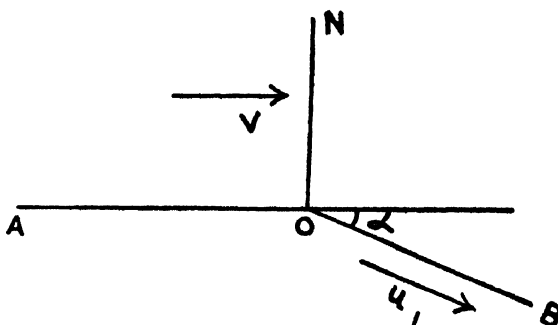
1. INTRODUCTION

Prandtl's problem of the expansion of a uniform two-dimensional stream of gas flowing around a corner at a supersonic speed had been studied (Durand, 1935) under adiabatic conditions and it had been shown that the velocity perpendicular to the radius vector from the corner is always equal to the speed of sound at the local conditions of pressure and density. Bernoulli's equation had been used to find the components of velocity in terms of the angle of deflection and also to find the relation between the pressure and the same angle.

In this paper the same problem has been studied starting with non-adiabatic conditions, conductivity and viscosity taken into account but it is found that the equations lead to adiabatic conditions and that the transverse velocity is equal to the velocity of sound as before. The equations can be straight integrated giving the velocity in terms of deflection. Expressions for the components of velocity and for pressure are analytically different from those obtained by the other method and are of some interest.

2. EQUATIONS OF MOTION

A uniform stream of gas flows parallel to a rigid boundary AO . At O the boundary makes an angle α with AO produced and then takes a straight course OB . It is proposed to investigate the flow around the corner O in the angle between ON and OB where ON is perpendicular to AO at O .



Take O as origin, (u, v) be components of velocity along and perpendicular to the radius vector from O .

Components of strain are

$$2 \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{2}{r} \frac{\partial v}{\partial \theta} + \frac{2u}{r}, \quad e_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}. \quad (1)$$

The divergence is

$$\Delta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta}. \quad (2)$$

The stress components are

$$\left. \begin{aligned} p_{rr} &= -p + \lambda \Delta + \mu e_{rr}, & p_{\theta\theta} &= -p + \lambda \Delta + \mu e_{\theta\theta} \\ \text{and} & & p_{r\theta} &= \mu e_{r\theta} \end{aligned} \right\} \quad \dots \quad (3)$$

where p is the pressure.

The equations of motions are

$$\left. \begin{aligned} \rho \left(u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} \right) &= \frac{\partial}{\partial r} p_{rr} + \frac{1}{r} \frac{\partial}{\partial \theta} p_{r\theta} + \frac{1}{r} (p_{rr} - p_{\theta\theta}), \\ \text{and } \rho \left(u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{uv}{r} \right) &= \frac{\partial}{\partial r} p_{r\theta} + \frac{1}{r} \frac{\partial}{\partial \theta} p_{\theta\theta} - \frac{2}{r} p_{r\theta}, \end{aligned} \right\} \quad \dots \quad (4)$$

where ρ is the density.

The equation of continuity is

$$\frac{\partial}{\partial r} (r\rho u) + \frac{\partial}{\partial \theta} (\rho v) = 0. \quad \dots \quad (5)$$

The equation of energy is

$$\begin{aligned} &\rho \left(u \frac{\partial i}{\partial r} + \frac{v}{r} \frac{\partial i}{\partial \theta} \right) - \left(u \frac{\partial p}{\partial r} + \frac{v}{r} \frac{\partial p}{\partial \theta} \right) \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\mu}{\sigma} \cdot r \frac{\partial i}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{\mu}{\sigma} \cdot \frac{\partial i}{\partial \theta} \right) + \Phi \quad \dots \quad (6) \end{aligned}$$

where i , called the enthalpy, is the heat content, σ the Prandtl number, and Φ , dissipation function given by

$$4\Phi = \lambda(e_{rr} + e_{\theta\theta})^2 + 2\mu(e_{rr}^2 + e_{\theta\theta}^2 + 2e_{r\theta}^2). \quad \dots \quad (7)$$

Also the equation of state for a perfect gas is

$$i\rho = \frac{\gamma}{\gamma-1} p \quad \dots \quad (8)$$

where γ is the ratio of specific heats at constant pressure and at constant volume.

We further assume that μ varies as some power of absolute temperature, and therefore as i^n ,

$$\text{i.e. } \frac{\mu}{i^n} = \text{const.} \quad \dots \quad (9)$$

where n is usually positive and less than unity.

These are all the equations. To solve these we make some simplifications.

3. SIMPLIFICATIONS OF THE EQUATIONS

In the present problem we assume that the velocity, density and pressure are constant along a radius, hence the above equations reduce to the following:—

$$\begin{aligned} \frac{\rho v}{r} \left(\frac{du}{d\theta} - v \right) &= -\frac{\lambda + 2\mu}{r^2} \left(u + \frac{dv}{d\theta} \right) + \frac{1}{r^2} \frac{d}{d\theta} \left\{ \mu \left(\frac{du}{d\theta} - v \right) \right\}, \\ \frac{\rho v}{r} \left(\frac{dv}{d\theta} + u \right) &= -\frac{3\mu}{r^2} \left(\frac{du}{d\theta} - v \right) - \frac{1}{r} \frac{dp}{d\theta} + \frac{1}{r^2} \frac{d}{d\theta} \left\{ (\lambda + 2\mu) \left(u + \frac{dv}{d\theta} \right) \right\} \end{aligned} \quad (4a)$$

$$\rho u + \frac{d}{d\theta}(\rho v) = 0, \quad \dots \dots \dots (5a)$$

$$\Phi = \frac{\lambda + 2\mu}{r^2} \left(u + \frac{dv}{d\theta} \right)^2 + \frac{\mu}{r^2} \left(\frac{du}{d\theta} - v \right)^2, \quad \dots \dots \dots (7a)$$

$$\frac{v}{r} \left(\rho \frac{di}{d\theta} - \frac{dp}{d\theta} \right) = \frac{1}{r^2} \frac{d}{d\theta} \left(\frac{\mu}{\sigma} \frac{di}{d\theta} \right) + \Phi. \quad \dots \dots \dots (6a)$$

Again the coefficients of r in various equations must all be zero.
Therefore we get

$$\frac{du}{d\theta} - v = 0, \quad \dots \dots \dots (10)$$

$$\rho v \left(u + \frac{dv}{d\theta} \right) = - \frac{dp}{d\theta}, \quad \dots \dots \dots (11)$$

$$\lambda + 2\mu = 0,$$

$$\Phi = 0,$$

$$\frac{d}{d\theta} \left(\frac{\mu}{\sigma} \frac{di}{d\theta} \right) = 0, \quad \dots \dots \dots (12)$$

$$\rho \frac{di}{d\theta} = \frac{dp}{d\theta}. \quad \dots \dots \dots (13)$$

It is to be noted that the minimum value of $\lambda + 2\mu$ is also zero.

4. SOLUTIONS OF THE EQUATIONS

From (12), taking σ as constant, we have with the help of (9),

$$i^n \frac{di}{d\theta} = -c, \quad \dots \dots \dots (14)$$

where c is a positive constant, so that from (13),

$$\frac{dp}{d\theta} = - \frac{c\rho}{i^n}. \quad \dots \dots \dots (15)$$

From (8), we have by differentiation with respect to θ ,

$$i \frac{d\rho}{d\theta} + \rho \frac{di}{d\theta} = \frac{\gamma}{\gamma-1} \frac{dp}{d\theta} = - \frac{\gamma}{\gamma-1} \frac{c\rho}{i^n}, \quad \dots \dots \dots \text{from (15)}$$

$$\text{or} \quad i^{n+1} \frac{d\rho}{d\theta} + \rho i^n \frac{di}{d\theta} = - \frac{\gamma}{\gamma-1} c\rho$$

$$\text{or} \quad i^{n+1} \frac{d\rho}{d\theta} = - \frac{1}{\gamma-1} c\rho, \quad \dots \dots \dots \text{from (14)}$$

$$\text{or} \quad (\gamma-1) \frac{d\rho}{\rho} = - \frac{cd\theta}{i^{n+1}} = \frac{di}{i}, \quad \dots \dots \dots \text{from (14)}$$

Hence

$$i = \alpha \rho^{\gamma-1} \text{ where } \alpha \text{ is a constant,}$$

so that

$$\frac{\gamma}{\gamma-1} p = i \rho = \alpha \rho^{\gamma}$$

or

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0} \right)^{\gamma}, \quad \dots \quad \dots \quad \dots \quad \dots \quad (16)$$

where suffix 0 denotes value when at rest.

This is the adiabatic law.

From equations (11) and (15), we have

$$\rho v \left(u + \frac{dv}{d\theta} \right) = \frac{c\rho}{i^n}, \quad \dots \quad \dots \quad \dots \quad \dots \quad (17)$$

which with the help of (5a) gives

$$v \left\{ -\frac{d}{d\theta}(v\rho) + \rho \frac{dv}{d\theta} \right\} = \frac{c\rho}{i^n}$$

$$\text{or} \quad v^2 \frac{d\rho}{d\theta} = -\frac{c\rho}{i^n}.$$

But we have already seen that

$$\frac{d\rho}{d\theta} = -\frac{c\rho}{(\gamma-1)i^{n+1}},$$

hence

$$v^2 = (\gamma-1)i = \gamma \frac{p}{\rho} = a^2, \quad \dots \quad \dots \quad \dots \quad \dots \quad (18)$$

where a is the local velocity of sound.

Again from (17), with the help of (14), we get

$$v \left(u + \frac{dv}{d\theta} \right) = -\frac{di}{d\theta}$$

or

$$u \frac{du}{d\theta} + v \frac{dv}{d\theta} = -\frac{di}{d\theta},$$

giving,

$$u^2 + v^2 = c_1 - 2i, \quad \dots \quad \dots \quad \dots \quad \dots \quad (19)$$

where c_1 is a constant.

Also from (14) we have

$$i^{n+1} = i_0^{n+1} - (n+1)c\theta$$

therefore (19) gives

$$q^2 = u^2 + v^2 = c_1 - 2i_0 \left\{ 1 - \frac{(n+1)c\theta}{i_0^{n+1}} \right\}^{\frac{1}{n+1}}. \quad \dots \quad \dots \quad (20)$$

Measuring θ from the line ON at which

$$u = 0, v = V$$

we have

$$V^2 = c_1 - 2i_0$$

so that

$$c_1 = V^2 + 2i_0 \quad \dots \quad \dots \quad \dots \quad \dots \quad (21)$$

where

$$i_0 = \frac{\gamma}{\gamma-1} \frac{p_0}{\rho_0}.$$

The flow will be purely radial and uniform along the boundary OB if when

$$\theta = \frac{\pi}{2} + \alpha, v = 0, u = u_1, \text{ say}$$

then
$$u_1^2 = V^2 + 2i_0 - 2i_0 \left\{ 1 - \frac{(n+1)c\left(\frac{\pi}{2} + \alpha\right)}{i_0^{n+1}} \right\}^{\frac{1}{n+1}} \quad \dots \quad (22)$$

Also
$$u^2 = q^2 - v^2 = V^2 - v^2 + 2i_0 - 2i_0 \left\{ 1 - \frac{(n+1)c\theta}{i_0^{n+1}} \right\}^{\frac{1}{n+1}}$$

$$= V^2 - \gamma \frac{p}{\rho} + \frac{2\gamma}{\gamma-1} \frac{p_0}{\rho_0} \left[1 - \left\{ 1 - \frac{(n+1)c\theta}{i_0^{n+1}} \right\}^{\frac{1}{n+1}} \right]. \quad \dots \quad (23)$$

Evidently $p = 0$ on the boundary OB . OB may be regarded as the free surface of the stream.

In the particular case when μ is constant, so that $n = 0$, we get

$$\left. \begin{aligned} q^2 &= V^2 + 2c\theta \\ u_1^2 &= V^2 + 2c\left(\frac{\pi}{2} + \alpha\right) \\ \therefore q^2 &= V^2 + \frac{u_1^2 - V^2}{\frac{\pi}{2} + \alpha} \cdot \theta \\ u^2 &= V^2 - \gamma \frac{p}{\rho} + \frac{u_1^2 - V^2}{\frac{\pi}{2} + \alpha} \cdot \theta \end{aligned} \right\} \quad \dots \quad (24)$$

and

In this particular case, we have from (15)

$$\frac{dp}{d\theta} = -c\rho$$

$$= -c\rho_0 \left(\frac{p}{p_0} \right)^{\frac{1}{\gamma}}, \quad \dots \quad \dots \quad \dots \quad \text{from (16)}$$

which gives on integration

$$\left(\frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}} = \frac{c\rho_0}{p_0} \cdot \frac{\gamma-1}{\gamma} \left(\frac{\pi}{2} + \alpha - \theta \right), \quad \dots \quad \dots \quad (25)$$

as $p = 0$ when $\theta = \frac{\pi}{2} + \alpha$, i.e. on OB .

This gives the relation between the pressure and θ .

Thus when

$$\theta = 0,$$

$$\begin{aligned} \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} &= \frac{c\rho_0}{p_0} \frac{\gamma-1}{\gamma} \left(\frac{\pi}{2} + \alpha\right) \\ &= \frac{\gamma-1}{2\gamma} \frac{\rho_0}{p_0} (u_1^2 - V^2). \end{aligned}$$

This may be taken as the pressure of the uniform stream parallel to AO .

REFERENCE

Durand, W. F. (1935). Aerodynamic Theory, 3, 243.

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FRAGMENTATION OF GLASS PLATES

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This paper is concerned with the random fragmentation of glass plates. The purpose is to discover the basic features of such a fragmentation process. The mathematical theory of the random fragmentation (division) of a line into a finite number of N parts has been discussed by several authors (Auluck and Kothari, 1954). It has its application in assessing the randomness of radioactive disintegrations and cosmic ray events. The average number $N(x)$ of fragments equal to or greater than x is given by (Feller, 1940)

$$N(x) = N \left(1 - \frac{x}{l}\right)^{N-1} \quad \dots \quad (1)$$

It is difficult to discuss the general problem of the fragmentation of an area, and, so far as known to the author, no such discussion has been given. However, it is easy to treat an idealised case of the fragmentation of a rectangle into smaller rectangles. Let us consider a rectangle of area $\Sigma = l_1 l_2$ and suppose it to be fragmented into $N_0 = N_1 N_2$ rectangles by placing at random N_1 lines parallel to the length of the rectangle and N_2 lines parallel to the breadth of the rectangle. The number of fragments of area equal to or greater than S is, using (1), given by (Auluck and Kothari, 1954)

$$N(S) = \frac{N_1 N_2 (N_1 - 1) (N_2 - 1)}{l_1 l_2} \iint \left(1 - \frac{x}{l_1}\right)^{N_1-2} \left(1 - \frac{y}{l_2}\right)^{N_2-2} dx dy \quad xy > S$$

or approximately

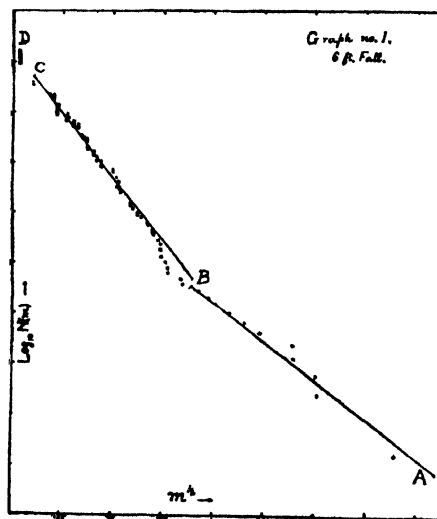
$$N(S) \sim -2N_0 \left(\frac{S}{S_0}\right)^{\frac{1}{2}} K_1 \left[2 \left(\frac{S}{S_0}\right)^{\frac{1}{2}}\right] \quad \dots \quad (2)$$

where $K_1(Z)$ is the Bessel function of imaginary argument, $N_0 = N_1 N_2$ is the total number of fragments, and S_0 is the average area of a fragment, $S_0 = \frac{\Sigma}{N_0}$.

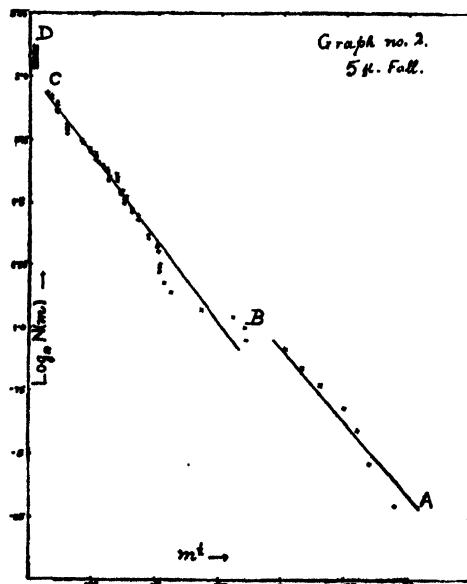
In view of the idealised conditions under which (2) has been derived it is not expected that it would describe exactly an actual area-fragmentation process, but all the same it is likely to illustrate the essential features of this process. This is borne by the experimental results described below.

A number of experiments were performed on fragmentation of glass plates of sheet glass. Experiments were done with plates of varying sizes. The plates were held horizontally between the fingers at a given height from an even hard concrete surface (the laboratory floor) and then released. These fragmented on hitting the floor. Fragments of each plate were collected separately and then weighed to an accuracy of about 10^{-3} gms. Particles below a mass of 5×10^{-3} gms. were lumped together and weighed, and their average mass was obtained. Particles below about 10^{-4} gms. were ignored. Data for some of the plates are given overleaf:

- Group—A. Sheet glass plate 8.3 cm. \times 8.3 cm. \times 0.2 cm.
 Mass 25 gms. approximately.
 Heights fallen through : 10 ft., 5 ft. and 2 ft.
- Group—B. Sheet glass plate 16.6 cm. \times 8.3 cm. \times 0.2 cm.
 Mass 50 gms. approximately.
 Height fallen through : 6 ft.



Graph No. 1 showing the relation between $\log_{10} N(m)$ and $m^{1/2}$. $N(m)$ is the number of fragments of mass equal to or greater than m . m is the mass of the fragments in grams. Height of fall = 6 ft.



Graph No. 2 showing the relation between $\log_{10} N(m)$ and $m^{1/2}$. $N(m)$ is the number of fragments of mass equal to or greater than m . m is the mass of the fragments in grams. Height of fall = 5 ft.

For fragments of each plate a graph between $\log_{10} N(m)$ and $m^{\frac{1}{2}}$ was plotted, where $N(m)$ is the number of fragments of mass equal to or greater than m . Assuming that the surface density of the plate is uniform, the relation between $N(m)$ and m will be the same as between $N(m)$ and S . In all cases irrespective of the size of the plate or the height through which it fell the graphs obtained were always of the same pattern, and it was apparent from the graphs (plot of $\log_{10} N(m)$ against $m^{\frac{1}{2}}$, m in gm.) that these fragments could be classified in three groups:—(i) the big ones, (ii) the small ones, and (iii) minute ones. Two typical groups are appended with the paper. The group (i) comprises of roughly ten fragments, and leaving aside the biggest, the others lie nearly on a straight line given by

$$\log_{10} N(m) = 0.75 m^{\frac{1}{2}} + 1.75 \quad \dots \quad (3)$$

The number in this group increases with the height of fall.

The group (ii) comprises particles which lie on another straight line: line BC in the figure. Almost all the smaller fragments lie on this part of the graph, i.e. between B and C . This part of the straight line may be represented roughly by the equation

$$\log_{10} N(m) = m^{\frac{1}{2}} + 2 \quad \dots \quad (4)$$

In this group also the number of fragments increases with the height of fall.

We now come to the portion CD in the graph. This constitutes our group (iii), i.e. minute fragments. The graph is here almost parallel to the $\log_{10} N(m)$ axis. Partly this is due to our lumping together fragments of mass smaller than 5×10^{-3} gms. Also in this region secondary fragments, i.e. fragments produced by fragmentation of fragments already once formed, may play a part.

In the following table we summarise some of the results. 1st column gives the description of the plate. 2nd, 3rd and 4th columns give the number of fragments under groups (i), (ii) and (iii). The remaining two columns give the slopes of the lines AB , corresponding to (i) and BC corresponding to group (ii).

TABLE

Description of the plate. cm. ³				Ht. of fall, in feet.	Total number of fragments.			Slope of line for	
					Group (i)	Group (ii)	Group (iii)	Group (i)	Group (ii)
1.	8.3 × 8.3 × 0.2	2	4	38	29	0.50	1.04
2.	do.	2	6	12	27	0.26	0.90
3.	do.	5	8	81	45	1.14	1.35
4.	do.	5	7	44	46	0.92	1.00
5.	do.	6	9	80	45	0.86	0.93
6.	do.	10	8	72	107	0.77	1.03
7.	16.6 × 8.3 × 0.2	6	13	129	58	0.72	1.25
8.	do.	6	7	127	26	0.77	1.35
9.	do.	6	11	145	75	0.68	1.72

The fact that the particles in each of the groups (i) and (ii) lie on straight lines (the lines are plots of $\log_{10} N(m)$ against $m^{\frac{1}{2}}$) shows that the fragmentation under each group is broadly in accordance with this theory, i.e. equation 2. It is, however, not clear in the present investigation why the groups (i) and (ii) do not lie on the same straight line. This would obviously need a closer study of this problem and it is proposed to take it up subsequently.

I am grateful to Prof. D. S. Kothari and Dr. F. C. Auluck for their guidance and to Principal Kapur for his kind interest and facilities given to me.

SUMMARY

This paper is concerned with the 'Random fragmentation of glass plates'. A number of experiments were performed with plates of varying sizes. The results are broadly in accordance with the theory of fragmentation given by Dr. F. C. Auluck and Dr. D. S. Kothari.

REFERENCES

- Auluck, F. C., Kothari, D. S. (1954). Random Fragmentation. *Nature*, **174**, 565-6.
Goudsmit, S. (1945). Random distribution of lines in a plane. *Revs. Mod. Phys.*, **17**, 321-2.
Feller, W. (1940). On the time distribution of so-called random events. *Phys. Rev.*, **47**, 906-8.

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EFFECT OF THE VAN DER WAALS' CORRECTIONS ON THE TRANSITION TEMPERATURE OF He³-He⁴ MIXTURES

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INTRODUCTION

The effect of the addition of liquid He³ on the transition temperature of liquid He⁴ has been studied in great detail in recent years. In order to investigate this change in the transition temperature Heer and Daunt (1951) treated liquid He⁴ and liquid He³, respectively, as perfect Bose-Einstein and Fermi-Dirac gases in smoothed potential wells. Assuming the liquids to form statistically independent assemblies in the mixture they showed that the transition temperatures could be predicted with fairly good accuracy. However, the agreement between theory and experiment as reported by them is, more or less, qualitative. It is also well-known that such perfect gas models do not very well represent the actual situation in the pure liquids. Recently, Ziro Mikura (1954) has modified Heer and Daunt's model by introducing a gap in the lowest portion of the one-particle-energy spectrum. But the transition temperatures obtained on the basis of this 'modified Bose-Einstein theory' are too low, for mixtures with He³ concentration greater than about 50 per cent. In this paper we shall show that the transition temperatures can be predicted with somewhat better accuracy when gas 'imperfections' are taken into account.

The imperfections are introduced in a qualitative way as corrections to the perfect gas model so that the liquids may formally be regarded as systems of 'quasi-free particles' obeying Bose and Fermi statistics. Thus, we assume that the total volume accessible to a 'particle' is somewhat less than the actual volume of the liquid (Van der Waals' correction). Next, we attribute to each particle an effective mass somewhat greater than the corresponding atomic mass and introduce in the free energy, terms arising from Lennard-Jones Devonshire type interactions. We shall not discuss here the details of this model. The aim of this paper is to bring out the importance of the Van der Waals' correction so far as the transition temperatures are concerned.

PURE LIQUID HE⁴

Applying the model to liquid He⁴ we write for the free energy the expression (with $y = \epsilon/kT$),

$$F_4^0 = -kT \left[-N_4 \ln A_4^0 - N_4 \ln(1 - A_4^0) - \frac{2\pi(2m_4^* kT)^{\frac{3}{2}}}{h^3} V_{4s} \int_0^\infty \ln\{1 - A_4^0 \exp(-y)\} \times \right. \\ \left. \times y^{\frac{1}{2}} dy \right] - \frac{a_4 N_4^2}{V_4^2} + \frac{b_4 N_4^5}{V_4^4}, \quad \dots \dots \dots (1)$$

where m_4^* is the effective mass, V_{4s} the accessible and V_4 the actual volume and a_4 , b_4 are constants to be determined by comparison with observations. Again,

A_4^0 is to be determined by the general condition that the total number of particles is fixed. The conservation of the total number of particles requires that

$$\left(\frac{\partial F_4^0}{\partial A_4^0} \right)_{T, V_4, V_{4s}} = 0, \quad \dots \dots \dots (2)$$

and this equation will evaluate A_4^0 . Next, we introduce the parameter, x , for the fraction of non-condensed particles, i.e. $x = N_4^*/N_4$, and put with London (1938)

$$A_4^0 = 1 - \frac{1}{(1-x)N_4} \quad \dots \dots \dots (3)$$

From eqs. (1), (2) and (3) we obtain

$$x N_4 = \left(\frac{2\pi m_4^* k T}{h^2} \right)^{\frac{3}{2}} V_{4s} \zeta\left(\frac{3}{2}\right) \quad \dots \dots \dots (4)$$

The transition temperature, T_λ^0 is defined by the fact that almost all the particles are non-condensed, i.e. $x = 1$. Hence eq. (4) gives:

$$T_\lambda^0 = \frac{h^2}{2\pi m_4^* k} \left(\frac{N_4}{V_{4s} \zeta\left(\frac{3}{2}\right)} \right)^{\frac{2}{3}} \quad \dots \dots \dots (5)$$

Next, using for the molar volume of liquid He⁴ the value 27.6 c.c. and recalling that the Van der Waals' correction term is 19.2 c.c. per mole we obtain for the accessible volume the value 8.4 c.c. per mole. Comparing eq. (5) with the observed value 2.186° K., of the transition temperature we get

$$m_4^* \approx 3.2 m_4.$$

We shall not, however, discuss the nature of the effective mass in detail because as will be seen below, the transition temperature of the mixture (relative to T_λ^0) is independent of the choice of m_4^* .

Mixtures of He⁴ and He³ and their transition temperatures

In the mixture, interactions between particles of one liquid and those of the other are neglected and therefore we regard the liquids as forming statistically independent systems. Evidently, the mixing process will not affect the nature of the interactions. The actual and accessible volumes of the mixture are written, as for ideal mixtures, in the form

$$V = V_4 + V_3 = N_4 V_4^0 + N_3 V_3^0, \\ V_s = V_{4s} + V_{3s} = N_4 V_{4s}^0 + N_3 V_{3s}^0,$$

respectively. Here V_4^0 , V_3^0 are the actual and V_{4s}^0 , V_{3s}^0 the accessible volumes per particle of liquid He⁴ and liquid He³, respectively.

According to the present model the free energy of liquid He³ is written as

$$F_3^0 = -kT \left[-N_3 \ln A_3^0 + \frac{4\pi(2m_3^*kT)^{\frac{3}{2}}}{h^3} V_{3e} \int_0^\infty \ln \{ 1 + A_3^0 \exp(-y) \} y^{\frac{1}{2}} dy \right] - \frac{a_3 N_3^3}{V_3^2} + \frac{b_3 N_3^5}{V_3^4}, \quad \dots \dots \dots (6)$$

whence we obtain for the free energy of the mixture

$$\begin{aligned} F &= F_3^0 + F_4^0 + \Delta F \\ &= -kT \left[-N_4 \ln A_4 - N_4 \ln(1 - A_4) - \frac{2\pi(2m_4^*kT)^{\frac{3}{2}}}{h^3} V_{4e} \int_0^\infty \ln \{ 1 - A_4 \exp(-y) \} y^{\frac{1}{2}} dy \right] - \\ &\quad -kT \left[-N_3 \ln A_3 + \frac{4\pi(2m_3^*kT)^{\frac{3}{2}}}{h^3} V_{3e} \int_0^\infty \ln \{ 1 + A_3 \exp(-y) \} y^{\frac{1}{2}} dy \right] - \\ &\quad \left(a_4 N_4^3 + \frac{a_3 N_3^3}{V^2} \right) + \left(\frac{b_4 N_4^5}{V^4} + \frac{b_3 N_3^5}{V^4} \right). \quad \dots \dots \dots (7) \end{aligned}$$

A_4 and A_3 being determined by the conservation conditions:

$$\begin{aligned} \left(\frac{\partial F}{\partial A_4} \right)_{T, A_3, V_4, V_{4e}, N_4} &= 0, \\ \left(\frac{\partial F}{\partial A_3} \right)_{T, A_4, V_3, V_{3e}, N_3} &= 0. \end{aligned}$$

Then, proceeding as before we obtain for the transition temperature, T_λ , of liquid He⁴ in the mixture:

$$T_\lambda = \frac{h^2}{2\pi m_4^* k} \left(\frac{N_4}{V_{4e} \zeta(\frac{3}{2})} \right)^{\frac{2}{3}}, \quad \dots \dots \dots (8)$$

which on combination with eq. (5) gives:

$$\frac{T_\lambda}{T_\lambda^0} = \left(\frac{N_4 V_{4e}^0}{N_4 V_{4e}^0 + N_3 V_{3e}^0} \right)^{\frac{2}{3}}, \quad \dots \dots \dots (9)$$

which is independent of m_4^* . This expression for T_λ/T_λ^0 is to be compared with that obtained by Heer and Daunt, namely:

$$\frac{T_\lambda}{T_\lambda^0} = \left(\frac{N_4 V_4^0}{N_4 V_4^0 + N_3 V_3^0} \right)^{\frac{2}{3}} \quad \dots \dots \dots (9')$$

Eqs. (9) and (9') have been used to compute the values of T_λ/T_λ^0 for various He³ concentrations. The values 37.6 c.c. and 13.8 c.c. were used for the actual and the accessible volumes of liquid He³ since the Van der Waals' correction term is 23.8 c.c. per mole. The results are plotted in Fig. 1 where the full curve represents

our results (eq. (9)) and the dotted curve those of Heer and Daunt (eq. (9')). For the sake of comparison we have included in Fig. 1 the curve (— · — · —) obtained by Ziro Mikura (1954). We have also included the observed λ -temperatures due to Daunt and Heer (1950) and Abraham *et al.* (1949). It is clear that the present model gives, more or less, the best agreement with experiment.

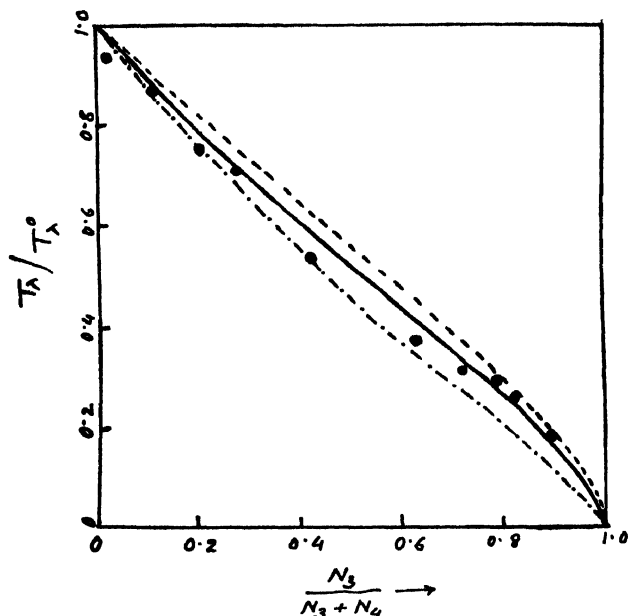


FIG. 1

Plot of T_λ/T_λ^0 against He^3 concentration: (X_3).

— curve: Present theory.

--- curve: Theory of Heer and Daunt (1951).

— · — · — curve: Mikura's Modified Bose-Einstein Liquid Theory.

The dark circles show the observed values due to Daunt and Heer (1950) and Abraham *et al.* (1949).

Next, we shall consider the slope of the T_λ vs. $X_3 (= N_3/(N_3 + N_4))$ curve at very low He^3 concentrations. Thus we have (taking $T_\lambda^0 = 2.186^\circ\text{K}$),

$$\text{Lt } \frac{\partial T_\lambda}{\partial X_3} = -\frac{2}{3} T_\lambda^0 \frac{V_3^0}{V_4^0} \dots \quad (10)$$

$$= -2.41,$$

from eq. (9) and

$$\text{Lt } \frac{\partial T_\lambda}{\partial X_3} = -\frac{2}{3} T_\lambda^0 \frac{V_3^0}{V_4^0} \dots \quad (10')$$

$$= -1.986,$$

from eq. (9'), in comparison to the observed value ~ -2.8 .*

* Recently Keller and others (unpublished results) have found that the slope of the T_λ vs. X_3 curve for very low concentrations is about -1.5 . It seems, therefore, that the modifications introduced here would be of importance only for rather high concentrations.

The above features clearly bring out the importance of the imperfections in the behaviour of He^3 - He^4 mixtures, especially when it is realised that our expression for T_λ/T_λ^0 does not involve any arbitrary constants. Evidently, Van der Waals' correction terms alone are important in modifying T_λ/T_λ^0 . It is surprising that this admittedly qualitative modification leads to, more or less, correct results. The details of this model and its further applications have been worked out and will be dealt with elsewhere.

In conclusion, it is my pleasant task to thank Professor D. S. Kothari for his kind interest and Dr. F. C. Auluck for his valuable guidance. I am also thankful to Dr. P. K. Katti for many useful discussions.

ABSTRACT

Semi-quantitative 'imperfect' gas models for liquid He^4 and liquid He^3 are proposed. Further, it is shown that the transition temperature of He^3 - He^4 mixtures can be predicted with fairly good accuracy on the basis of merely Van der Waals' type corrections to the perfect gas model.

REFERENCES

- Abraham, B. M., Weinstock, B., and Osborne, D. V. (1949). Lambda-temperatures of He^3 - He^4 Mixtures. *Phys. Rev.*, **76**, 864.
Daunt, J. G., and Heer, C. V. (1950). Lambda-temperatures of Solutions of He^3 in He^4 below 1°K . *Phys. Rev.*, **79**, 46-51.
Heer, C. V., and Daunt, J. G. (1951). A Contribution to The Theory of Bose-Einstein Liquids. *Phys. Rev.*, **81**, 447-454.
London, F. (1938). On The Bose-Einstein Condensation. *Phys. Rev.*, **54**, 947-954.
Mikura, Z. (1954). On The Bose-Einstein Liquid Model for Liquid Helium. *Prog. Theor. Phys.* (Japan), **11**, 25-52.

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DIFFERENTIAL THERMAL ANALYSIS OF Na_2SO_4

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(Communicated by K. Banerjee, F.N.I.)

(Received February 2; read May 27, 1955)

The existence of anhydrous sodium sulphate in five polymorphic forms provides an interesting problem in X-ray crystallography. From the researches of many a worker and Kracek (1929), the existence of anhydrous sodium sulphate in three forms was found with certainty. These forms are (a) Thenardite or $\text{Na}_2\text{SO}_4\text{V}$, which separates above 32.5°C . from aqueous solution saturated at atmospheric pressure, (b) the form $\text{Na}_2\text{SO}_4\text{III}$, which appears when $\text{Na}_2\text{SO}_4\text{V}$, heated to high temperature, is cooled down to room temperature and (c) the hexagonal high temperature form $\text{Na}_2\text{SO}_4\text{II}$, which persists only at high temperature. Kracek and Gibson (1930), from the study of the salt in contact with its saturated solution in a pressure dilatometer, established the stable inversion.



The inversion of $\text{Na}_2\text{SO}_4\text{IV}$ to $\text{Na}_2\text{SO}_4\text{V}$ could not be located with certainty because of the very small volume effect accompanying this change and that is why the 185°C . inversion actually represented the change $\text{Na}_2\text{SO}_4\text{III} \rightarrow \text{Na}_2\text{SO}_4\text{V}$. From the study of the heating and cooling curves of anhydrous Na_2SO_4 , Kracek could identify the following phases:

Temperature $^\circ\text{C}$.			
Break	Phase Change	Heating	Cooling
A	$\text{V} \rightarrow \text{IV}$	197 ± 2	
B	$\text{IV} \rightarrow \text{III}$	210 ± 2	
C	$\text{V} \rightarrow \text{III}$	217 ± 2	
D	$\text{IV} \rightleftharpoons \text{II}$	230 ± 2	210, Irreversible in neutral.
E, K	$\text{II} \rightleftharpoons \text{I}$	238 ± 2	234.5
F (H)	$\text{III} \rightleftharpoons \text{I}$	244 ± 250	—
L	$\text{II} \rightleftharpoons \text{III}$...	228

In the heating and cooling curves of Na_2SO_4 , as obtained by Kracek, all the breaks can be seen there clearly.

Kracek and Kasanda (1930) could identify the high temperature hexagonal form of Na_2SO_4 , i.e. $\text{Na}_2\text{SO}_4\text{II}$ and also $\text{Na}_2\text{SO}_4\text{III}$ from their X-ray diffraction study. But up to this time no definite information about the crystal structure of $\text{Na}_2\text{SO}_4\text{III}$ was known. However, we (1953, 1954), by the application of Hesse (1945) and Stosick's (1949) method, have shown that $\text{Na}_2\text{SO}_4\text{III}$ definitely belongs to the tetragonal crystal class with 16 molecules in the unit cell. While studying the crystal structure of $\text{Na}_2\text{SO}_4\text{III}$ by X-ray diffraction method, we had carried out the differential thermal analysis of Na_2SO_4 , the results of which are described below.

EXPERIMENTAL AND DISCUSSION

Sodium sulphate, used in this work, was prepared from Baker's 'Analyzed' anhydrous Na_2SO_4 . The salt, dissolved in water, was precipitated again by pouring the warm solution into warm 95% alcohol. The solution was then filtered and the precipitate was freed from alcohol and water. Na_2SO_4 , thus prepared, was then crushed into fine powders in an agate mortar. This powdered Na_2SO_4 was then used for differential thermal analysis.

The apparatus, used for this analysis, consisted of two micromaxes. One recorded the actual temperature of the furnace by means of a thermocouple inserted into a blank container, usually containing Al_2O_3 , which do not transform even at high temperature. The other recorded any endothermic or exothermic reaction, occurring in the sample under investigations by means of another thermocouple, one end of which was dipped in the sample and the other end being dipped in the Al_2O_3 container. The furnace temperature was varied at regular interval, usually by 10°C . per minute and was kept constant by means of a powerstat. The powdered sample of Na_2SO_4 was placed in the second container. Both the Al_2O_3 and Na_2SO_4 containers were placed in the furnace. The thermo-e.m.f. developed due to the difference of temperature, which is also due to the heat of reaction, at the two ends of the thermocouple, was recorded by the usual potentiometric method. In our investigation, we actually used chromel-alomel thermocouple. The differential thermal curves during heating, cooling and reheating of Na_2SO_4 are given below.

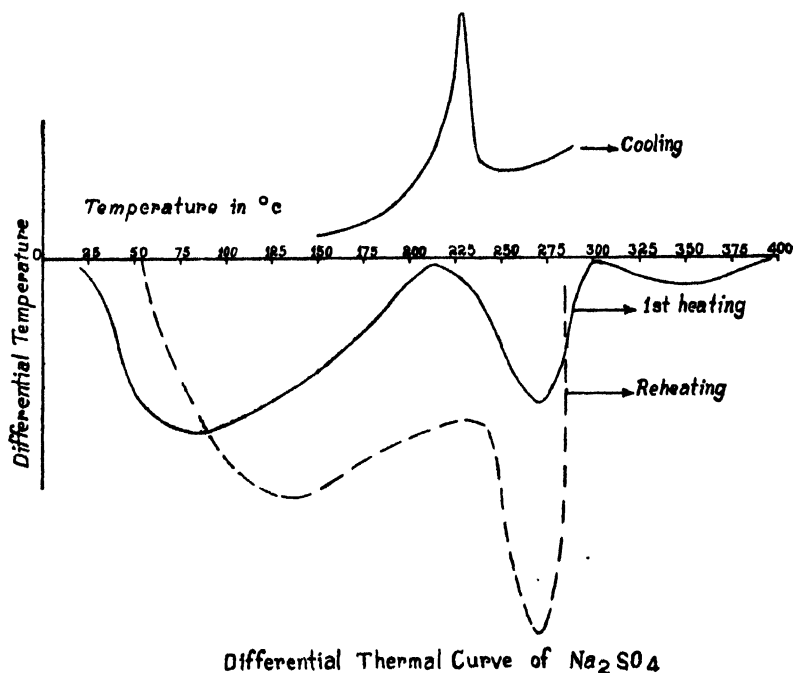


FIG. 1

As it can be seen from the above Fig. 1, one peak at 90°C . was found during the first heating of Na_2SO_4 . This peak was definitely due to the expulsion of occluded water. No breaks at 197°C . and 210°C . were found in this case. This was due to the fact that the first reaction in our case overlapped the other two.

The break which indicated the reaction $\text{Na}_2\text{SO}_4\text{V} \rightarrow \text{Na}_2\text{SO}_4\text{III}$ was also found by us at 215°C . During cooling, the peak, giving indications for the reaction $\text{Na}_2\text{SO}_4\text{II} \rightarrow \text{Na}_2\text{SO}_4\text{III}$ was also observed by us at 225°C . The curve, during reheating, is different from that found by Kracek (1929). In the present curve, the first peak was at 140°C . This peak may be due to the reaction $\text{Na}_2\text{SO}_4\text{III} \rightarrow \text{Na}_2\text{SO}_4\text{IV}$, which was not observed by Kracek, during reheating. The break at 230°C . clearly gave indications for the reaction of $\text{Na}_2\text{SO}_4\text{IV} \rightarrow \text{Na}_2\text{SO}_4\text{II}$.

ACKNOWLEDGEMENTS

The author wishes to express his sincerest gratitude to Prof. K. Banerjee, D.Sc., F.N.I., and to Dr. A. Bose, D.Sc., for their keen interest in the work. He is also grateful to the authorities of the Central Glass and Ceramic Research Institute for kindly permitting him to carry out the differential thermal analysis in their laboratories and to the Council of National Institute of Sciences of India for awarding him a research fellowship.

ABSTRACT

The results of the differential thermal analysis of anhydrous sodium sulphate are described. The transition temperatures for the various transformations, such as $\text{Na}_2\text{SO}_4\text{—Na}_2\text{SO}_4\text{III}$, $\text{Na}_2\text{SO}_4\text{II—Na}_2\text{SO}_4\text{III}$, $\text{Na}_2\text{SO}_4\text{III—Na}_2\text{SO}_4\text{IV}$ and $\text{Na}_2\text{SO}_4\text{IV—Na}_2\text{SO}_4\text{II}$, have been found out clearly from the breaks of the curves.

REFERENCES

- Dasgupta, D. R. (1953). Crystal Class of $\text{Na}_2\text{SO}_4\text{III}$. *J. Chem. Phys.*, **21**, 2097.
——— (1954). An X-ray study of $\text{Na}_2\text{SO}_4\text{III}$. *Acta. Cryst.*, **7**, 275.
Hesse, R. (1948). Indexing of powder photograph of tetragonal, hexagonal and orthorhombic crystal. *Acta. Cryst.*, **1**, 200.
Kracek, F. C. (1929). The polymorphism of sodium sulphate I. Thermal analysis. *J. Phys. Chem.*, **33**, 1284.
Kracek, F. C. and Gibson, R. E. (1929). The polymorphism of sodium sulphate II. Density of anhydrous sodium sulphate at 25°C . *J. Phys. Chem.*, **33**, 1304.
Kracek, F. C. and Kasanda, C. J. (1930). The polymorphism of sodium sulphate IV. X-ray analysis. *J. Phys. Chem.*, **34**, 1741.
Stosick (1949). A method of Indexing powder photographs using linear diophantine equations and some test for crystal class. *Acta. Cryst.*, **2**, 271.

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QUANTUM MECHANICAL CALCULATION ON THE REACTIVITY OF AZOLES AND AZINES

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(Communicated by P. B. Sarkar, F.N.I.)

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INTRODUCTION

It has been shown by Wheland and Pauling (1935) that the calculated permanent charge distribution for a number of homocyclic and heterocyclic compounds, using molecular orbital method, can be satisfactorily correlated with the chemical reactivity of these compounds. Coulson and co-worker (1947, 1949) have extended this calculation to a large number of polynuclear hydrocarbons with hetero-atoms by essentially the same method (with modification in some cases), in correlating the reactivity with permanent charge distribution. In the present paper is reported the electron density calculations on azoles and azines and its correlation with chemical reactivity. These five and six membered hetero-rings bear to pyrrol and pyridine same relation as pyridine and pyrrol bears to benzene and corresponding hypothetical five membered homocyclic ring. These compounds are of interest to organic chemist because of their peculiar chemical reactivity and to bio-chemist for their interesting physiological properties.

METHOD OF CALCULATION

The method of calculation is too well known to reproduce here. The values of coulomb integral used in the present calculation are $E_0 + 2\beta$, $E_0 + 0.25\beta$ and E_0 for nitrogen, carbon bonded to nitrogen and carbon bonded to carbon respectively, β being the resonance integral which was taken to be equal for all bonds. The energy equations higher than second were solved by Pinch method correct up to

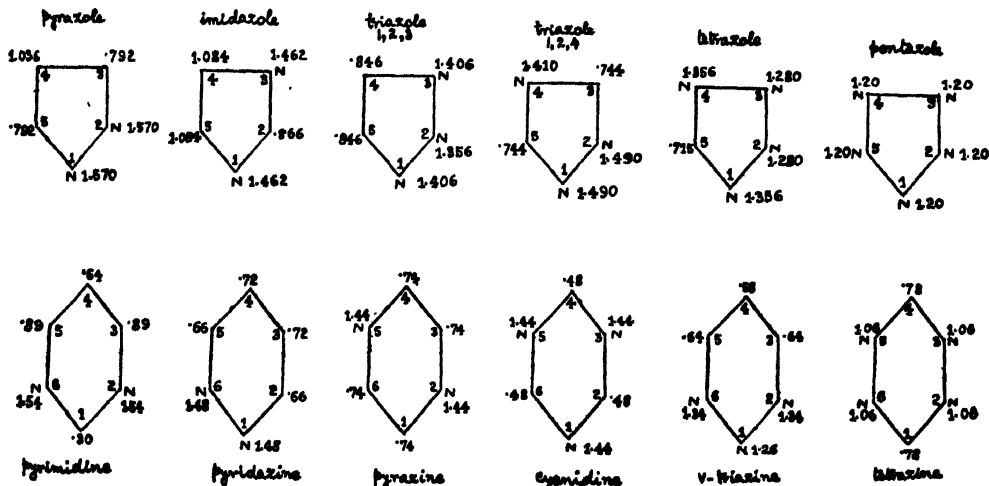


FIG. 1

first place. Π -electron densities for these compounds are shown in Fig. 1, against each atom. The numbering of the atoms is also given inside the ring.

DISCUSSION

It may be observed that the Π -electron density is rather high at 4 position in pyrazole so this position is highly reactive. Halogenation, nitration, and sulfonation take place at this position. Lesser electron density at the nitrogen atom makes this less basic than pyrrol. Imidazol, similarly, is less basic than pyrazole. Nitration and sulfonation proceed with equal ease at 4 and 5 positions of this compound. Bromination can cause substitution at all the carbon atoms. The substitution reaction at 2 position is most probably a secondary reaction, i.e. migration of the halogen atom from nitrogen to the carbon atom at 2 position (Morton, 1946). The reason for the assumption is two-fold. Firstly the hydrogen at 1 position can be replaced by bromine giving a tetrabromo compound. And secondly a methyl group from nitrogen does migrate to 2 position. 1, 2, 3-triazole is weakly basic. Rather low electron densities on carbon atoms in triazole makes this compound highly resistant to substitution reaction. Methyl groups at these positions are easily oxidized to give triazole carboxylic acid. Tetrazole is weakly acidic. Electron density at 5 position in tetrazole is extremely low, so the compound is easily attacked by alkali at this position. Pentazole, if it exists, will be extremely unreactive.

Compared to pyridine diazines are less basic, while triazines are less basic than diazines. Tetrazines are neutral or slightly acidic. Calculation agrees fairly well with this behaviour with the exception of pirimidine. High electron densities at the nitrogen atoms should make pirimidine basic. Actually the compound is a mono-basic acid. Extremely low electron density at 1 position may cause easy dissociation of the hydrogen atom at this position and this may account for its acidic character (Richter, 1923).

In pyrazine low electron densities at 2 and 5 positions make the compound susceptible to anionoid attack. Methyl substitution at these positions are easily oxidized to pyrazine carboxylic acid. In pirimidine, too, methyl substitution at 1 position is easily oxidized. Halogen atom at this position is replaced by NH_2 group by reaction with ammonia or sodamide. The mobility of halogen at 1 position in pirimidine is so great that it can be replaced by SCN by reaction with KSCN. Low electron densities at 2, 4 and 6 positions in cyanidine also makes the halogen atoms at these positions easily replaceable by NH_2 group. Methyl groups at these positions are easily oxidized to carboxylic acid. Low electron densities at the carbon atoms in azines make these compounds resistant to nucleophilic substitution, so these compounds do not undergo nitration, bromination or sulfonation.

SUMMARY

A quantum mechanical calculation of the electron densities at different positions of azole and azine molecules have been made and an effective correlation has been obtained with the chemical reactivity of the respective position.

REFERENCES.

- Longuet-Higgins, H. C., and Coulson, C. A. (1947). A theoretical Investigation of the distribution of electrons in some heterocyclic molecules containing nitrogen. *Trans. Faraday Soc.*, **43**, 87.
- (1949). The electronic structure of some Aza-derivatives of Naphthalene, Anthracene and Phenanthrene. *J. Chem. Soc.*, 971.
- Morton, A. A. (1946). The chemistry of heterocyclic compounds. McGraw-Hill Book Co., 362.
- Richter, V. (1923). Organic Chemistry, Vol. III, Kegan Paul—Tech. Trubeng & Co., Ltd.
- Wheland, G. W. and Pauling, L. (1935). A Quantum Mechanical Discussion of the orientation of substituents in Aromatic molecules. *J. Amer. Chem. Soc.*, **57**, 2087.

AN X-RAY STUDY OF NETTLE FIBRES

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Samples of nettle fibres grown in the wild tracts of Darjeeling, Sikkim and Nepal were lately available for study. These samples are classified under two local names Pooah and Sishnu. Pooah has been identified as *Boehmeria frutescens* and Sishnu as *Boehmeria girardinia heterophylla* by the botanists. Fibres from Pooah are soft and flexible but fibres from Sishnu are definitely coarser than those from Pooah. In this communication Pooah is referred to as sample A and Sishnu as B.

The physical properties of these A and B varieties, as studied earlier (Deb and Sen, 1949), reveal that—

- (1) the density of these fibres are nearly that of cotton and much higher than that of jute;
- (2) when fully retted and degummed they produce spinnable long filaments finer than the jute filaments;
- (3) they possess small amount of pore space;
- (4) unlike jute fibres when these fibres are bleached white, the wet strength of the filaments appears to be higher than the dry filaments; and
- (5) the tensile strength of these fibres are greater than jute.

So it appears from the earlier studies that the nettle fibres are in many respects better than jute variety. To prove this and to study them fully X-ray pictures of the samples A and B were taken. A small bunch of fibres of 1 mm. thickness was mounted in a cylindrical fibre camera and X-ray pictures were taken in Phillips sealed tube in Cu-radiation. Usually exposures of 4 to 4½ hours were given in each case. The angular co-ordinates of θ and μ of the spots were obtained in the usual way from the X-ray pictures and the glancing angle was calculated from the relation $\cos \theta \cos \mu = \cos 2\theta_B$. The spots were identified on the basis of the unit cell of dimensions $a = 8.35 \text{ \AA}$, $b = 10.3 \text{ \AA}$, $c = 7.9 \text{ \AA}$ and $\beta = 84^\circ$, and the space group is P_{21} (Meyer and Mark, 1929), of cellulose which is the main constituent of these fibres. The details of the spots, their indices, intensities and characters are given below in Tables I and II and the photographs in Plate VIII.

TABLE I
Sample A: (Pooah variety)

Indices	Intensity	Character
101 ..	Sharp (strong)	Like a spot.
002 ..	Very strong	Extended into arc.
002 β ..	Medium strong	Extended into arc.
004 ..	Very weak	Extended into arc.
310 ..	Weak	Diffuse and extended into arc.
021 ..	Medium	Like a spot.
131 or 130 ..	Weak	Diffuse.
230 ..	Weak	Diffuse and extended into arc.

TABLE II
Sample B: (Sishnu variety)

Indices	Intensity	Character
101 ..	Strong	Broad spot-like.
002 ..	Very strong	Sharp and extended into arc.
002 β ..	Medium strong	Sharp and extended into arc.
004 .	Weak	Diffuse.
310 ..	Weak	Bent into arc.
021 ..	Medium	Like a spot.
221 ..	Weak	Broad and diffuse.
130 or 131 ..	Weak	Broad and diffuse.
230 ..	Medium	Broad, bent into arc.
040 ..	Weak	Bent into arc.
042 ..	Weak	Bent into arc. Diffuse.

In the Pooah variety (sample A) the long axis of cellulose unit cell is parallel to the fibre axis. From the fairly sharp characters of the different planes like 101, 002, etc., appearing in the picture it appears that the degree of crystallinity is sufficiently good for a fibre diagram. Clearness of the background indicate that unlike the jute fibres the crystallites in the micellar regions are more closely packed and the amorphous part if present at all must be extremely small.

In the Sishnu variety (sample B) the long axis of the cellulose unit cell is also parallel to the fibre axis. The number of planes appearing in the picture is more than the Pooah fibres. The widths of the spots in this case are practically like the spots in sample A.

The general analyses of these fibres indicate that out of these two types Sishnu (B) shows sharper and stronger spots. However, if all these X-ray diagrams are compared with the standard X-ray diagrams of ramie (Mark and Meyer, 1929) and jute (Banerjee, K. and Roy, A. K., 1941), it becomes apparent that the nettle fibres, as far as general X-ray diagrams and the packing of crystallites in the micellar regions are concerned, are as sharp and clear as ramie, and definitely much sharper and clearer than jute. This conclusion is thus quite in accord with the observations on tensile strength and other physical properties studied by Deb and Sen (1949).

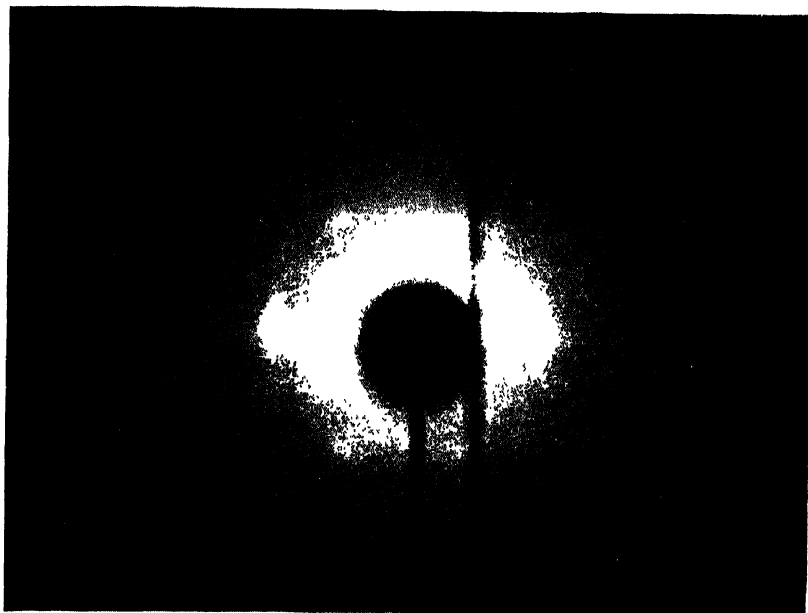
Thanks are due to Prof. K. Banerjee, D.Sc., F.N.I., and Prof. B. N. Srivastava, D.Sc., F.N.I., for their keen interest in the work. Thanks are also due to Dr. B. K. Banerjee for valuable discussions and Dr. S. Deb for supplying the fibres.

ABSTRACT

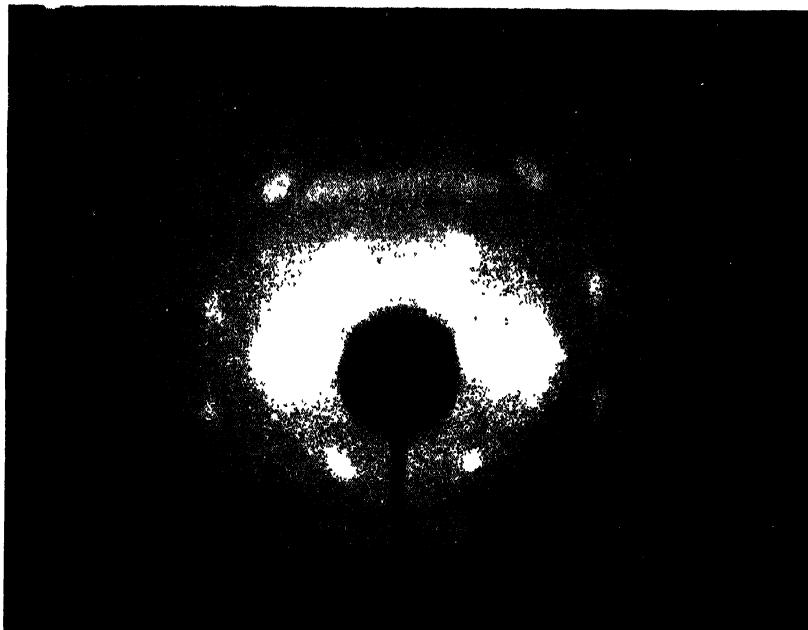
Pooah and Sishnu types of nettle fibres were studied by X-ray method. The general X-ray diagrams were analysed. It is proved that the Sishnu type yields slightly sharper and clearer X-ray diagram than the other, and that the X-ray diagrams of both the types of nettle fibres are nearly as good as ramie and definitely much sharper and clearer than the jute fibres, indicating large crystallite size and low proportion of amorphous content.

REFERENCES

- Banerjee, B. K. (1947). An X-ray Study of some vegetable Fibres. *Ind. Journal of Physics*, 21, 259.
 Banerjee, K. and Roy, A. K. (1941). The Structure of Jute Fibre by X-ray Diffraction Method. *Proc. Nat. Inst. Sci.*, 8, 377.
 Deb, S. and Sen, K. R. (1949). Physical quality of fibre from wild Nettle of Darjeeling. *Jour. Science Club*, 3, 15.
 Mark, H. and Meyer, K. H. (1929). Über den Bau des Kristallisierten Anteils der Cellulose II, 2, 115.



A



B

X ray diagrams of (A) Pooah Fibres and (B) Sishnu Fibres

SEMIQUANTITATIVE EVALUATION OF RESONANCE INTEGRAL AND OVERLAP INTEGRAL FOR C-N BOND IN S-TRIAZINE

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(Communicated by B. N. Ghosh, F.N.I.)

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A complete theoretical computation has been made on benzene molecule according to MO method by Mayer and Sklar (1938). The only empirical value used by them was 1.40 Å for the C-C bond length. In the present paper a semi-empirical method has been used, i.e. the theoretical calculation has been combined with spectroscopic data in evaluating the resonance integral for C-N bond in S-triazine (Fig. 1). The method is essentially the same as the one used by Mulliken (1948) for barozol. Calculations on benzene are given here side by side in order to show the correspondence between the two molecules and the applicability of similarity transformation.

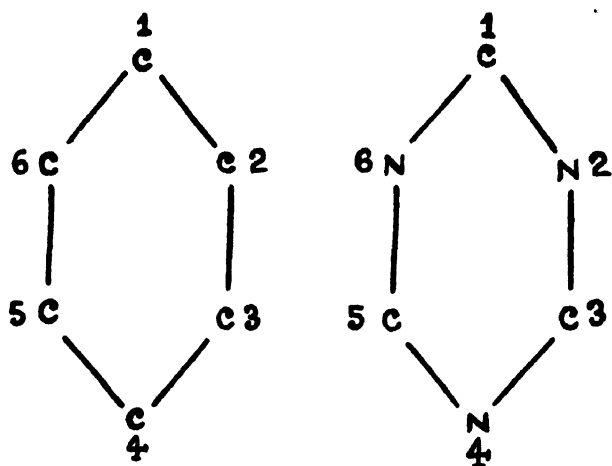


FIG. 1. Benzene and S-triazine

Neglecting non-neighbouring atom interactions we may write the secular determinants for the six approximate MOs as

$$\begin{vmatrix} A & 1 & 0 & 0 & 0 & 1 \\ 1 & A & 1 & 0 & 0 & 0 \\ 0 & 1 & A & 1 & 0 & 0 \\ 0 & 0 & 1 & A & 1 & 0 \\ 0 & 0 & 0 & 1 & A & 1 \\ 1 & 0 & 0 & 0 & 1 & A \end{vmatrix} = 0 \text{ (benzene)}$$

$$\begin{vmatrix} A' & 1 & 0 & 0 & 0 & 1 \\ 1 & A'' & 1 & 0 & 0 & 0 \\ 0 & 1 & A' & 1 & 0 & 0 \\ 0 & 0 & 1 & A'' & 0 & 0 \\ 0 & 0 & 0 & 1 & A' & 0 \\ 1 & 0 & 0 & 0 & 1 & A'' \end{vmatrix} = 0 \text{ (S-triazine)}$$

where $A = \frac{\alpha - E}{\gamma - ES}$, $A' = \frac{\alpha - E}{\gamma^* - ES^*}$, $A'' = \frac{\alpha^1 - E}{\gamma^* - ES^*}$

$$\alpha = \int \bar{\Psi}_c^1 H \Psi_c^1 d\tau, \quad \alpha^1 = \int \bar{\Psi}_N^2 H^* \Psi_N^2 d\tau$$

$$\gamma = \int \bar{\Psi}_c^1 H \Psi_c^2 d\tau, \quad \gamma^* = \int \bar{\Psi}_c^1 H^* \Psi_N^2 d\tau$$

$$S = \int \bar{\Psi}_c^1 \Psi_c^2 d\tau \quad \text{and} \quad S^* = \int \bar{\Psi}_c^1 \Psi_N^2 d\tau.$$

The Ψ 's are atomic $2p_x$ orbitals, the subscripts refer to the kind of atom and the superscripts to the location in the ring. H and H^* are the one electron Hartree field Hamiltonian operator for benzene and S-triazine respectively.

Substituting

$$\alpha - E = -X$$

$$\gamma - \alpha S = \beta$$

$$\gamma^* - \alpha S^* = \beta^*$$

and

$$\alpha^1 - \alpha = \delta$$

we get as a first approximation

$$A = \frac{-X}{\beta - SX}, \quad A' = \frac{-X}{\beta^* - XS^*} \quad \text{and} \quad A'' = \frac{\delta - X}{\beta^* - XS^*}$$

where α , γ and β are known to be negative quantities. (The α 's are electronegativity parameter.) By a suitable similarity transformation the secular determinants can be reduced to the forms:

$$\begin{array}{cccccc} A & 2 & 0 & 0 & 0 & 0 \\ 2 & A & 0 & 0 & 0 & 0 \\ 0 & 0 & A & 1 & 0 & 0 \\ 0 & 0 & 1 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 1 \\ 0 & 0 & 0 & 0 & 1 & A \end{array} = 0 \text{ (benzene)} \quad \begin{array}{cccccc} A' & 2 & 0 & 0 & 0 & 0 \\ 2 & A' & 0 & 0 & 0 & 0 \\ 0 & 0 & A' & 1 & 0 & 0 \\ 0 & 0 & 1 & A' & 0 & 0 \\ 0 & 0 & 0 & 0 & A' & 1 \\ 0 & 0 & 0 & 0 & 1 & A' \end{array} = 0 \text{ (S-triazine)}$$

The molecular orbitals corresponding to the roots $A = -2$ and $A = -1$ (twice) in benzene and the corresponding orbitals in S-triazine are each occupied in the ground state by two electrons. The average height above the ground state of the first excited state, corresponding in benzene to the excitation of one electron from one of the $A = -1$ orbitals to one of the $A = +1$ orbitals is given by the difference $(X_2 - X_1)$, of the two roots of

$$A^2 - 1 = 0, \text{ i.e. of } X^2 - (\beta - SX)^2 = 0$$

namely,

$$X_2 - X_1 = -2\beta/(1 - S^2) \text{ for benzene,}$$

and for S-triazine, of

$$A' A'' - 1 = 0, \text{ i.e. } \frac{-X}{(\beta^* - XS^*)} \cdot \frac{\delta - X}{(\beta^* - XS^*)} - 1 = 0$$

or

$$X^2(1 - S^*) + X(2S^*\beta^* - \delta) - \beta^{*2} = 0$$

$$\text{namely,} \quad X_2 - X_1 = -\frac{\beta^*(4+d^2-4dS^*)^{\frac{1}{2}}}{1-S^*}$$

$$\text{where } d = \frac{\delta}{\beta^*}.$$

The overlap integral S and S^* can easily be evaluated if we use Slater's (1930) $2p_\pi$ orbitals. The results are

$$S = \left(1 + x + \frac{2}{5}x^2 + \frac{1}{15}x^3\right)e^{-x}$$

for like atoms, and

$$S^* = \left\{ \frac{(ab)^{\frac{1}{2}}}{\frac{1}{2}(a+b)} \right\} \frac{1}{\eta^2} \left\{ \cosh x\eta + \frac{1}{\eta} \sinh x\eta + 3 \left(1 - \frac{1}{\eta^2}\right) \frac{1+x}{x^2} \left(\cosh x\eta - \frac{1}{x\eta} \sinh x\eta \right) \right\} e^{-x}$$

for unlike atoms, where

$$\begin{aligned} X &= Z'R/n^*, \text{ for like atoms,} \\ &= \frac{1}{2}(a+b)R, \text{ for unlike atoms,} \\ a &= Z'/n^*, b = Z''/n^*, \\ Z' &= \text{effective nuclear charge on carbon} = 3.25, \\ Z'' &= \text{effective nuclear charge on nitrogen} = 3.90, \\ n^* &= \text{effective quantum number,} \\ &= 2 \text{ for } 2p_\pi \text{ orbitals,} \end{aligned}$$

$$\eta = \frac{(a-b)}{(a+b)} \text{ (nuclear asymmetry)}$$

and R = internuclear distance in atomic unit. (Atomic unit of distance is 0.528 Å.)

The actual internuclear distance was taken as 1.41 Å for both C-C and C-N. With these values we get $S = 0.25$ and $S^* = 0.235$ and the mean energies from the ground state to the first excited state

$$\begin{aligned} \text{benzene: } E_2 - E_1 &= -2.31\beta, \\ \text{S-triazine: } E_2 - E_1 &= 1.31\beta^* (4+d^2-0.96d)^{\frac{1}{2}}. \end{aligned}$$

The first excited state in benzene is 4.9 ev above the ground state; with this value we get β (spec.), the spectroscopically determined β as

$$\beta_{\text{spec.}} = -4.9/2.13 = -2.30 \text{ ev.}$$

For S-triazine, first excited state is 5.29 ev above ground state (Halverson and Hirst, 1951). So

$$\beta_{\text{spec.}} = \frac{5.29 \times .76}{2} = -2.01 \text{ ev}$$

if $d = 1$. This gives $\delta = -2.01$ ev about 0.3 ev lower than that calculated by Mulliken for nitrogen in borazol.

It may be observed that the values of resonance integral and overlap integral for C-N bond is slightly less than those for C-C bond.

Calculation was not extended to other excited states of S-triazine, since only one absorption band has been experimentally detected for this compound.

SUMMARY

Theoretical MO calculation has been combined with the spectroscopic data in calculating the overlap integral and resonance integral for C-N bond in S-triazine. The results are 0.235 and -2.01 ev, respectively and slightly lower than C-C bond.

REFERENCES

- Mayer, M. and Sklar, A. L. (1938). Calculation of the lower excited levels of Benzene. *J. Chem. Phys.*, **6**, 645.
- Roothaan, C. C. and Mulliken, R. S. (1948). Molecular orbital treatment of ultraviolet spectra of benzene and borazole. *J. Chem. Phys.*, **16**, 118.
- Slater, J. C. (1930). Atomic Shielding Constants. *Phys. Rev.*, **36**, 57.
- Halverson, F. and Hirst, R. C. (1951). Near ultraviolet solution spectra of diazines. *J. Chem. Phys.*, **19**, 711.

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PARTITIONS INTO PRIMES

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1. Let $P(n)$ denote the number of distinct ways in which a positive integer n can be expressed as a sum of the prime numbers 2, 3, 5, 7, 11, . . . repeated any number of times.

Thus $5 = 5, 2+3$; and $8 = 3+5, 2+3+3, 2+2+2+2$;

so that $P(5) = 2$ and $P(8) = 3$.

We take $P(0) = 1$.

In this paper, we give a table of values of $P(n)$ for $n \leq 300$. The calculations were made by us independently and discrepancies corrected. After the check, which Dr. H. Gupta has applied, there can be no doubt about the correctness of our results.

2. $P(n)$ may be evaluated in two different ways:

First method.

Denote * by $P(n, p)$ the number of those of the $P(n)$ partitions of n into primes, which contain the prime p as the largest summand.

Then

$$P(n) = \sum_p P(n, p) \quad \dots \quad (1)$$

where p runs through all primes less than or equal to n .

It may be noticed that

$$P(p, p) = 1.$$

Denoting the r th prime by p_r , with $p_1 = 2, p_2 = 3$ and so on, we have

$$P(n, p_r) = \sum_{i=1}^r P(n - p_r, p_i) \quad \dots \quad (2)$$

because if from a partition of n into primes with p_r as the largest summand, we take away p_r , we are left with a partition of $n - p_r$ with p_r or a smaller prime as the largest summand. From (2), we readily obtain the recurrence formula

$$\begin{aligned} P(n, p_r) &= P(n - p_r, p_r) + \sum_{i=1}^{r-1} P(n - p_r, p_i) \\ &= P(n - p_r, p_r) + P(n - p_r + p_{r-1}, p_{r-1}). \quad \dots \quad (3) \end{aligned}$$

Evidently, for $p_r > \frac{n}{2}$,

$$P(n, p_r) = P(n - p_r) \quad \dots \quad (4)$$

* $P(n, p_r)$ is the coefficient of x^{n-p_r} in the expansion of $\prod_{i=1}^r (1 - x^{p_i})^{-1}$.

As an illustration, we write down the table for $P(n, p)$ for values of n up to 9 and calculate the values for $n = 10$.

$\begin{smallmatrix} p \\ n \end{smallmatrix}$	2	3	5	7	$P(n)$
1	0				0
2	1				1
3	0	1			1
4	1	0			1
5	0	1	1		2
6	1	1	0		2
7	0	1	1	1	3
8	1	1	1	0	3
9	0	2	1	1	4
10	1	1	2	1	5

$$P(10, 2) = P(8, 2) = 1;$$

$$P(10, 3) = P(7, 3) + P(9, 2) = 1 + 0 = 1;$$

$$P(10, 5) = P(5, 5) + P(8, 3) = 1 + 1 = 2;$$

$$P(10, 7) = P(3, 7) + P(8, 5) = 0 + 1 = 1.$$

and

Hence

$$P(10) = P(10, 2) + P(10, 3) + P(10, 5) + P(10, 7) = 5.$$

Second method.

$P(n)$ is the coefficient of x^n in the expansion of

$$f(x) \equiv \prod_{r=1}^{\infty} (1 - x^{p_r})^{-1}.$$

$$\text{Now } * \{f(x)\}^{-1} = \prod_{r=1}^{\infty} (1 - x^{p_r}) = 1 - x^2 - x^3 + x^8 + x^9 - x^{11} + x^{16} - x^{19} - x^{23} +$$

Hence

$$1 = (1 - x^2 - x^3 + x^8 + x^9 - x^{11} + \dots) \sum_{n=0}^{\infty} P(n)x^n$$

so that

$$P(n) = P(n-2) + P(n-3) - P(n-8) - P(n-9) + P(n-11) - \dots$$

Thus

$$P(10) = P(8) + P(7) - P(2) - P(1) = 3 + 3 - 1 = 5.$$

* See H. Gupta, 'Partitions into distinct primes'.

n	$p(n)$	n	$p(n)$	n	$p(n)$
1	0	51	899	101	43709
2	1	52	987	102	46696
3	1	53	1083	103	49871
4	1	54	1186	104	53243
5	2	55	1298	105	56826
6	2	56	1420	106	60631
7	3	57	1552	107	64671
8	3	58	1695	108	68957
9	4	59	1850	109	73506
10	5	60	2018	110	78331
11	6	61	2198	111	83447
12	7	62	2394	112	88874
13	9	63	2605	113	94625
14	10	64	2833	114	1 00719
15	12	65	3079	115	1 07175
16	14	66	3344	116	1 14014
17	17	67	3630	117	1 21255
18	19	68	3936	118	1 28923
19	23	69	4268	119	1 37038
20	26	70	4624	120	1 45627
21	30	71	5007	121	1 54709
22	35	72	5419	122	1 64320
23	40	73	5861	123	1 74482
24	46	74	6336	124	1 85225
25	52	75	6845	125	1 96583
26	60	76	7393	126	2 08585
27	67	77	7979	127	2 21265
28	77	78	8608	128	2 34658
29	87	79	9282	129	2 48807
30	98	80	10003	130	2 63745
31	111	81	10776	131	2 79516
32	124	82	11603	132	2 96161
33	140	83	12488	133	3 13727
34	157	84	13435	134	3 32258
35	175	85	14445	135	3 51808
36	197	86	15527	136	3 72427
37	219	87	16681	137	3 94170
38	244	88	17914	138	4 17088
39	272	89	19232	139	4 41250
40	302	90	20636	140	4 66711
41	336	91	22134	141	4 93538
42	372	92	23732	142	5 21804
43	413	93	25436	143	5 51573
44	456	94	27251	144	5 82925
45	504	95	29186	145	6 15933
46	557	96	31246	146	6 50686
47	614	97	33439	147	6 87262
48	677	98	35772	148	7 25757
49	744	99	38257	149	7 66262
50	819	100	40899	150	8 08872

n	$p(n)$	n	$p(n)$	n	$p(n)$
151	8 53692	201	103 12927	251	914 77898
152	9 00827	202	108 01607	252	953 24698
153	9 50393	203	113 12080	253	993 24684
154	10 02502	204	118 45265	254	1034 83632
155	10 57278	205	124 02104	255	1078 07529
156	11 14849	206	129 83601	256	1123 02573
157	11 75344	207	135 90769	257	1169 75172
158	12 38904	208	142 24686	258	1218 31963
159	13 05679	209	148 86458	259	1268 79839
160	13 75815	210	155 77234	260	1321 25912
161	14 49471	211	162 98212	261	1375 77558
162	15 26812	212	170 50639	262	1432 42423
163	16 08014	213	178 35813	263	1491 28391
164	16 93247	214	186 55065	264	1552 43647
165	17 82712	215	195 09801	265	1615 96652
166	18 76598	216	204 01462	266	1681 96183
167	19 75108	217	213 31548	267	1750 51297
168	20 78460	218	223 01629	268	1821 71385
169	21 86867	219	233 13328	269	1895 66161
170	23 00576	220	243 68324	270	1972 45661
171	24 19812	221	254 68365	271	2052 20288
172	25 44843	222	266 15264	272	2135 00804
173	26 75925	223	278 10910	273	2220 98343
174	28 13326	224	290 57246	274	2310 24409
175	29 57342	225	303 56317	275	2402 90920
176	31 08265	226	317 10220	276	2499 10190
177	32 66409	227	331 21140	277	2598 94960
178	34 32097	228	345 91339	278	2702 58409
179	36 05666	229	361 23177	279	2810 14186
180	37 87467	230	377 19090	280	2921 76374
181	39 77861	231	393 81607	281	3037 59545
182	41 77239	232	411 13365	282	3157 78780
183	43 85994	233	429 17077	283	3282 49663
184	46 04537	234	447 95565	284	3411 88297
185	48 33306	235	467 51763	285	3546 11368
186	50 72740	236	487 88713	286	3685 36090
187	53 23313	237	509 09557	287	3829 80254
188	55 85505	238	531 17571	288	3979 62282
189	58 59833	239	554 16136	289	4135 01182
190	61 46816	240	578 08759	290	4296 16642
191	64 47003	241	602 99078	291	4463 28963
192	67 60967	242	628 90875	292	4636 59174
193	70 89299	243	655 88056	293	4816 28967
194	74 32618	244	683 94674	294	5002 60763
195	77 91567	245	713 14927	295	5195 77753
196	81 66824	246	743 53172	296	5396 03890
197	85 59069	247	775 13908	297	5603 63918
198	89 69035	248	808 01816	298	5818 83402
199	93 97474	249	842 21750	299	6041 88762
200	98 45164	250	877 78708	300	6273 07270

PARTITIONS INTO DISTINCT PRIMES

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1. In the preceding paper, Mr. Gupta and Miss Luthra have given a table for $P(n)$, the number of partitions of n into primes 2, 3, 5, 7, 11, etc., repeated any number of times, for values of n up to 300. While the work of calculation was going on, it was occasionally found that the two workers obtained different answers. Though finally the discrepancies were corrected by them, an element of doubt remained in my mind and I decided to check their work by finally applying the second method given in their paper. Miss Luthra had applied the check for n up to 100, but this was not enough when the table was extended to $n = 300$. For this purpose, I needed the first 301 coefficients in the expansion of

$$\prod_{p} (1 - x^p) \dots \dots \dots (1)$$

where p runs through primes not exceeding 300. Incidentally it led to the calculation of a table giving the number of partitions of n into distinct primes. Here I give not only this table of partitions but also the first 301 coefficients in the expansion of (1). Mr. Gupta computed independently the table of partitions into distinct primes for n up to 200 and our results agree.

2. I proceeded as follows. Let

$$\prod_{p > 2} (1 + x^p) = \sum_{n=0}^{\infty} Q(n)x^n, \quad Q(0) = 1; \quad \dots \dots (2)$$

so that $Q(n)$ is the number of partitions of n into distinct *odd* primes;

$$\prod_{p \geq 2} (1 + x^p) = \sum_{n=0}^{\infty} R(n)x^n, \quad R(0) = 1; \quad \dots \dots (3)$$

so that $R(n)$ is the number of partitions of n into distinct primes; and finally

$$\prod_{p \geq 2} (1 - x^p) = \sum_{n=0}^{\infty} S(n)x^n, \quad S(0) = 1. \quad \dots \dots (4)$$

Now since

$$\prod_{p > 2} (1 - x^p) = \sum_{n=0}^{\infty} (-1)^n Q(n)x^n,$$

we have

$$R(n) = Q(n) + Q(n-2), \quad \dots \dots \dots (5)$$

and

$$S(n) = (-1)^n \{Q(n) - Q(n-2)\}. \quad \dots \dots \dots (6)$$

The values of $R(n)$ are listed in Table 1, and those of $S(n)$ in Table 2. The value for $n = 193$ is, for example, given against 19 in the column headed 3.

I find that

$$\sum_{r=0}^n P(r) S(n-r) = 0, \text{ for } n = 300.$$

This sets at rest any doubts about the correctness of the tables presented here and in the preceding paper.

TABLE 2

Values of $S(n)$

$n \rightarrow$ \downarrow	0	1	2	3	4	5	6	7	8	9
0	1	0	1	-1	0	0	0	0	1	1
1	0	-1	0	0	0	0	1	0	0	-1
2	0	0	0	-1	1	1	0	-1	0	-1
3	0	-1	1	1	1	1	1	-1	-1	-1
4	2	0	1	-1	1	0	0	-3	2	1
5	1	-2	1	-2	1	-2	1	0	2	-3
6	3	-1	0	-2	4	-1	2	-4	1	-1
7	3	-5	4	1	2	-3	4	-4	3	-5
8	3	-1	4	-8	6	-1	2	-7	6	4
9	8	-6	3	-4	6	-10	8	-4	5	-6
10	10	-10	7	-10	7	4	10	-15	12	-6
11	7	-12	13	-12	14	-12	9	-10	13	-20
12	19	-10	8	-17	20	-16	19	-20	13	-12
13	23	-29	22	-15	17	-21	25	-28	26	-24
14	23	-19	28	-40	34	-21	21	-34	36	-36
15	39	-36	27	-28	44	-50	44	-36	31	-42
16	54	-50	51	-50	40	-39	56	-74	63	-41
17	50	-63	63	-74	72	-62	57	-59	78	-93
18	87	-67	60	-82	97	-91	96	-93	75	-78
19	110	-129	108	-89	93	-104	125	-134	126	-121
20	108	-111	141	-166	155	-117	117	-153	164	-169
21	181	-160	136	-155	193	-215	198	-167	166	-189
22	225	-234	219	-216	196	-192	251	-295	260	-216
23	226	-259	285	-303	306	-272	249	-279	328	-368
24	351	-296	286	-339	384	-387	391	-375	328	-352
25	444	-482	441	-388	390	-435	487	-525	509	-464
26	454	-472	547	-628	583	-496	504	-578	635	-661
27	668	-623	561	-613	737	-781	738	-674	652	-729
28	834	-859	836	-798	754	-781	927	-1026	951	-846
29	861	-955	1031	-1103	1099	-995	957	-1043	1183	-1278
30	1234									

FURTHER INVESTIGATIONS ON COLUMBIUM OXIDE BANDS

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INTRODUCTION

After the analysis of the three bands (1, 0), (0, 0) and (0, 1) of columbium oxide described in a previous paper has been obtained, it is found that besides the lines forming the *R* and *P* branches there are a number of fainter rotational lines which occur at the end nearer the head. A further study of these lines has indicated that they might form members of a satellite branch. In this paper the results of this study and a discussion of the electronic transition giving rise to this system of bands of CbO are given.

SATELLITE BRANCH

The higher resolution, sharply focussed photographs of the three bands reproduced in Plate IX revealed that the unassigned rotational lines mentioned above

TABLE I
Satellite branch in (1, 0) Band

Wave number	First difference	Wave number	First difference
22148.70	1.41	22121.78	..
147.29	1.50
145.79	1.62	116.89	..
144.17	1.73
142.44	1.82	111.49	2.61
140.62	1.68	108.88	..
138.94	1.87
137.07	1.98	103.14	..
135.09	2.30
132.79	1.84	096.85	2.69
130.95	2.03	094.16	3.12
128.92	2.38	091.04	3.20
126.54	2.24	087.84	3.23
124.30	2.52	084.61	

TABLE II
Satellite branch in (0, 0) Band

Wave number	First difference	Wave number	First difference
21312.97		21262.61	
	1.05		2.93
311.92		259.68	
	1.34		2.06
310.58		256.72	
	1.25		2.61
309.33		254.11	
	1.13		3.13
308.20		250.98	
	1.48		2.90
306.72		248.08	
	1.86		3.25
304.86		244.83	
	1.35		3.32
303.51		241.51	
	1.49		3.00
302.02		238.51At	
	1.55		3.33
300.47		235.18	
	1.61		3.52
298.86		231.66	
	1.71		3.30
297.15		228.36	
	..		3.58
..	..	224.78	
	..		3.55
..	..	221.23	
	..		3.58
..	..	217.65	
	..		3.88
..	..	213.77	

287.44		..	
	2.21		..
285.23		206.28	
	2.10		4.06
283.13		202.22	
	2.31		3.92
280.82At		198.30	
	2.17		4.02
278.65		194.28	
	2.22		4.10
275.43		190.18	
	2.83		3.88
272.60At		186.30	
	2.14		4.36
270.46		181.94	
	2.58		4.31
267.88		177.63	
	2.73		4.35
265.15		173.28	
	2.54		4.51
		168.77	

TABLE III
Satellite branch in (0, 1) Band

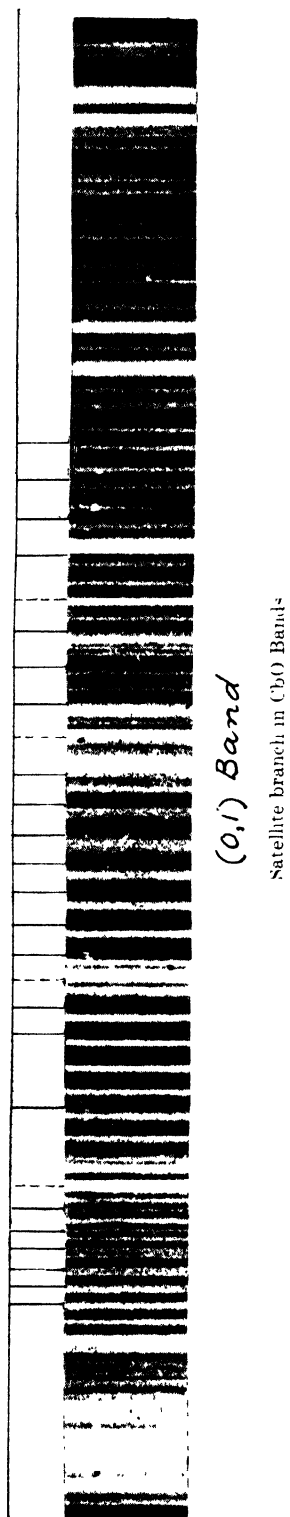
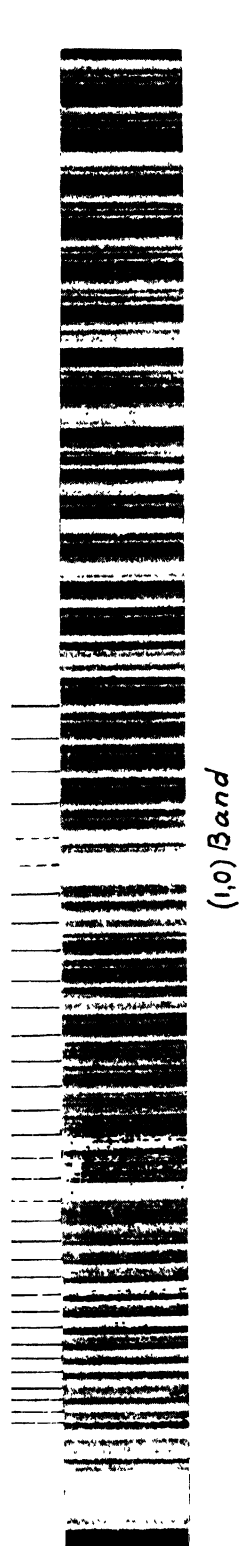
Wave number	First difference	Wave number	First difference
20323.53	1.39	20295.34	2.39
322.14	1.42	292.95	..
320.72	1.51
319.21	1.34	288.76	3.15
317.87	1.41	285.61	2.69
316.46At	1.67	282.92	2.51
314.79	..	280.41	2.71
..	..	277.70	2.93
310.18	..	274.77	2.75
..	..	272.02	2.48
..	..	269.54At	..
..
301.95	2.09	263.05	3.19
299.86	2.24	259.86	3.09
297.62	2.28	256.77	

show a gradual rise and fall in intensity. They occur in proximity to the members of the *P* and *R* branches. The interval from the respective *P* or *R* lines is found to be increasing as one proceeds away from the head. These could be grouped into a branch as shown in the Tables I, II and III, which give the wave numbers together with the first differences in the satellite branch in the three bands (1, 0), (0, 0) and (0, 1) respectively. These lines are shown marked for each band in Plate IX. The interpretation of the satellite branch is given in the discussion on the electronic transition.

ELECTRONIC TRANSITION

From the analysis given in the previous paper it is difficult to determine conclusively the electronic transition involved in the emission of this system. But a few inferences are drawn from the structure as far as it is identified and a general discussion is presented of various considerations bearing on the nature of this electronic transition. These considerations indicate that the transition is $\Sigma-\Sigma$ with a multiplicity of either two or a higher even value (probably four) for the two states.

As a first step, one may predict from theoretical considerations the possible electronic states of the molecule from the usual standpoints: either from the states of the separated atoms or from the probable electronic configurations of the molecule.



(1) The low-lying states of the columbium atom established by Meggers and Humphreys (1945) correspond to the configuration $4d^45s$ (ground configuration), giving the deepest term 6D and $4d^35s^2$, giving the term 4F ; oxygen has the normal configuration $2s^22p^4$ leading to the ground term 3P . In the formation of the molecule CbO from these two atoms, the following molecular states are obtained, taking $4d^45s$ as the normal configuration of columbium atom as definitely established by Meggers and Humphreys.

With the usual notation

$$S_1 = 5/2 \text{ and } S_2 = 1 \text{ so that } S = 7/2, 5/2, 3/2$$

$$\Lambda = 0, 1; {}^8\Sigma, {}^8\pi; {}^6\Sigma, {}^6\pi; {}^4\Sigma, {}^4\pi$$

$$= 0, 1, 2; {}^8\Sigma, {}^8\pi, {}^8\Delta; {}^6\Sigma, {}^6\pi, {}^6\Delta; {}^4\Sigma, {}^4\pi, {}^4\Delta$$

$$= 0, 1, 2, 3; {}^8\Sigma, {}^8\pi, {}^8\Delta, {}^8\phi; {}^6\Sigma, {}^6\pi, {}^6\Delta, {}^6\phi; {}^4\Sigma, {}^4\pi, {}^4\Delta, {}^4\phi$$

No doublet electronic states occur among the molecular states predicted above.

(2) Considering the electronic configuration of the molecule as a whole, the diatomic molecule CbO may give rise to the following states:—

$$\sigma^2\sigma^2\sigma^2\pi^4\pi \quad 2\pi \quad (1)$$

$$\sigma^2\sigma^2\sigma\pi^4\pi^2 \quad 4\Sigma \quad (2)$$

$$\sigma^2\sigma^2\sigma^2\pi^3\pi^2 \quad 4\pi \text{ or } 2\pi \quad (3)$$

$$\sigma^2\sigma^2\sigma\pi^4\pi\pi \quad 4\Sigma \quad (4)$$

and so on. Regarding the multiplicity of the expected electronic states it is evident that it must be even. The multiplicity may be two or four since higher multiplicity terms are not expected even from a general survey of the structure of the spectrum. But regarding the nature of the terms in the upper and lower states there are obviously several possibilities as noted from the above two methods.

On the basis of a resemblance between the electronic configurations of the CbO molecule and the yttrium atom (similar to what has been shown by Lowater (1929) to exist between TiO molecule and the calcium atom), Ramakrishna Rao (1950) tentatively assigned this system in CbO to the transition $4\pi-4\pi$. While in TiO, Christy (1929) suggested the transition $3\pi-3\pi$ from the rotational analysis, Ramakrishna Rao's conclusion about the system A of CbO as a $4\pi-4\pi$ transition was not based on rotational structure data.

Titanium and zirconium occur in the fourth left sub-group, vanadium and columbium in the fifth left sub-group and oxygen in the sixth right sub-group of the periodic table. A similarity may therefore be expected from the electronic transitions of titanium oxide and zirconium oxide and among vanadium oxide and columbium oxide. The latter may resemble the ionized molecule O_2^+ .

The rotational structure of the first and second negative bands of O_2^+ establishes the electronic transitions $2\pi_u-2\pi_g$ and $4\Sigma_g^--4\pi_u$ as characteristic of O_2^+ . If an analogy can be drawn between the three molecules which have the same number of outer electrons (11) one should expect the above transitions in vanadium oxide and columbium oxide also. The structure, relative intensities and energy terms in the case of band systems for a $4\Sigma-4\pi$ transition were calculated by Budo (1937) and also by Nevin (1938). As many as 48 branches may be expected from such a transition or at least 40, on account of the blending of the levels $F'_1(K)$ and $F'_4(K)$ and $F'_2(K)$ and $F'_3(K)$ of the 4Σ level. The existence of four components corresponding to the splitting in the 4π level should be a distinct feature of this system. Four such components were actually observed by Nevin at intervals of about 50 cm.^{-1} in the first negative bands of O_2^+ and each of these branches indicated also a separation of 0.45 cm.^{-1} associated with the spin fine structure of the 4Σ state.

Dealing with this band system of CbO, its analogy with O_2^+ led the writer in the early stages to detect, if possible, the existence of the structure described. Four components could not at all be identified. No doubt three closely situated heads right at the beginning of the band are observed in the earlier photographs but in the later higher resolution pictures, only one of these three heads appears to be a genuine one, the other two having a distinctly different, diffuse appearance, as described in the previous paper. Further, even three independent groups of branches could not be associated with these heads nor could an extensive Q branch be identified. The structure of the CbO bands on the whole was found relatively simpler than what might be expected of a $^4\Sigma-^4\pi$ transition.

A $^2\pi-^2\pi$ transition also does not seem to be consistent with the observed structure, as the expected structure for this transition for both case (a) and case (b) states can be predicted from the well-known structure of the β system of NO and the ultraviolet bands of O_2^+ . The appearance of double, double-headed system, characteristic of $^2\pi-^2\pi$ is not found in CbO. Further, a π state of any multiplicity whatsoever cannot perhaps be postulated at all since no evidence is found of Λ -type doubling even for high values of K , which normally varies quadratically with K .

For vanadium oxide Mahanti suggested $^2\Delta-^2\Delta$ as a probable transition concerned in the emission of the band system. This transition might be expected in the spectrum of CbO also since both occur in the same sub-group. But no positive evidence was given by Mahanti in support of his suggestion, such as the number of missing lines, the intensities or the variation of the R and P intervals with K . The last criterion has been studied by the author (unpublished) in VO and in CbO and the variation of the R and P intervals with K has been found to be distinctly linear in VO and nearly so in CbO (which can be seen to be the case theoretically for a $\Sigma-\Sigma$ transition), whereas for a $^2\Delta-^2\Delta$ transition one should expect, theoretically, a non-linear variation, as is observed in NiH bands (Pearse, 1935).

The possibility of ascribing this system (system A of CbO) to a $\Sigma-\Sigma$ transition in which both the Σ states are either doublets or quartets is next examined. It is well known that in a $^2\Sigma$ state there is a case (b) spin doubling of each rotational level but no Λ -type doubling. Six branches are to be expected from the selection rules— $P_1(J)$, $P_2(J)$, $^PQ_{12}(J)$, $^RQ_{21}(J)$, $R_1(J)$, $R_2(J)$. The first three branches lie close together and the last three form another close group. There will be a gap due to the absence of one line $K' = 0 \rightarrow K'' = 0$. The P and R branches are of relatively high and equal intensities, while the two satellite branches are feeble and rapidly fall off in intensity as K increases.

Comparing this structure with the one observed in CbO there is no doubt a good agreement as far as the number of the main branches is concerned. But the observed satellite branch or branches are relatively rather too strong for $^2\Sigma-^2\Sigma$ transition. Further, it can also be predicted that in a $^2\Sigma-^2\Sigma$ transition the relative disposition of the satellites with respect to the main branches will be as shown on page 248 (Herzberg, 1951). The $^PQ_{12}$ occurs to the violet side of the P lines while the $^RQ_{21}$ occurs on the red side of the R lines.

In the present case whether the electronic transition is $^2\Sigma-^2\Sigma$ cannot be definitely decided since (a) the two predicted satellite branches are not identified (only one satellite branch is observed and the other that is expected if the transition is $^2\Sigma-^2\Sigma$ may have been missed on account of superposition) and (b) the position of the observed satellite branch with respect to the main branch lines, that is, whether it occurs on the short wavelength or the long wavelength side of it, could not be ascertained because the numbering could not be definitely arrived at.

For examining the possibility of $^4\Sigma$ states a detailed study is made of this transition since in the existing literature there does not seem to be any band system so far observed that is ascribed to a $^4\Sigma-^4\Sigma$ transition. The energy level diagram and the expected branches are given in a previous paper (1954), which is drawn up on

the basis of the theoretical calculations of the energy expressions derived by Budo for the $^4\Sigma$ state. For this state F_1 and F_4 levels lie close together and similarly F_2 and F_3 are close so that the electronic state can be regarded in effect as consisting of only two levels. According to the J, K and $+$ \rightleftharpoons $-$ selection rules 18 branches will be obtained. They form into two groups, namely, an R -form group consisting of $R_1, R_4; R_2, R_3, {}^RQ_{32}; {}^RQ_{43}, {}^RP_{42}; {}^RQ_{21}, {}^RP_{31}$ and a P -form group made up of $P_1, P_4; P_2, P_3, {}^PQ_{23}; {}^PR_{24}, {}^PQ_{34}; {}^PQ_{12}, {}^PR_{13}$. No main Q branches will be obtained according to the $+$ \rightleftharpoons $-$ selection rule. On account of the blending of the levels in a $^4\Sigma$ state referred to above, the nine R form components and the nine P form ones may give rise to actually four branches in each group as shown below.

R form Group	P form Group
R_1, R_4	P_1, P_4
$R_2, R_3, {}^RQ_{32}$	$P_2, P_3, {}^PQ_{23}$
${}^RQ_{43}, {}^RP_{42}$	${}^PQ_{34}, {}^PR_{24}$
${}^RQ_{21}, {}^RP_{31}$	${}^PQ_{12}, {}^PR_{13}$

To indicate this on the diagram the branches which form a blend are marked close together. Effectively eight branches can be expected in a $^4\Sigma-^4\Sigma$ transition. Even this eight branch structure can perhaps be observed only for high K values. For the lowest K values all the R form lines may merge into one line and the P form lines into another line thus giving rise to just two series of strong lines near the head—an appearance that closely corresponds to the observed feature in all the three bands of CbO .

For high values of K , the R form and the P form blends may be separated from the respective main components and one or more satellite branches may be resolved depending on the resolution. Actual observation and identification has shown only one satellite branch and the numbering is not definite. Hence it is not possible to draw any definite conclusion as to whether the system belongs to the transition $^4\Sigma-^4\Sigma$ or not, from this standpoint. Regarding the nature of the observed satellite branch the following remarks may be made.

(a) Considering this in the first instance as an R -form satellite branch (either as ${}^RP_{31}({}^RQ_{21})$ or as ${}^RP_{42}({}^RQ_{43})$ by reference to the transition diagram of $^4\Sigma-^4\Sigma$ given in a previous paper) all possible sets of frequency differences with the four main branch lines are formed and compared with the combination differences already arrived at from the main branches for the upper and the lower states for checking up the combination relations which are as follows:

$$\begin{aligned}
 &{}^RP_{31}(K)({}^RQ_{21}) - {}^PR_{24}(K)({}^PQ_{34}) = R_2(K) - P_2(K) = \Delta_2 F'_2(K) \text{ or} \\
 &{}^RP_{42}(K)({}^RQ_{43}) - {}^PR_{13}(K)({}^PQ_{12}) = R_1(K) - P_1(K) = \Delta_2 F'_1(K) \text{ and} \\
 &{}^RP_{31}(K-1)({}^RQ_{21}) - {}^PR_{24}(K+1)({}^PQ_{34}) = R_1(K-1) - P_1(K+1) = \Delta_2 F'_1(K) \text{ or} \\
 &{}^RP_{42}(K-1)({}^RQ_{43}) - {}^PR_{13}(K+1)({}^PQ_{12}) = R_2(K-1) - P_2(K+1) = \Delta_2 F'_2(K)
 \end{aligned}$$

It is found that in all the three bands for a particular numbering of the satellite branch, the above relations are satisfied under the assumption that one of the four main branches has to be also regarded as a satellite branch (either as ${}^PR_{24}({}^PQ_{43})$ or as ${}^PR_{13}({}^PQ_{12})$ as the case may be). The K -numbering of the satellite branch determined from the above combination relations is given below for the three bands:

(1, 0) Band: Satellite branch $\nu 22147.29$ ($K = 32$) Main branch $\nu 22102.72$
 ($K = 32$) (on which a
 satellite branch is superposed to be superposed).

- (0, 0) Band: Satellite branch $\nu 21256.72$ ($K = 32$) Main branch $\nu 21302.79$ ($K = 32$) (on which a satellite branch is superposed to be superposed).
- (0, 1) Band: Satellite branch $\nu 20323.53$ ($K = 33$) Main branch $\nu 20276.58$ ($K = 33$) (on which a satellite branch is superposed to be superposed).

Thus, if the transition is a $^4\Sigma-^4\Sigma$, the satellite branches identified in the three bands may be either (1) $^R P_{31}$ ($^R Q_{21}$) in which case the satellite branch $^P R_{24}$ ($^P Q_{34}$) should be assumed to be superposed on a main branch, or (2) $^R P_{42}$ ($^R Q_{43}$) in which case the satellite branch $^P R_{13}$ ($^P Q_{12}$) should be assumed to be superposed on a main branch, with the numbering given above. But it has not been possible to decide between the two alternatives. Further it should be mentioned that, if this numbering is correct, these branches could be traced in all the three bands up to K values as high as nearly 60.

(b) The observed satellite branch may also be regarded as either $^P R_{24}$ ($^P Q_{34}$) or $^P R_{13}$ ($^P Q_{12}$) and attempts to check up the combination relations have not been possible with this interpretation because it is not possible to obtain the values of the combination differences for members corresponding to small K values. It may, however, be stated that on the assumption that this is a P -form R branch, the maximum observed value of K would be much less than 60.

The transition cannot be a $^6\Sigma-^6\Sigma$ or a $^8\Sigma-^8\Sigma$ since the number of main branches, besides the satellite branches, will be greater than the number of observed branches in CbO , in spite of the blending of levels with the same $|J-K|$ (six in the former and eight in the latter transition), whereas the number of observed branches is five out of which four are main branches. Even under a higher resolution than has been used in these investigations it is not likely that the number of branches may be greater than six at best.

These considerations indicate that the transition is $\Sigma-\Sigma$ with a multiplicity of either two or more probably four for the two states.

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ABSTRACT (SUMMARY)

A further study of sharply focussed photographs of the (1, 0), (0, 0) and (0, 1) bands of system A of columbium oxide led to the detection of a satellite branch in each band, in addition to the main P and R branches reported previously. A discussion is given of the electronic transition of the band system. This is indicated to be probably a $^4\Sigma-^4\Sigma$, corresponding to which there is no band system as yet known.

REFERENCES

- Budo, A. (1937). Rotationsstruktur von $^4\Sigma-^4\Sigma$ -Banden, *Zeit. f. Phys.*, **105**, 73.
 Herzberg, G. (1951). *Molecular Spectra and Molecular Structure*, Vol. I, Second Edition. *Diatomic Molecules*, 248.

- Humphreys, C. J. and Meggers, W. F. (1945). Term analyses of the first two spectra of columbium. *Bur. Stand. J. Res.*, **34**, 477.
- Lowater, F. (1929). The band system of titanium oxide. *Proc. Phy. Soc. (Lond.)*, **41**, 557.
- (1932). The band system of zirconium oxide. *Proc. Phy. Soc. (Lond.)*, **44**, 51.
- Mahanti, P. C. (1935). The band spectrum of vanadium oxide. *Proc. Phy. Soc. (Lond.)*, **47**, 433.
- Nevin, T. E. (1938). Rotational analysis of the first negative band spectrum of oxygen. *Phil. Trans. Roy. Soc.*, **237A**, 471.
- Pearse, R. W. B., and Gaydon, A. G. (1935). Band spectrum of nickel hydride. *Proc. Roy. Soc.*, **148**, 312.
- Ramakrishna Rao, V. (1950). The complex band spectrum of columbium oxide (The diatomic molecule CbO). *Ind. J. Phys.*, **24**, 35.
- Suryanarayana Rao, K. (1953). Intensities in $4\Sigma \rightarrow 4\Sigma$ transition. *Ind. J. Phys.*, **27**, 368.
- (1954). Rotational analysis of the columbium oxide bands. *Proc. Nat. Inst. Sci. (India)*. *In press*.

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ON THE SOLUTION OF THE SYSTEM OF EQUATIONS IN INTERNAL BALLISTICS

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1. INTRODUCTION

The aim of the present paper is to present a solution of the system of Equations in Internal Ballistics of a conventional gun in an improved and modified form over that given by Billard (1948). Billard has assumed that the shot-start pressure is zero. But with zero shot-start pressure, the results obtained do not tally with, rather are far from, the actual observed results. Hence the shot-start pressure is assumed not to be zero. However, the solution is applicable to those cases only in which the Charbonnier's Form-Function $\phi(z)$ —defined as the ratio of surface of emission of the propellant grain (or grains) at time t to the original surface of emission—can be expressed as $\sqrt{1-Kz}$. In most of the cases of propellants in Service use in India, the form-function $\phi(z)$ can be expressed, exactly or very nearly, in this form.

As is shown below, the Billard's Z -function, when the shot-start pressure is not zero, involves incomplete Beta-functions for which we shall require the 'Incomplete Beta-functions Tables' edited by Karl Pearson (1934). The labour involved is considerable. In this paper as far as the calculation of this particular Z -function is concerned, we have assumed that the shot-start pressure is zero and we have obtained the solution without the help of any table. It is found that the error involved thus is very small (of the order of less than 1.0% in most of the cases) and the working becomes shorter and easier.

For finding a first approximation to maximum pressure, the co-volume correction is neglected but by applying successive approximation method, the results can be obtained to any degree of accuracy. It has been found that the results obtained after two and three approximations do not generally differ from one another, at least up to three places of decimal; moreover, they closely tally with those obtained by G.M. II method.

Billard (1948) has used his method in the case of arms of small calibres in which the mean density of loading varies practically between 0.8 and 0.9. But it has been found that the method is equally applicable to the cases of arms of high calibres in which the mean density of loading may vary from 0.2 to 0.7.

The method presented here is different from that of Sugot described in *Internal Ballistics* (1951; pp. 103–105).

This paper is developed in the following way:—

- | | |
|-----------|---|
| Section 1 | . Introduction. |
| Section 2 | . Certain assumptions regarding combustion are made. |
| Section 3 | . Principal notations used are explained. |
| Section 4 | . The classical equations of Internal Ballistics are given. |
| Section 5 | . A solution of the above equations is obtained. |
| Section 6 | . An expression for maximum pressure is deduced. |
| Section 7 | . The position of all-burnt is discussed. |

- Section 8 .. Expressions for 'after the all-burnt position' (in particular for muzzle velocity) deduced.
- Section 9 .. Some applications of the method. Results for maximum pressure and muzzle velocity are obtained:—
- (i) by assuming different values for the shot-start pressure but using the same cordite;
 - (ii) with different types of propellants in cordite form;
 - (iii) with different types of propellants in tubular form;
 - (iv) with different web sizes of the same propellant, both in cordite and tubular forms.
- Section 10 .. The method is extended to investigate the variations in maximum pressure and muzzle velocity when a tubular propellant is inhibited on the outer surface or on the inner surface. For this purpose, it is shown first that in the different cases of inhibition, the form-function $\phi(z)$ can be expressed very nearly as $\sqrt{1-Kz}$.

2. ASSUMPTIONS REGARDING COMBUSTION

Burning of powder is a very complex phenomenon. With a view to getting a tangible solution, we shall make the following assumptions as done by Billard (1948) also:—

- (i) That all the grains of a charge are of the same geometrical form, of the same dimensions and of the same chemical composition.
- (ii) That all the grains of the charge are homogeneous in character.
- (iii) That all the grains are ignited at the same time throughout their surface and that they burn according to the Piobert's law, i.e. in parallel layers so that the surface always remains parallel to the initial surface at every instant of burning.
- (iv) That the velocity of linear combustion u , at any instant is proportional to the pressure p at that instant and to a certain parameter, dependent on the physico-chemical nature of the powder charge.
- (v) That the variations in temperature of the various adjacent layers produce negligible effects on the velocity of combustion.

3. PRINCIPAL NOTATIONS

The following principal notations have been used here:—

- F .. The Force Constant of the propellant charge.
- C .. The original mass of the propellant charge.
- A .. Area of the cross-section of the parallel portion of the bore including the area of the grooves when the bore is rifled. In the absence of a precise value, it is taken as $\frac{1}{4}\pi d^2 \times 1.02$, where d is the calibre of the bore.
- D .. The smallest linear dimension of a grain of the unburnt propellant.
- K_0 .. Cubic capacity of the chamber when the breech is closed and the projectile is at the original position.
- K_3 .. Total cubic capacity of the bore, including the chamber and the parallel portion.
- d .. Calibre, i.e. the diameter of the parallel portion of the bore.
- w .. Mass of projectile.
- w_1 .. Equivalent mass moved $= 1.06 w + \frac{C}{2}$.

- b .. Co-volume of the propellant gases.
 β .. Rate of burning coefficient.
 γ .. Ratio of specific heats of propellant gases.
 n .. $\frac{\gamma-1}{2}$.
 δ .. Density of solid propellant.
 θ .. Form coefficient of propellant.
 Δ .. Loading density = $\frac{C}{K_0}$ in metric units and = $27.68 \frac{C}{K_0}$ in British units when C is in lbs. and K_0 in cubic inches.
 Q .. A parameter called 'vivacity' or 'quickness' of the propellant
 $= \frac{\beta}{D}(1+\theta)$.
 l .. Equivalent length of initial air-space in chamber = $\frac{K_0 - C/\delta}{A}$.
 $'$.. $-\frac{nA^2}{Q^2 F C w_1}$.
 $\phi(z)$.. The Charbonnier Form-Function, defined as the ratio of surface of emission of a powder grain at any time t to the initial surface of emission of the grain = $\frac{s}{s_0}$.
 K .. Parameter of the form-function $\phi(z) = \sqrt{1 - Kz}$.
 λ .. $= \frac{1}{n(1+K/4r)}$.
 z .. Fraction of mass of charge burnt at time t .
 f .. Fraction of D remaining unburnt at time t .
 p .. Mean pressure of the propellant gases at time t .
 v .. Velocity of shot at time t .
 x .. Shot-travel at time t .
 ξ .. Volume of bore (including the chamber) up to the base of the projectile, i.e. = $Ax + K_0$, at time t .

Generally, suffix 0 indicates initial values when the shot starts; suffix 1, values when the pressure is maximum (except w_1); suffix 2, values at all-burnt position; suffix 3, values as the projectile passes the muzzle.

4. THE CLASSICAL EQUATIONS OF INTERNAL BALLISTICS

The Classical Equations of Internal Ballistics as given in *Internal Ballistics* (1951) are:—

$$p \left\{ A(x+l) - Cz \left(b - \frac{1}{\delta} \right) \right\} + \frac{1}{2}(\gamma-1)w_1 v^2 = FCz \quad \dots \quad (1)$$

$$w_1 v \frac{dv}{dx} = Ap \quad \dots \quad (2)$$

$$z = (1-f)(1+\theta f) \quad \dots \quad (3)$$

$$D \frac{df}{dt} = -\beta p \quad \dots \quad (4)$$

Equation (1) can easily be written as

$$p \left\{ \left(\xi - \frac{C}{\delta} \right) - \left(b - \frac{1}{\delta} \right) Cz \right\} + mw_1 v^2 = FCz \quad \dots \quad (1-i)$$

where

$$\xi = Ax + K_0.$$

From equations (3) and (4), we can deduce, provided θ is a constant, the Charbonnier-Schmitz equation

$$\begin{aligned} \frac{dz}{dt} &= \frac{\beta}{D(1+\theta)} \sqrt{1 - \frac{4\theta}{(1+\theta)^2} z} \cdot p \\ &= Q \cdot p \cdot \phi(z) \quad \dots \quad (5) \end{aligned}$$

where

$$K = \frac{4\theta}{(1+\theta)^2} \text{ and } \phi(z) = \sqrt{1 - Kz} \quad \dots \quad (5-i)$$

Thus the equations for solution become

$$p \left\{ \left(\xi - \frac{C}{\delta} \right) - \left(b - \frac{1}{\delta} \right) Cz \right\} + mw_1 v^2 = FCz \quad \dots \quad (6)$$

$$w_1 \frac{dv}{dt} = Ap \quad \dots \quad (7)$$

and

$$\frac{dz}{dt} = Q \cdot p \cdot \phi(z) \quad \dots \quad (8)$$

5. SOLUTION OF THE EQUATIONS

Case I :—When $K \neq 0$ and $z_0 \neq 0$.

From equation (5),

$$\int_{z_0}^z \frac{dz}{\phi(z)} = Q \int_{t_0}^t p \, dt = X, \text{ say.}$$

Thus

$$\begin{aligned} X &= \int_{z_0}^z \frac{dz}{\sqrt{1 - Kz}} \\ &= \frac{2}{K} \left[K' - \sqrt{1 - Kz} \right] \quad \dots \quad (9) \end{aligned}$$

where

$$K' = \sqrt{1 - Kz_0} \quad \dots \quad (9-i)$$

From (7) and (8), we get

$$\begin{aligned} \frac{dz}{dv} &= \frac{Qw_1}{A} \phi(z) \\ \therefore X &= \int_{z_0}^z \frac{dz}{\phi(z)} = \frac{Qw_1}{A} \int_0^v dv \\ \text{i.e., } X &= \frac{Qw_1 v}{A} \quad \dots \quad (10) \end{aligned}$$

and
$$\frac{w_1 v^2}{F K_0} = \frac{C}{n K_0} r X^2 \quad \dots \dots \dots (10-i)$$

giving v as a function of z and z_0 .

From (6), we get

$$\frac{p}{F} = \frac{z - r X^2}{\left(\frac{\xi}{C} - \frac{1}{\delta}\right) - \left(b - \frac{1}{\delta}\right)z} \quad \dots \dots \dots (11)$$

giving p as a function of z , z_0 and shot-travel x .

From (7) or (2), we get

$$w_1 v \frac{dv}{dx} = A p$$

which with the help of (10) and (11) gives the following linear differential equation of the first order:—

$$\frac{d}{dz} \left[\left(\frac{\xi}{C} - \frac{1}{\delta} \right) - \left(b - \frac{1}{\delta} \right) z \right] - \frac{r}{n} \frac{X}{(z - r X^2) \phi(z)} \cdot \left\{ \left(\frac{\xi}{C} - \frac{1}{\delta} \right) - \left(b - \frac{1}{\delta} \right) z \right\} = - \left(b - \frac{1}{\delta} \right) \quad \dots \dots \dots (12)$$

The integrating factor for (12) is

$$Y = e^{-\frac{r}{n} \int_{z_0}^z \frac{X}{(z - r X^2) \phi(z)} \cdot dz}$$

where
$$\log Y = -\frac{r}{n} \int_{z_0}^z \frac{X}{(z - r X^2) \phi(z)} dz \quad \dots \dots \dots (13)$$

$$= \log \left(1 - \frac{K' - \sqrt{1 - Kz}}{a} \right)^{a\lambda_1} \left(1 + \frac{K' - \sqrt{1 - Kz}}{b} \right)^{b\lambda_1}$$

Hence
$$Y = \left(1 - \frac{K' - \sqrt{1 - Kz}}{a} \right)^{a\lambda_1} \left(1 + \frac{K' - \sqrt{1 - Kz}}{b} \right)^{b\lambda_1} \quad (14)$$

where
$$a - b = \frac{2K' \cdot \frac{K}{4r}}{1 + \frac{K}{4r}}; \quad ab = \frac{\frac{K}{4r} \cdot Kz_0}{1 + \frac{K}{4r}} \quad \dots \dots \dots (14-i)$$

i.e.
$$a = \frac{\frac{K}{4r}}{1 + \frac{K}{4r}} [\sqrt{1 + 4rz_0} + \sqrt{1 - Kz_0}] \quad \dots \dots \dots (14-ii)$$

$$b = -\frac{\frac{K}{4r}}{1 + \frac{K}{4r}} [\sqrt{1 + 4rz_0} - \sqrt{1 - Kz_0}] \quad \dots \dots \dots (14-iii)$$

and
$$\lambda_1 = \frac{\lambda}{a + b} \quad \dots \dots \dots (14-iv)$$

and from equations (14-ii), (14-iii) and (14-iv), it follows that

$$a\lambda_1 = \frac{\lambda}{2} \left[1 + \frac{\sqrt{1-Kz_0}}{\sqrt{1+4rz_0}} \right] \text{ and } b\lambda_1 = \frac{\lambda}{2} \left[1 - \frac{\sqrt{1-Kz_0}}{\sqrt{1+4rz_0}} \right] \quad \dots (14-v)$$

Clearly $Y = 1$ initially, since $z = z_0$.

The solution of (12) is then

$$\left[\left(\frac{\xi}{C} - \frac{1}{\delta} \right) - \left(b - \frac{1}{\delta} \right) z \right] Y - \left[\left(\frac{K_0}{C} - \frac{1}{\delta} \right) - \left(b - \frac{1}{\delta} \right) z_0 \right] = - \left(b - \frac{1}{\delta} \right) Z \quad \dots (15)$$

where

$$Z = \int_{z_0}^z Y dz. \quad \dots \dots \dots (15-i)$$

Thus

$$Z = \frac{2}{K} a^{-a\lambda_1} b^{-b\lambda_1} \int_0^\alpha (a-\alpha)^{a\lambda_1} (b+\alpha)^{b\lambda_1} (K'-\alpha) d\alpha$$

where

$$\alpha = K' - \sqrt{1-Kz}.$$

If

$$\alpha = a - (a-b)u,$$

$$\begin{aligned} Z = \frac{2}{K} a^{-a\lambda_1} b^{-b\lambda_1} (a+b)^{\lambda+1} & \left[a' B_u \{ (a\lambda_1+1), (b\lambda_1+1) \} \right. \\ & - a' B_{\frac{a}{a+b}} \{ (a\lambda_1+1), (b\lambda_1+1) \} - (a+b) B_u \{ (a\lambda_1+2), (b\lambda_1+1) \} \\ & \left. + (a+b) B_{\frac{a}{a+b}} \{ (a\lambda_1+2), (b\lambda_1+1) \} \right] \quad \dots \dots \dots (16) \end{aligned}$$

where

$$u = \frac{a-\alpha}{a+b} = \frac{a - \{ K' - \sqrt{1-Kz} \}}{a+b}; a' = a - K' \quad \dots (16-i)$$

and $B_u(l, m)$ is the Incomplete Beta-function defined as

$$B_u(l, m) = \int_0^u u^{l-1} (1-u)^{m-1} du,$$

which can be evaluated, though after some labour, with the help of 'Incomplete Beta-Function Tables' edited by Karl Pearson (1934).

Thus the equations (15), (10) and (11) give the shot-travel $x \left(= \frac{\xi - K_0}{A} \right)$, velocity v and pressure p in terms of z and z_0 as follows :—

$$\left(\frac{\xi}{C} - \frac{1}{\delta} \right) - \left(b - \frac{1}{\delta} \right) z = \frac{\left(\frac{K_0}{C} - \frac{1}{\delta} \right) - \left(b - \frac{1}{\delta} \right) (Z+z_0)}{Y} \quad \dots \dots (17)$$

$$v = \frac{A}{Qw_1} \cdot X \quad \dots \dots \dots (18)$$

and

$$\frac{p}{F} = \frac{(z-rX^2)Y}{\left(\frac{K_0}{C} - \frac{1}{\delta} \right) - \left(b - \frac{1}{\delta} \right) (Z+z_0)} \quad \dots \dots (19)$$

where X , Y and Z are given in terms of z and z_0 by (9), (14) and (16).

The value of z_0 in terms of shot-start pressure p_0 is given by (1) or (19) as

$$z_0 = \frac{\frac{p_0}{F} \left(\frac{K_0}{C} - \frac{1}{\delta} \right)}{1 + \frac{p_0}{F} \left(b - \frac{1}{\delta} \right)} \quad \dots \dots \dots (20)$$

Case II:—When $K = 0$ and $z_0 \neq 0$.

In this case, equations (5-i), (9), (14) and (16) for determining X , Y and Z become

$$\phi(z) = 1 \quad \dots \dots \dots (21)$$

$$X = z - z_0; \quad K' = 1 \quad \dots \dots \dots (22)$$

$$Y = \left(1 + \frac{X}{a^*} \right)^{a^* \lambda^*} \left(1 - \frac{X}{b^*} \right)^{b^* \lambda^*} \quad \dots \dots \dots (23)$$

$$Z = (a^*)^{-a^* \lambda^*} (b^*)^{-b^* \lambda^*} (a^* + b^*)^{\frac{n+1}{n}} \left[B \frac{b^*}{a^* + b^*} \{ (b^* \lambda^* + 1), (a^* \lambda^* + 1) \} \right. \\ \left. - B \frac{b^* - (z - z_0)}{a^* + b^*} \{ (b^* \lambda^* + 1), (a^* \lambda^* + 1) \} \right] \quad \dots \quad (24)$$

where

$$\left. \begin{aligned} a^* b^* &= \frac{z_0}{r}, \quad b^* - a^* = \frac{1}{r} \\ a^* &= \frac{\sqrt{1 + 4rz_0} - 1}{2r} \\ b^* &= \frac{\sqrt{1 + 4rz_0} + 1}{2r} \\ \text{and} \quad \lambda^* &= \frac{1}{n(a^* + b^*)} \end{aligned} \right\} \quad \dots \dots \dots (24-i)$$

Thus equations (17), (18) and (19) give shot-travel x , velocity v and pressure p in terms of z , z_0 and X , Y , Z where X , Y , Z are given in terms of z and z_0 by (22), (23) and (24).

Case III:—When $K \neq 0$ and $z_0 = 0$.

Here the equations (9), (14) and (16) giving X , Y , Z in terms of z become

$$X = \frac{2}{K} [1 - \sqrt{1 - Kz}] \quad \dots \dots \dots (25)$$

$$Y = \left(\frac{a - 1 + \sqrt{1 - Kz}}{a} \right)^\lambda \quad \dots \dots \dots (26)$$

$$\text{where} \quad a = \frac{K/2r}{1 + K/4r} \quad \dots \dots \dots (26-i)$$

$$Z = \frac{(\gamma - 1)\lambda}{2r(\lambda + 1)(\lambda + 2)} \left[\gamma\lambda - Y^{\frac{\lambda+1}{\lambda}} \{ (\lambda + 1)\sqrt{1 - Kz} - 1 + (\gamma - 1)\lambda \} \right] \quad \dots \quad (27).$$

and the equations (17), (18) and (19) for determining the shot-travel x , velocity v and pressure p become

$$\left(\frac{x}{C} - \frac{1}{\delta}\right) - \left(b - \frac{1}{\delta}\right)z = \frac{\left(\frac{K_0}{C} - \frac{1}{\delta}\right) - \left(b - \frac{1}{\delta}\right)Z}{Y} \quad \dots \quad (28)$$

$$v = \frac{A}{Qw_1} \cdot X \quad \dots \quad (29)$$

$$\text{and} \quad \frac{p}{F} = \frac{(z - rX^2)Y}{\left(\frac{K_0}{C} - \frac{1}{\delta}\right) - \left(b - \frac{1}{\delta}\right)Z} \quad \dots \quad (30)$$

This is the case considered by Billard (1948).

Case IV:—When $K = 0$ and $z_0 = 0$.

The equations for determining, X , Y , Z become

$$X = z \quad \dots \quad (31)$$

$$Y = (1 - rz)^{\frac{1}{n}} \quad \dots \quad (32)$$

$$\left. \begin{aligned} Z &= \frac{n}{r(n+1)} \left[1 - (1 - rz)^{\frac{n+1}{n}} \right] \\ &= \frac{1}{r} \cdot \frac{\gamma - 1}{\gamma + 1} \left[1 - Y^{\frac{\gamma+1}{2}} \right] \end{aligned} \right\} \quad \dots \quad (33)$$

The values of the shot-travel x , velocity v and pressure p at any instant are then determined by equations (28), (29) and (30).

6. DETERMINATION OF MAXIMUM PRESSURE

At the position of maximum pressure, equation (19) gives

$$\frac{p_1}{F} = \frac{(z_1 - rX_1^2)Y_1}{\left(\frac{K_0}{C} - \frac{1}{\delta}\right) - \left(b - \frac{1}{\delta}\right)(Z_1 + z_0)} \quad \dots \quad (34)$$

(the suffix 1 denoting the values of the variables at the position of maximum pressure).

Also differentiating (19), putting $dp = 0$ and with the help of (12) and (34), we get

$$\frac{\gamma - 1}{2\gamma} \left[1 + \left(b - \frac{1}{\delta}\right) \frac{p_1}{F} \right] = r \frac{X_1}{\phi_1} \quad \dots \quad (35)$$

From (34) and (35), we get

$$r \frac{X_1}{\phi_1} = \frac{\gamma - 1}{2\gamma} \left[1 + \frac{\left(b - \frac{1}{\delta}\right)(z_1 - rX_1^2)Y_1}{\left(\frac{K_0}{C} - \frac{1}{\delta}\right) - \left(b - \frac{1}{\delta}\right)(Z_1 + z_0)} \right] \quad \dots \quad (35A)$$

This equation determines the value z_1 of z at which maximum pressure occurs. For this purpose, a graphical method can be applied. A first approximation to z_1 is obtained by neglecting the term containing $\left(b - \frac{1}{\delta}\right)$ in (35A); three or four values of z in the neighbourhood of this z_1 , are taken; the values of X_1 , ϕ_1 , Y_1 , and Z_1 corresponding to these values of z , are calculated and then the curves

$$f_1(z) = r \frac{X_1}{\phi_1}$$

$$\text{and } f_2(z) = \frac{\gamma-1}{2\gamma} \left[1 + \frac{\left(b - \frac{1}{\delta}\right) \left(z_1 - rX_1^2\right) Y_1}{\left(\frac{K_0}{C} - \frac{1}{\delta}\right) - \left(b - \frac{1}{\delta}\right) \left(Z_1 + z_0\right)} \right]$$

are plotted. The intersection of these two curves would give us the required value of z_1 , corresponding to which the values of X_1 , Y_1 and Z_1 are calculated and then the value of p_1 is determined from equation (34).

A practical method for Maximum Pressure.—A practical and easier method for calculating the maximum pressure to any degree of accuracy, without resorting to a graphical method, is to use the method of 'successive approximation'. Calculations have been made using this approximation method and the results obtained thereby for second approximation tally closely with the actual observed results and with the results obtained by the other methods, say, for example, G.M.II method. Moreover the results of second and third approximations do not generally differ from one another up to third place of decimal.

The method can be explained as follows:—

(i) To find a first approximation to z_1 , we neglect the co-volume term $\left(b - \frac{1}{\delta}\right) \frac{p_1}{F}$ in (35) and get

$$r \frac{X_1}{\phi_1} = \frac{\gamma-1}{2\gamma} \dots \dots \dots (36)$$

giving when $K \neq 0$,

$$z_1 = \frac{1}{K} \left[1 - \left\{ \frac{K'}{1 + \frac{\gamma-1}{\gamma} \cdot \frac{K}{4r}} \right\}^2 \right] \dots \dots \dots (37-i)$$

and when $K = 0$,

$$z_1 = z_0 + \frac{\gamma-1}{2\gamma} \cdot \frac{1}{r} \dots \dots \dots (37-ii)$$

(ii) The values of X_1 and Y_1 are then calculated from equations (9) and (14) when $K \neq 0$ and from equations (22) and (23) when $K = 0$.

(iii) The value of Z_1 is calculated with the help of (27) instead of (16) when $K \neq 0$ and with the help of (33) instead of (24) when $K = 0$.

It has been found that this supposition of $z_0 = 0$ in the determination of Z_1 only, produces a very small error in the final result but decreases the labour for calculating the value of Z_1 , to a great extent.

(iv) Then the value $p_{1,1}$ of p_1 , to the first approximation, is determined from (34).

(v) To find a second approximation to z_1 , the value $p_{1,1}$ of p_1 obtained in (iv) is substituted in (35); the value of z_1 is again determined from

$$z_1 = \frac{1}{K} \left[1 - \left\{ \frac{K'}{1 + \frac{\gamma-1}{\gamma} \cdot \frac{K}{4r} (1+E)} \right\}^2 \right] \quad \dots \quad (37\text{-iii})$$

when $K \neq 0$,

$$\text{and from} \quad z_1 = z_0 + \frac{\gamma-1}{2\gamma} \cdot \frac{1}{r} (1+E) \quad \dots \quad (37\text{-iv})$$

when $K = 0$,

$$\text{where} \quad E = \left(b - \frac{1}{\delta} \right) \frac{p_{1,1}}{F} \quad \dots \quad (37\text{-v})$$

New values of X_1 , Y_1 and Z_1 corresponding to the new value of z_1 , are calculated as previously and finally the value of p_1 .

The determination of ξ_1 , (giving the position of the shot) and v_1 , at the instant when the maximum pressure occurs presents no difficulty as they can be determined from (17) and (18), once the values of z_1 , X_1 , Y_1 and Z_1 are determined.

Restriction on the values of r .

From equation (37-iii) it is obvious that since $z_1 \leq 1$,

$$\left. \begin{aligned} r &> \frac{\gamma-1}{4\gamma} \left[\frac{K\sqrt{1-K}}{\sqrt{1-Kz_0} - \sqrt{1-K}} \right] (1+E) \\ &\quad \text{when } K > 0. \\ \text{Similarly from (37-iv),} \\ r &> \frac{\gamma-1}{2\gamma} \cdot \frac{1+E}{1-z_0}, \quad \text{when } K = 0. \end{aligned} \right\} \quad \dots \quad (37\text{-vi})$$

For values of r less than this value, maximum pressure occurs at the position of all-burnt. For the first approximation, however, E is taken as equal to zero.

7. POSITION OF ALL-BURNT

At the position of all-burnt, $z = 1$; then we have from (17), (18) and (19) the following formulae giving the shot-travel x_2 , velocity v_2 and pressure p_2 :-

$$\left(\frac{\xi_2}{C} - b \right) = \frac{\left(\frac{K_0}{C} - \frac{1}{\delta} \right) - \left(b - \frac{1}{\delta} \right) (Z_2 + z_0)}{Y_2} \quad \dots \quad (38)$$

$$v_2 = \frac{A}{Qw_1} \cdot X_2 \quad \dots \quad (39)$$

$$\frac{p_2}{F} = \frac{(1-rX_2^2)Y_2}{\left(\frac{K_0}{C} - \frac{1}{\delta} \right) - \left(b - \frac{1}{\delta} \right) (Z_2 + z_0)} = \frac{1-rX_2^2}{\left(\frac{\xi_2}{C} - b \right)} \quad \dots \quad (40)$$

where X_2 , Y_2 and Z_2 are determined from (9), (14) and (16) when $K \neq 0$ as follows:—

$$X_2 = \frac{2}{K} \left[K' - \sqrt{1-K} \right] \dots \dots \dots (41)$$

$$Y_2 = \left[1 - \frac{K' - \sqrt{1-K}}{a} \right]^{a\lambda_1} \left[1 + \frac{K' - \sqrt{1-K}}{b} \right]^{b\lambda_1} \dots \dots (42)$$

$$\begin{aligned} Z_2 = \frac{2}{K} a^{-a\lambda_1} b^{-b\lambda_1} (a+b)^{\lambda+1} & \left[a' B_{u_2}(\overline{a\lambda_1+1}, \overline{b\lambda_1+1}) \right. \\ & - a' B_{a/a+b}(\overline{a\lambda_1+1}, \overline{b\lambda_1+1}) - (a+b) B_{u_2}(\overline{a\lambda_1+2}, \overline{b\lambda_1+1}) \\ & \left. + (a+b) B_{a/a+b}(\overline{a\lambda_1+2}, \overline{b\lambda_1+1}) \right] \dots \dots \dots (43) \end{aligned}$$

and when $K = 0$, these quantities are determined from equations (22), (23) and (24) by putting $z = 1$.

For the determination of Z_2 , we may, however, use the formula (27) when $K \neq 0$ and formula (33) when $K = 0$, as explained in the last section.

Then

$$Z_2 = \frac{(\gamma-1)\lambda}{2r(\lambda+1)(\lambda+2)} \left[\gamma\lambda - Y_2^{\frac{\lambda+1}{\lambda}} \{(\lambda+1)\sqrt{1-K} - 1 + (\gamma-1)\lambda\} \right] \dots (44)$$

when $K \neq 0$,

$$\text{and} \quad = \frac{1}{r} \cdot \frac{\gamma-1}{\gamma+1} \left[1 - Y_2^{\frac{\gamma+1}{2}} \right], \quad \text{when } K = 0. \quad \dots \dots (44A)$$

8. AFTER THE ALL-BURNT POSITION

Since after the all-burnt position, $z = 1$, the equation (11) gives

$$\begin{aligned} \frac{p}{F} &= \frac{1-rX^2}{\left(\frac{\xi}{C} - b \right)} \\ \therefore \frac{p}{p_2} &= \frac{1-rX^2}{1-rX_2^2} \left(\frac{\frac{\xi_2}{C} - b}{\frac{\xi}{C} - b} \right) \dots \dots \dots (45) \end{aligned}$$

But after the position of all-burnt, the expansion may be regarded as very nearly adiabatic.

$$\therefore p \left(\frac{\xi}{C} - b \right)^\gamma = p_2 \left(\frac{\xi_2}{C} - b \right)^\gamma$$

or

$$\frac{p}{p_2} = \frac{1-rX^2}{1-rX_2^2} \cdot \left(\frac{\frac{\xi_2}{C} - b}{\frac{\xi}{C} - b} \right) = \left(\frac{\frac{\xi_2}{C} - b}{\frac{\xi}{C} - b} \right)^\gamma \dots \dots (46)$$

whence

$$\frac{1-rX^2}{1-rX_2^2} = \left(\frac{\frac{\xi_2}{C} - b}{\frac{\xi}{C} - b} \right)^{\gamma-1} \quad \dots \quad \dots \quad \dots \quad (46-i)$$

Also from (10-i),

$$1-rX^2 = 1 - \frac{nw_1}{FC} v^2. \quad \dots \quad \dots \quad \dots \quad (47)$$

Hence after the position of all-burnt, the velocity v of the projectile and pressure p at any instant are given by

$$\left(1 - \frac{nw_1}{FC} v^2 \right) = (1-rX_2^2) \left(\frac{\frac{\xi_2}{C} - b}{\frac{\xi}{C} - b} \right)^{\gamma-1} \quad \dots \quad \dots \quad (48)$$

and

$$\frac{p}{F} = (1-rX_2^2) \frac{\left(\frac{\xi_2}{C} - b \right)^{\gamma-1}}{\left(\frac{\xi}{C} - b \right)^{\gamma}} \quad \dots \quad \dots \quad \dots \quad (49)$$

Hence the muzzle velocity v_3 is given as

$$v_3^2 = \frac{FC}{nw_1} \left[1 - (1-rX_2^2) \left(\frac{\frac{\xi_2}{C} - b}{\frac{\xi_3}{C} - b} \right)^{\gamma-1} \right]. \quad \dots \quad \dots \quad (50)$$

This formula resembles in form with Sugot's formula given in *Internal Ballistics* (1951; p. 105).

9. SOME APPLICATIONS

(i) Calculations for muzzle velocity and maximum pressure have been made by using the above formulae with different shot-start pressures p_0 in the case of 6" B.L. gun Mk VII with the following data:—

Propellant charge SC 103 cordite; charge weight = 23.31 lbs.

$$F = 1964; \quad \beta = 0.997; \quad \gamma = 1.249;$$

$$\frac{1}{\delta} = 17.64; \quad b - \frac{1}{\delta} = 8.26;$$

$$w_1 = 113.77 \text{ lbs.}; \quad \frac{K_0}{C} = 73.788074.$$

The results of the calculations have been shown in Table I.

Corresponding results, obtained by G.M.II method, are also given for a comparative study. In all the calculations, velocity and pressure have been given in feet/sec. and tons/sq. in. respectively.

TABLE I

p_0	G.M. II Method		Present Method		
	M.V.	Max. Pr.	M.V.	Max. Pressure	
				1st approximation	2nd approximation
0	2536.67	17.4193	2539.43	17.5175	17.53275
0.5	2552.80	18.0788	2558.10	18.1336	18.15087
1.0	2564.55	18.6656	2571.20	18.6366	18.65552
1.5	2573.97	19.2010	2582.59	19.1010	19.12877
2.0	2582.80	19.6990	2592.72	19.5436	19.56541

(ii) In Table II, results for a comparative study of maximum pressure and muzzle velocity for different propellants, SC, AN, WM and W, are given using the same gun data of 6" B.L. gun, the same propellant mass 23.31 lbs. and the same web size $D = 0.103$ ", for the cordite shape. In this as well as in the following tables, results for maximum pressure have been given up to second approximation and the required data for AN, WM and W propellants have been taken from *Internal Ballistics* (1951).

TABLE II

 $(\theta = 1)$

p_0	SC Cordite		AN Cordite		WM Cordite		W Cordite	
	M.V.	Max. Pr.	M.V.	Max. Pr.	M.V.	Max. Pr.	M.V.	Max. Pr.
0.5	2558.1	18.1509	2326.9	12.6906	2640.1	21.5718	2627.4	20.3975
1.0	2571.2	18.6555	2349.8	13.2325	2650.0	22.0938	2638.5	20.8835
1.5	2582.6	19.1288	2369.0	13.7339	2658.6	22.4801	2648.1	21.3334

(iii) Table III gives the M.V. and Max. Pr. for tubular shape ($\theta = 0$) of the same web size $D = 0.075$ " and of the same mass 23.31 lbs. for different types of propellants, supposing $p_0 = 1.5$ tons/sq. in. with the same gun 6" B.L. Mk VII.

TABLE III

 $(\theta = 0)$

Propellant	M.V. (ft./sec.)	Max. Pr. (tons/sq. in.)
SC	2529.6	17.82020
AN	2293.3	10.82501
WM	2614.7	22.46612
W	2600.1	20.86682

(iv) Table IV gives the results for max. pressure and muzzle velocity for different web sizes but the same mass 23.31 lbs. of SC cordite on the assumption of shot-start pressure $p_0 = 1.5$ tons/sq. in. while Table V gives the same results for SC/T propellant for another set of web size, using the same 6" B.L. gun.

TABLE IV

 $(\theta = 1)$

Web size D''	M.V. (ft./sec.)	Max. Pr. (tons/sq. in.)
0.100	2592.9	19.71457
0.102	2586.0	19.31706
0.103	2582.6	19.12877
0.104	2579.1	18.92925
0.106	2571.8	18.55102

TABLE V

 $(\theta = 0)$

Web size D''	M.V. (ft./sec.)	Max. Pr. (tons/sq. in.)
0.070	2557.3	19.85329
0.072	2546.3	19.03218
0.074	2534.9	18.20765
0.075	2529.6	17.82019
0.076	2523.0	17.42127
0.078	2510.7	16.63204
0.080	2497.8	15.85624
0.082	2484.5	15.12398
0.103	2307.0	10.01940

The results for Tables I, II, IV and V are shown graphically also.

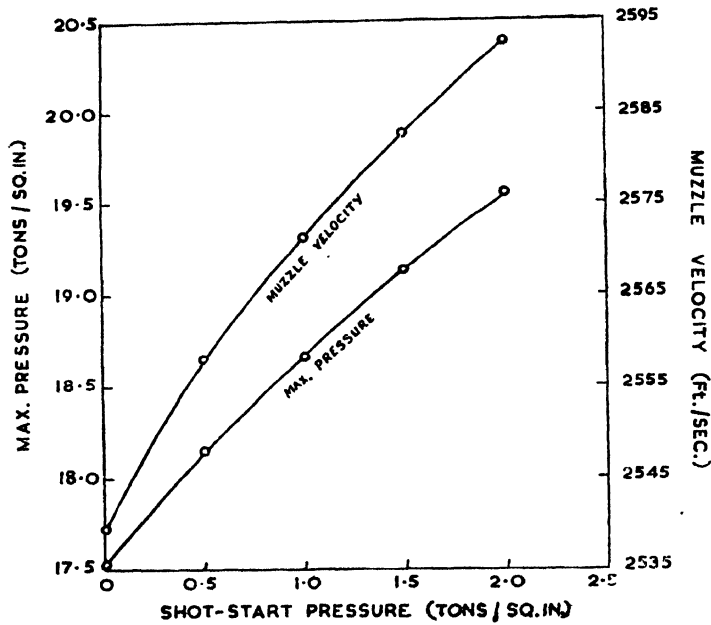


FIG. (TABLE I)

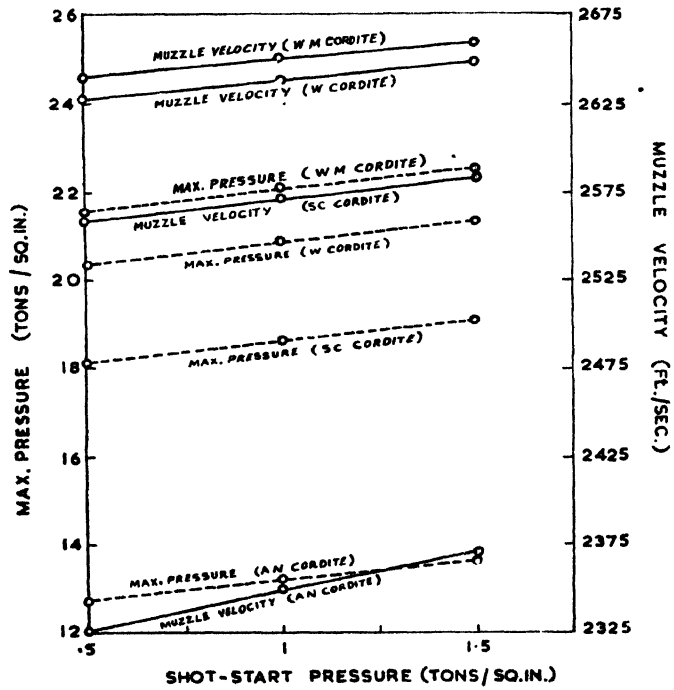


FIG. (TABLE II)

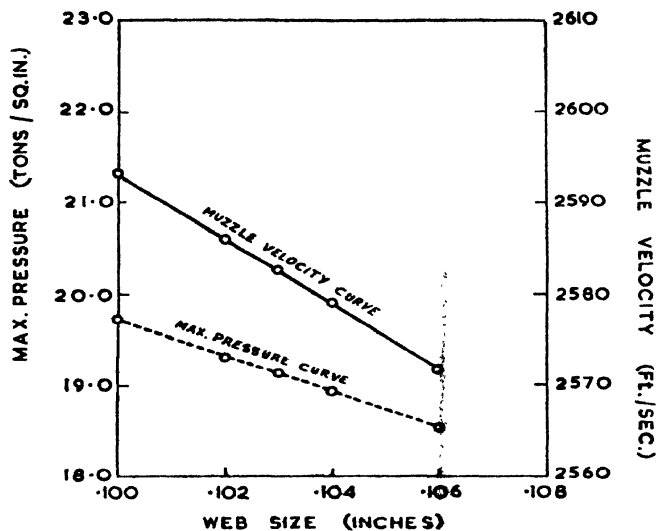


FIG. (TABLE IV)

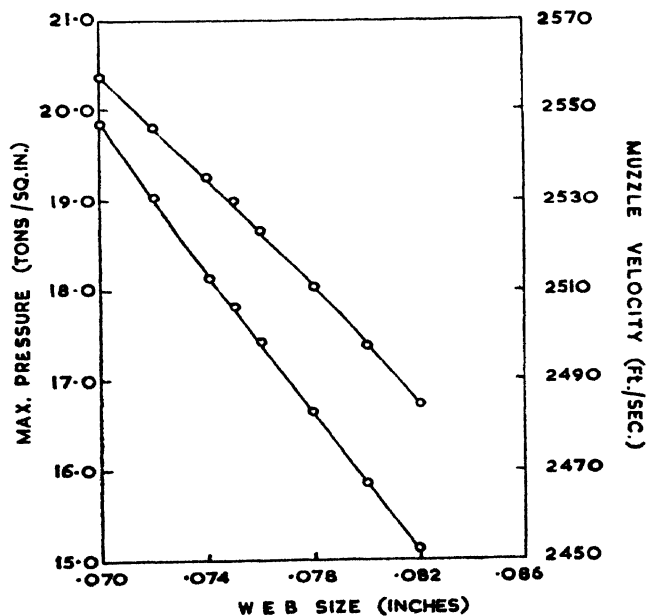


FIG. (TABLE V)

10. EFFECT OF INHIBITION ON A TUBULAR PROPELLANT

Inhibited propellants have been found very useful, specially in rockets; it is therefore worth studying the effect of inhibiting (coating) a propellant. Here we shall consider the case of Tubular Propellants under three heads: (i) uninhibited, (ii) inhibited inside, (iii) inhibited outside.

(i) *Tubular propellant (uninhibited):*—

Let the initial dimensions of a propellant grain be

r_0 = internal radius, R_0 = external radius, H_0 = length.

At the end of a certain time t , let a thickness ϵ be burnt; then the dimensions are

$$R = R_0 - \epsilon, \quad r = r_0 + \epsilon, \quad H = H_0 - 2\epsilon.$$

Then if v_0 , v , s_0 and s represent the original volume, volume at time t , original surface of emission and the surface of emission at time t respectively, we have

$$v_0 = \pi(R_0^2 - r_0^2)H_0$$

$$v = \pi(R_0 - r_0 - 2\epsilon)(R_0 + r_0)(H_0 - 2\epsilon)$$

$$s_0 = 2\pi(R_0 + r_0)(H_0 + R_0 - r_0)$$

$$s = 2\pi(R_0 + r_0)(H_0 + R_0 - r_0 - 4\epsilon)$$

$$\begin{aligned} \therefore z &= \frac{v_0 - v}{v_0} \\ &= \frac{2(H_0 + R_0 - r_0)}{H_0(R_0 - r_0)} \epsilon - \frac{4}{H_0(R_0 - r_0)} \epsilon^2 \quad \dots \quad \dots \quad (51) \end{aligned}$$

$$\text{and} \quad \phi(z) = \frac{s}{s_0} = 1 - \frac{4}{H_0 + R_0 - r_0} \epsilon. \quad \dots \quad \dots \quad \dots \quad (52)$$

Eliminating ϵ between (51) and (52), we get

$$\phi(z) = \sqrt{1 - Kz} \quad \dots \quad \dots \quad \dots \quad \dots \quad (53)$$

$$\text{where} \quad K = \frac{4H_0(R_0 - r_0)}{(H_0 + R_0 - r_0)^2} \quad \dots \quad \dots \quad \dots \quad \dots \quad (54)$$

Thus in this case, $\phi(z)$ can be expressed exactly as $\sqrt{1 - Kz}$, a result which is generally found in a number of text-books and also given by Billard (1948).

(ii) *Tubular propellants inhibited inside.*

Taking the same initial dimensions, here we shall have

$$R = R_0 - \epsilon, \quad H = H_0 - 2\epsilon, \quad r = r_0 = \text{constant}.$$

Then

$$v_0 = \pi(R_0 + r_0)(R_0 - r_0)H_0$$

$$v = \pi[(R_0^2 - r_0^2)H_0 - 2\epsilon(R_0H_0 + R_0^2 - r_0^2) + \epsilon^2(H_0 + 4R_0) - 2\epsilon^3]$$

$$s_0 = 2\pi[R_0H_0 + R_0^2 - r_0^2]$$

$$s = 2\pi[(R_0H_0 + R_0^2 - r_0^2) - \epsilon(H_0 + 4R_0) + 3\epsilon^2]$$

$$\therefore z = \frac{2(R_0H_0 + R_0^2 - r_0^2)}{H_0(R_0^2 - r_0^2)} \epsilon - \frac{H_0 + 4R_0}{H_0(R_0^2 - r_0^2)} \epsilon^2 + \frac{2}{H_0(R_0^2 - r_0^2)} \epsilon^3 \quad \dots \quad (55)$$

and
$$\phi(z) = 1 - \frac{H_0 + 4R_0}{R_0 H_0 + R_0^2 - r_0^2} \epsilon + \frac{3}{R_0 H_0 + R_0^2 - r_0^2} \epsilon^2 \quad \dots \quad \dots \quad (56)$$

In terms of f the fraction of the web size $D (= R_0 - r_0)$, remaining at time t , z can be expressed as

$$z = (1-f) \left[1 + \frac{(H_0 + 4r_0)(R_0 - r_0)}{H_0(R_0 + r_0)} f + \frac{2(R_0 - r_0)^2}{H_0(R_0 + r_0)} f^2 \right] \quad \dots \quad \dots \quad (57A)$$

since $\epsilon = D(1-f)$.

Here z is a cubic in f . Usually, however, the grains are long compared with their web size $D (= R_0 - r_0)$ and the term $\frac{2(R_0 - r_0)^2}{H_0(R_0 + r_0)} f^2$ is small. If we neglect this term, we can write

$$z \sim (1-f)(1+\theta f) \quad \dots \quad \dots \quad \dots \quad (57B)$$

where
$$\theta \sim \frac{(H_0 + 4r_0)(R_0 - r_0)}{H_0(R_0 + r_0)} \quad \dots \quad \dots \quad \dots \quad (57C)$$

Here z and $\phi(z)$ both are expressed in terms of ϵ ; in order to get $\phi(z)$ in terms of z we should eliminate ϵ between (55) and (56). But this is impracticable. We can, however, replace $\phi(z)$ by a function $\Phi(z) = \sqrt{1-Kz}$, following Billard (1948), without producing much error, by choosing K in such a way that the velocity of the projectile at end of combustion is equal to that which would result, had we chosen the form-function $\phi(z)$. The numerical example given below justifies our assumption.

Thus

$$X = \int_{z_0}^z \frac{dz}{\Phi(z)} = \frac{2}{K} [\sqrt{1-Kz_0} - \sqrt{1-Kz}] \quad \dots \quad \dots \quad (58A)$$

$$\therefore X_2 = \frac{2}{K} [\sqrt{1-Kz_0} - \sqrt{1-K}] \quad \dots \quad \dots \quad (58B)$$

Now
$$dz = \frac{s}{v_0} d\epsilon \quad (\text{as can easily be seen})$$

and
$$\phi(z) = \frac{s}{s_0},$$

$$\therefore X = \int_{z_0}^z \frac{dz}{\phi(z)} = \frac{s_0}{v_0} \int_{\epsilon_0}^{\epsilon} d\epsilon = \frac{s_0}{v_0} (\epsilon - \epsilon_0).$$

$$\therefore \text{at all-burnt, } X_2 = \frac{s_0}{v_0} (R_0 - r_0 - \epsilon_0) \quad \dots \quad \dots \quad (59)$$

since $\epsilon_2 = R_0 - r_0$.

Hence from equations (58B) and (59), it follows that

$$\frac{2}{K} [\sqrt{1-Kz_0} - \sqrt{1-K}] = \frac{s_0}{v_0} (R_0 - r_0 - \epsilon_0)$$

which on simplification, gives

$$K^2 + 8\alpha^2(1+z_0)K - 16[\alpha^2 - \alpha^4(1-z_0)^2] = 0 \quad \dots \quad \dots \quad (60)$$

where
$$\alpha = \frac{v_0}{s_0(R_0 - r_0 - \epsilon_0)} \quad \dots \quad \dots \quad \dots \quad (61)$$

The quantity ϵ_0 corresponding to any value of z_0 can be determined from the cubic equation (55) by applying the Horner's method of solving an equation. It may also be noted from the values of s_0 and s that the surface of emission is degressing and hence K and θ would be positive.

(iii) *Tubular propellant inhibited outside on the curved surface.*

Here

$$R = R_0 = \text{constant}, \quad r = r_0 + \epsilon, \quad H = H_0 - 2\epsilon.$$

Then

$$\begin{aligned} v_0 &= \pi(R_0^2 - r_0^2)H_0 \\ v &= \pi[H_0(R_0^2 - r_0^2) - 2(r_0H_0 + R_0^2 - r_0^2)\epsilon - (H_0 - 4r_0)\epsilon^2 + 2\epsilon^3] \\ s_0 &= 2\pi[r_0H_0 + R_0^2 - r_0^2] \\ s &= 2\pi[(r_0H_0 + R_0^2 - r_0^2) + (H_0 - 4r_0)\epsilon - 3\epsilon^2] \\ \therefore z &= \frac{2(r_0H_0 + R_0^2 - r_0^2)}{H_0(R_0^2 - r_0^2)}\epsilon + \frac{H_0 - 4r_0}{H_0(R_0^2 - r_0^2)}\epsilon^2 - \frac{2\epsilon^3}{H_0(R_0^2 - r_0^2)} \quad \dots \quad \dots \quad (62) \end{aligned}$$

and
$$\phi(z) = 1 + \frac{H_0 - 4r_0}{r_0H_0 + R_0^2 - r_0^2}\epsilon - \frac{3}{r_0H_0 + R_0^2 - r_0^2}\epsilon^2 \quad \dots \quad \dots \quad (63)$$

Here also it can easily be seen that if f be the fraction of the web size $D (= R_0 - r_0)$, z can be written as

$$z = (1-f) \left[1 - \frac{(H_0 - 4R_0)(R_0 - r_0)}{H_0(R_0 + r_0)}f - \frac{2(R_0 - r_0)^2}{H_0(R_0 + r_0)}f^2 \right] \quad \dots \quad \dots \quad (64A)$$

since $\epsilon = D(1-f)$.

In this case z is cubic in f . Usually, however, the grains are long compared with their web size $(R_0 - r_0)$, therefore, the term $\frac{2(R_0 - r_0)^2}{H_0(R_0 + r_0)}f^2$ is very small. If we neglect this term, we can write, for long grains

$$z \sim (1-f)(1+\theta f) \quad \dots \quad \dots \quad \dots \quad (64B)$$

where

$$\theta \sim \frac{(H_0 - 4R_0)(R_0 - r_0)}{H_0(R_0 + r_0)} \quad \dots \quad \dots \quad \dots \quad (64C)$$

The elimination of ϵ from equations (62) and (63) is impracticable. We, however, replace $\phi(z)$ by the function

$$\Phi(z) = \sqrt{1 - Kz} \quad \dots \quad \dots \quad \dots \quad (65)$$

where, proceeding as in the last paragraph, we shall have

$$K^2 + 8\alpha'^2(1+z_0)K + 16[\alpha'^4(1-z_0)^2 - \alpha'^2] = 0 \quad \dots \quad \dots \quad (66)$$

where

$$\alpha' = \frac{v_0}{s_0(R_0 - r_0 - \epsilon_0)} \quad \dots \quad \dots \quad \dots \quad (67)$$

The following example will show that the two functions $\phi(z)$ and $\Phi(z)$ do not differ much from one another. This result can be verified by taking other dimensions, whatsoever, provided the length of the propellant grain is sufficiently great in comparison with the web size ($R_0 - r_0$).

Example: SC propellant with tubular form, having the dimensions $R_0 = 0.100''$, $r_0 = 0.025''$ and $H_0 = 200(R_0 - r_0) = 15''$, is used in a 6-inch B.L. gun mark VII. Mass of the charge is 23.31 lbs. and the assumed shot-start pressure p_0 is 1.5 tons/sq. in. Then from equation (20), $z_0 = 0.042614$.

Case I: Uninhibited propellant.

Here from equation (54), $K = 0.0198015$.

Case II: Propellant inhibited inside.

The equation (55) becomes, with $z_0 = 0.042614$,

$$14.2222 \epsilon_0^3 - 109.5111 \epsilon_0^2 + 21.4666 \epsilon_0 - 0.042614 = 0.$$

Solving it by Horner's method, we get $\epsilon_0 = 0.002006$, (the other values can easily be neglected).

Then

$$\alpha = 0.6381875$$

and

$$K^2 + 3.397114 K - 4.083841 = 0$$

giving

$$K = 0.9413175 \text{ (neglecting the negative value } -4.338431).$$

We also get

TABLE VI

ϵ	0	0.015	0.030	0.045	0.060	0.075
z	0	0.297408	0.545824	0.745536	0.897032	1
$\phi(z)$	1	0.847404	0.695702	0.544895	0.394982	0.245963
$\Phi(z)$	1	0.848555	0.697286	0.546089	0.394471	0.242244

We may note that even if we assume $p_0 = 1$ ton/sq. in.

then

$$z_0 = 0.0284689; \epsilon_0 = 0.0013353; K = 0.941321, \text{ and } \alpha = 0.6323768.$$

Also

ϵ	0	0.015	0.030	0.045	0.060	0.075
$\Phi(z)$	1	0.848554	0.697284	0.540087	0.394467	0.242238

And if we assume $p_0 = 0$, we get $K = 0.9413217$ showing that the different values of K do not generally differ from one another up to the fourth or fifth decimal place and produce negligible difference in the values of $\Phi(z) = \sqrt{1 - Kz}$. In any case the values of $\phi(z)$ and $\Phi(z)$ do not differ much. It may also be noted that with $K = -4.338431$, the values of $\Phi(z)$ are far from those of $\phi(z)$.

Case III: Propellant inhibited outside.

With $z_0 = 0.042614$, we have

$$14.2222 \epsilon_0^3 - 105.9555 \epsilon_0^2 - 5.4666 \epsilon_0 + 0.042614 = 0.$$

By Horner's method, we get $\epsilon_0 = 0.0068789$ (neglecting the other values).

Then $\alpha' = 2.685318$ from equation (66).

Equation (65) becomes $K^2 + 60.145757 K + 647.190112 = 0$ giving $K = -14.035799$ or -46.109958 .

With $K = -14.035799$ we get

TABLE VII

ϵ	0	0.015	0.030	0.045	0.060	0.075
z	0	0.105792	0.258976	0.459264	0.706368	1
$\phi(z)$	1	1.579707	2.155902	2.728585	3.297756	3.863414
$\Phi(z)$	1	1.576348	2.152890	2.728761	3.303701	3.877602

Thus we see that with $K = -14.035799$, the values of $\phi(z)$ and $\Phi(z)$ are very nearly equal. However, with $K = -46.109958$, the values are far from one another. We, therefore, neglect $K = -46.109958$ and take the former value.

In each of the two cases of inhibition (ii) and (iii), θ is calculated from

$$K = \frac{4\theta}{(1+\theta)^2},$$

giving two values of θ corresponding to each value of K . Out of these two values of θ , that value is taken which is nearly equal to that given by equations (57C) and (64C) respectively.

The above example shows how the value of K varies when the tubular propellant is coated inside or outside. Table VIII gives the results for max. pressure and muzzle velocity.

TABLE VIII

Propellant (Tubular)	M.V. (ft./sec.)	Max. Pr. (tons/sq. in.)
Uninhibited	2547.7	18.084
Inhibited inside ..	2624.0	21.894
Inhibited outside ..	2498.6	17.862

SUMMARY

In this paper a solution of the system of equations in Internal Ballistics of a conventional gun has been given with the Charbonnier's Form-Function expressed as $\phi(z) = \sqrt{1-Kz}$. The shot-start pressure is assumed to be some finite quantity different from zero and thus the approach is different from that of Billard (1948) who has taken zero shot-start pressure. The solution involves the application of Incomplete Beta-Function Tables for the evaluation of the function-Z. However, it has been found that if for the evaluation of this particular function Z, the shot-start pressure is taken as zero (but not in other cases) no tables are necessary and the error involved is generally of the order of 1.0%.

Some calculations based on the method have been made to show the effects on maximum pressure and muzzle velocity of different loading conditions and for different values of shot-start pressure.

The method has been extended to investigate the variations in maximum pressure and muzzle velocity when a tubular propellant is inhibited on the outer or on the inner surface.

ACKNOWLEDGEMENT

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REFERENCES

- Billard, M. A. (1948). *Memorial de l'Artillerie française*, tome 22, 3^e fasc., 619-648.
Internal Ballistics (1951). His Majesty's Stationery Office Publication.
Pearson, Karl (1934). Incomplete Beta-function Tables, Published by Biometrika Office, University College, London.

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ROTATIONAL ANALYSIS OF THE COLUMBIUM OXIDE BANDS

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(Communicated by K. Rangadhama Rao, F.N.I.)

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NATIONAL INSTITUTE OF SCIENCES OF INDIA

Re : *Cost of Publication*

In view of the heavy cost of Printing and Publication, the Council desires to urge upon Authors of Papers the absolute necessity for *brevity of statement* and for *restricting to a minimum* the number of Plates, Text-figures and Tables.

all. Further, as observed by him, while the systems *C* and *B* give evidence of multiplicity, in system *A* 'the multiplet structure is complicated by the existence of rotational structure'.

The determination, if possible, of the rotational structure of the bands is considered necessary in the light of the above observations. Such a study would also be interesting in view of the fact that spectra of only a few of the oxides of the elements of transition groups have so far been analysed, namely, titanium oxide, zirconium oxide and vanadium oxide. The present paper describes the results obtained in an attempt in this direction.

EXPERIMENTAL

For preliminary studies, all the sources mentioned by Ramakrishna Rao have been employed using spectrographs of comparatively low dispersion: (1) A Hilger two prism glass Littrow instrument, with which an exposure of about 15 minutes was sufficient, (2) the first order of a 10' concave grating with 15,000 lines per inch (dispersion 5.7 \AA/mm.) giving exposures of about 10 to 30 minutes, and (3) final pictures for the measurements of the bands were taken in the first order of a 21' concave grating with 30,000 lines per inch (dispersion 1.2 \AA/mm.) and having a 6-inch ruled surface set up in the Paschen mounting. An exposure of about 2

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INTRODUCTION

The spectrum of columbium oxide has been first obtained in this laboratory by Ramakrishna Rao (1950) in the course of an investigation of the line spectrum of columbium. They have been obtained in three different sources: (1) in the flame of a D.C. arc between columbium electrodes run at 220 volts and 3 amperes, (2) in the flame of a D.C. arc between graphite rods fed by columbium pentachloride, and (3) in a heavy current discharge from 2 kW. D.C. Generator with columbium pentachloride in the quartz discharge tube. The band spectrum consists of well-marked, red-degraded heads extending from about λ 4200 towards the red to the limit of sensitivity of the panchromatic plate. The bands have been divided into three systems, designated as *A*, *B* and *C* in the respective spectral regions λ 4200–5100, λ 5000–6300 and λ 6000 upwards. The systems *B* and *C* considerably overlap each other. Ramakrishna Rao has suggested a vibrational analysis of the three systems and has indicated a general correspondence between the bands of columbium oxide and the analogous α , β and γ systems in titanium oxide (Lowater, 1929) and zirconium oxide (Lowater, 1932). For the vanadium oxide molecule (Mahanti, 1935) only one system has so far been identified.

Describing the features of the three systems of columbium oxide Ramakrishna Rao has observed that the structure of the bands conspicuously differs from group to group. There is an apparent partially open structure in system *A*; the structure is much less open in system *B* while in system *C* there is no rotational structure at all. Further, as observed by him, while the systems *C* and *B* give evidence of multiplicity, in system *A* 'the multiplet structure is complicated by the existence of rotational structure'.

The determination, if possible, of the rotational structure of the bands is considered necessary in the light of the above observations. Such a study would also be interesting in view of the fact that spectra of only a few of the oxides of the elements of transition groups have so far been analysed, namely, titanium oxide, zirconium oxide and vanadium oxide. The present paper describes the results obtained in an attempt in this direction.

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hours on Ilford S.R. Panchromatic plates was sufficient to get the bands well developed. The last spectra were taken using the flame of a D.C. arc between columbium electrodes run at 220 volts and 4 amperes. The electrodes were of the 'Specpure' variety supplied specially by Johnson, Matthey & Co. This 21' grating is the one mounted in the Applied Physics department of the University of Calcutta.

In addition to the plates mentioned above, three plates of the (1, 0), (0, 0) and (0, 1) bands have been made available to the author by Dr. V. R. Rao who has taken the arc spectra of columbium on the 21' grating mounted in the Yerkes observatory, U.S.A. The author is much indebted to him for these plates. The spectra have been measured employing usual iron arc international standards.

DESCRIPTION OF THE BANDS

(0, 0) *Band*: The photographs have shown three clear and closely situated heads (ν 21322.7, ν 21321.3 and ν 21319.8) of gradually increasing intensity, the one at the long wavelength side being the most intense. The resolution is not good in the immediate vicinity of these heads. A little away from the head on the red side a series of strong lines starts. Each of these strong lines breaks up into two close lines one of which (the one on the violet side) is more intense than the other. These pairs of close lines continue up to a short distance when the more intense component line in each pair will be resolved into two very close lines and thus we get groups of three close lines. Still farther, these groups of three lines become groups of four lines.

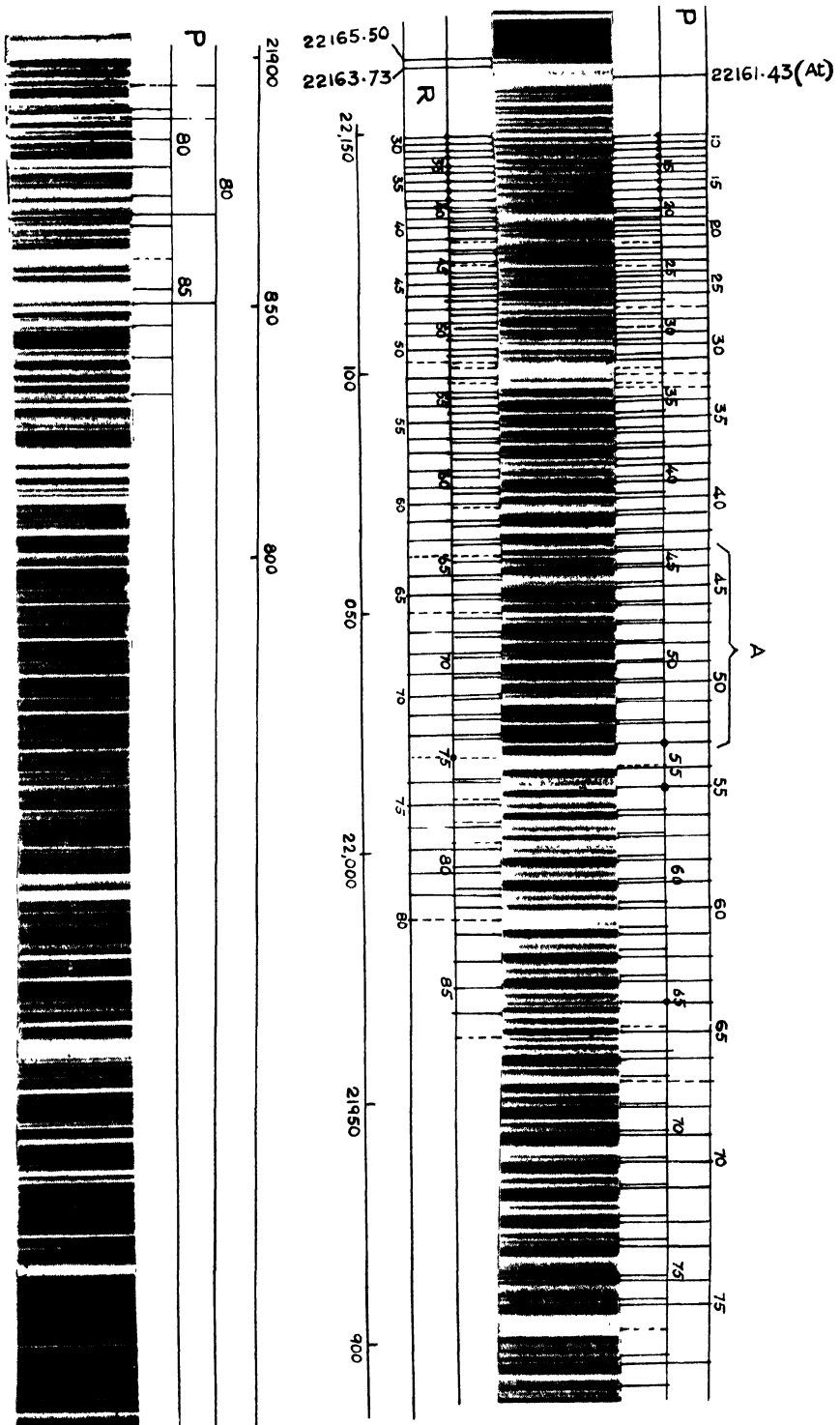
Another point that is brought out clearly in the spectrograms is the nature of the three component heads mentioned above. The least refrangible component at ν 21319.8 is sharp and gives the impression of head formed due to the accumulation of close lines. The other two components are diffuse and headless and their appearance* is such as to justify a conclusion that these may not form part of this band at all.

(0, 1) *Band*: This also exhibits clearly three heads (ν 20342.2, ν 20340.4 and ν 20339.0). A little further to the red at ν 20328.9 is a band which apparently looks like another head. Immediately from this line follow a few lines among which there is first a slight increase and then a gradual fall in intensity. It is after these that a series of strong lines occurs, corresponding to the one observed in the (0, 0) band mentioned above. These strong lines become broad and diffuse and after a short distance, each line of the series gets resolved into a close group of four component lines. These close groups get wider further off from the band and the structure is broadened and a short region presents a clearly open structure. This helped considerably in serving as a starting point for the identification of the branches. At about ν 20210 there is the appearance of a head which indicates that the (1, 2) band may start here. This is evident from the low dispersion spectrograms. Thus there is a complication of the structure due to overlapping by the (1, 2) band.

(1, 0) *Band*: In this band two heads (ν 22165.5 and ν 22163.7) can be clearly discerned, the red side one being more intense than the other. It is quite likely that the third head might have been masked by the strong atomic line at ν 22161.4 occurring close to the above-mentioned two heads. The resolution is not enough very near the heads and a little away from the heads commence two clear series of lines and continue up to a short distance where due to an increase in the resolution there is a fall in intensity and a complexity in the rotational structure. This region is followed by a very distinct region in which groups, each of four distinct component lines, are prominently present. Even a cursory examination of the spectrogram points out the members of the four branches distinctly, which could

* This feature is common to (0, 1) and (1, 0) bands also.

A0.



be followed up on either side and the complete branches identified. From about ν 22018.6 which could be clearly noticed in the low dispersion pictures commences the (2, 1) band due to which there is a complication in the structure.

All the features mentioned above can be seen from the reproduction of the bands in Plates X, XI and XII. There is still a considerable amount of confusion due to the occurrence of a number of atomic lines. These have been carefully sifted out by comparison with the list due to Humphreys and Meggers (1945).

ANALYSIS

Columbium occurs in the same vertical group of the periodic table as vanadium. Hence the rotational structure of the bands of columbium oxide and the electronic transition involved in their emission may be expected to be similar to that of vanadium oxide. The similarity between the two band systems is evident from an examination of the general structure and appearance of the bands. In vanadium oxide although the electronic transition has not been conclusively established, four distinct series of lines constituting branches of the type R_1 and R_2 and P_1 and P_2 have been identified by Mahanti (1935) in the (0, 0), (1, 0) and (0, 1) bands and rotational analysis has been suggested consistent with the combination relationships expected from the assignments of the bands. The presence of what is considered as a short and strong Q branch led Mahanti to suggest that ${}^2\Delta - {}^2\Delta$ is the transition concerned in the emission of the system.

In analysing the rotational structure of the columbium oxide bands the method adopted has been first to identify the four different branches R and P analogous to those of vanadium oxide. As has already been mentioned in the description of the bands, a few distinct groups, each consisting of four lines, stand out prominently in all the three bands (1, 0), (0, 0) and (0, 1). (Cf. regions marked A , B and C in Plates X, XI and XII respectively.) These have provided a clue. The four lines in these groups are taken as constituting the four branches, R_1 , R_2 , P_1 and P_2 and when they are extrapolated towards the head, the groups of four lines become groups of three lines, one of the lines in each group being common to two of the branches. Extrapolating further towards the head, it is found that all the four branches merge giving rise to a series of strong lines—a feature which is very prominent in all the three bands. Thus four branches are picked out in each of the bands. In this arrangement the successive members (of the series of strong lines near the head) have formed into a branch.

As a next step in the analysis, a comparison is made of the B values of the oxide molecules of all elements round about columbium in the periodic table from beryllium to molybdenum. From that comparison the B value of columbium oxide molecule is taken to lie between 0.15 and 0.6 cm^{-1} . By counting the lines almost up to the head and allowing for a small deviation because of lack of resolution at the head, the lower limit of the combination difference for a chosen line is calculated using the above lower limit for B value and a value for K which is definitely less than the number of lines between the head and that particular line, while the upper limit of the combination difference is determined by an upper limit to the increase in combination difference with K , consistent with the upper limit of the B value given above. Then for all pairs of the branches (which would be twelve in number for each band, each branch being treated as an R and a P branch since the nature of the branches is not known) all possible sets of frequency differences are formed in the three bands between the limits of the combination difference mentioned above. Altogether about 400 sets of frequency differences are written down and a thorough attempt at checking up the combination relations among the above sets of frequency differences in the three bands is made systematically. Ultimately, for only one particular relative numbering of the branches in the three bands, the combination relations between the bands are found to be satisfied with

TABLE I
(1, 0) Band

* in all these tables indicates rotational lines overlapped by columbium atomic lines.

Columns 4 and 5 in these tables give the mean value of $\Delta_2 F_1(K)$ and $\Delta_2 F_2(K)$ for the upper and lower states respectively.

K	R	P	$-\frac{\Delta_2 F'(K)}{R(K) - P(K)}$	$\frac{\Delta_2 F''(K)}{= R(K-1) - P(K+1)}$
11		22149.59 146.63		
12		148.14 145.00		
13		146.63 143.29		
14		145.00 141.60		
15		143.29 139.74		
16		141.60 138.01		
17		139.74 136.07		
18		137.70* 134.11		
19		135.70 132.14		
20		133.70 130.43		
21		131.60 128.26		
22		129.39 125.94		
23		127.15 123.66		
24		124.86 121.25		
25		122.50 118.81		
26		120.05 116.29		
27		117.38 113.64*		
28		114.80 111.09		
29		22112.12 108.36		

TABLE I—(Contd.)

K	R	P	$\frac{\Delta_2 F''(K)}{= R(K) - P(K)}$	$\frac{\Delta_2 F''(K)}{= R(K-1) - P(K+1)}$
30		109.45 105.53		
31	22149.59 146.63	106.64 102.72	43.43	
32	148.14 145.00	103.74 099.44	44.77	49.23
33	146.63 143.29	100.81 096.95	46.08	50.76
34	145.00 141.60	097.82 093.80	47.49	52.31
35	143.29 139.74	094.78 092.52	48.87	53.76
36	141.60 138.01	091.65 087.44	49.76	55.11
37	139.74 136.07	088.63 084.18	51.50	56.83
38	137.70* 134.11	085.12 080.84	52.93	58.30
39	135.70 132.14	081.79 077.42	54.32	59.37
40	133.70 129.73	078.10 073.97	55.68	61.39
41	131.60 127.72	074.63 070.43	57.13	62.76
42	129.39 125.36	071.10 066.81*	58.42	64.36
43	127.15 123.12	067.46 063.15	59.83	65.84
44	124.86 120.68	063.69 059.38	61.24	67.39
45	122.50* 118.16	060.05 055.45	62.58	68.90
46	120.05 115.68	056.10 051.65	63.99	70.36
47	117.38 112.98	052.24 047.70	65.21	71.93
48	114.80 110.32	048.21 043.66	66.63	73.32
49	112.12 107.57	044.18 039.55	67.98	74.80
50	109.45 104.60	040.13 035.39	69.27	76.36

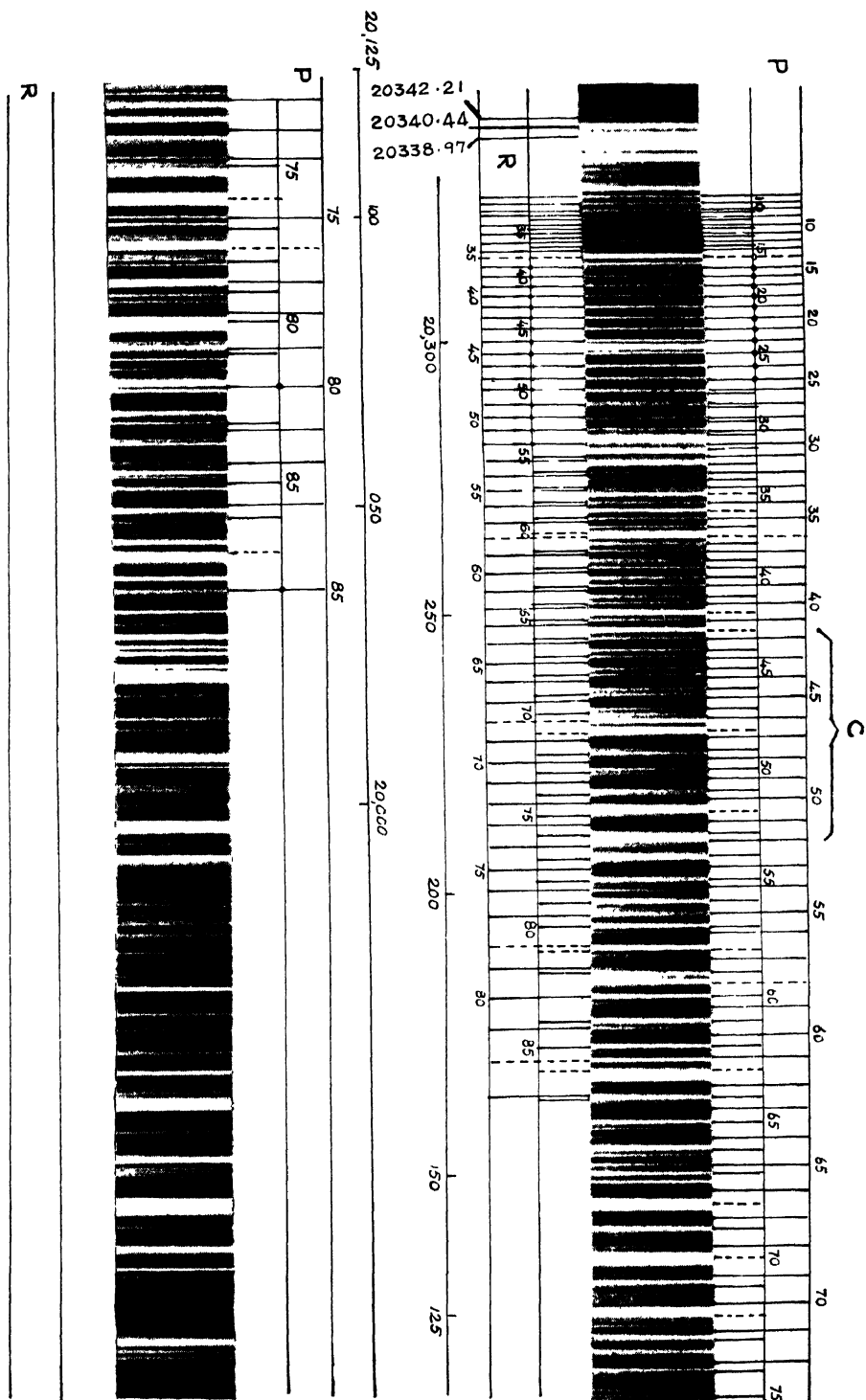


TABLE I—(Contd.)

K	R	P	$\Delta_2 F'(K)$ $= R(K) - P(K)$	$\Delta_2 F''(K)$ $= R(K-1) - P(K+1)$
51	22106.64 101.72	22035.81 031.16	70.70	77.68
52	103.74 098.83	031.57* 027.13	71.94	79.27
53	100.81 095.83	027.13 022.70	73.41	81.14
54	097.85 092.69	022.70 017.59	75.13	82.32
55	094.78 089.37	018.29 013.71	76.07	84.24
56	091.65 086.26	013.71 008.35	77.93	85.81
57	088.63 082.90	009.10 003.44	79.50	87.39
58	085.12 079.59	004.36 21998.77	80.79	88.93
59	081.79 076.09	999.63 994.05	82.10	90.38
60	078.38 072.58	994.89 989.06	83.51	91.46
61	075.15 069.01	989.96 984.95	84.63	92.99
62	071.77* 065.29	984.95 980.04	86.04	95.01
63	067.94 061.68	980.04 974.10	87.36	96.24
64	064.28 057.67	974.87 969.72	88.68	98.08
65	060.62 053.87	969.72 963.74	90.52	99.64
66	056.10 049.95*	964.55* 958.12	91.69	100.90
67	052.75* 045.83	959.36 953.33	92.95	102.36
68	049.01 041.76	953.33* 948.01	94.72	104.07
69	044.97 037.57	948.52 941.93	96.05	105.75
70	040.95 033.26*	943.05 936.32	97.42	106.96
71	036.78 029.01	937.47 931.16	98.58	108.93

TABLE I—(Concl'd.)

K	R	P	$\Delta_2 F'(K)$ $= R(K) - P(K)$	$\Delta_2 F''(K)$ $= R(K-1) - P(K+1)$
72	22032.59 024.65	21931.95 924.41	100.44	110.53
73	028.32 019.84*	925.33 919.51	101.66	112.11
74	024.05 015.01	920.52 912.51	103.02	113.51
75	019.84* 010.48	913.56 907.59	104.59	114.68
76	015.63 005.81	908.70 901.01	105.87	..
77	011.25* 001.15	902.03*	118.23
78	006.80* 21996.41	896.82 888.16	109.16	120.14
79	22002.33* 21991.59	890.57 881.55	110.80	..
80	21997.68 ..	884.50	122.76
81	993.29 981.74	879.16 869.25	113.31	..
82	988.23 977.74	872.64 863.67	114.83	126.12
83	982.64 ..	866.66 856.13	..	127.80
84	977.74 ..	859.98 850.39		
85	972.84 ..	853.33 ..		
86	967.59 .	846.73 ..		
87	962.28* ..	840.19 .		
88	833.06 ..		
89	826.14 ..		

TABLE II

(0, 0) Band

K	R	P	$\Delta_2 F''(K)$ $= R(K) - P(K)$	$\Delta_2 F''(K)$ $= R(K-1) - P(K+1)$
10		21307.46 305.82*		
11		305.82* 304.39		
12		304.39 302.79		
13		302.79 301.16		
14		301.16 299.46		
15		299.46 297.72		
16		297.72 295.87		
17		295.87 293.96		
18		293.96 291.95		
19		291.95 289.93		
20		289.93 287.93		
21		287.93 285.84		
22	21317.43 316.52	285.84 283.62	32.25	
23	316.52 315.61	283.62 281.34	33.59	36.81
24	315.61 314.70	281.34 279.10	34.94	38.19
25	314.70 313.60	279.10 276.66	36.27	39.69
26	313.60 312.55	276.66 274.27	37.61	41.14
27	312.55 311.26	274.27 271.76	38.89	42.59
28	311.26 310.03	271.76 269.22	40.16	44.05
29	310.03 308.73	269.22 266.50*	41.52	45.55

TABLE II—(Contd.)

K	R	P	$\Delta_2 F''(K)$ $= R(K) - P(K)$	$\Delta_2 F''(K)$ $= R(K-1) - P(K+1)$
30	308.73 307.46	266.50* 263.70	43.00	46.87
31	307.46 305.82*	263.70 261.33*	44.13	48.45
32	21305.82* 304.39	21260.83 258.46	45.46	49.83
33	304.39 302.79	257.98 255.65	46.78	51.23
34	302.79 301.16	255.02 252.73	48.10	52.65
35	301.16 299.46	252.10 249.78	49.37	54.09
36	299.46 297.72	249.09 246.68	50.71	55.58
37	297.72 295.87	245.96 243.51	52.06	57.04
38	295.87 293.96	242.75* 240.36	53.36	58.49
39	293.96 291.95	239.47 237.15	54.64	59.95
40	291.95 289.93	236.19 233.75	55.97	61.32
41	289.93 287.93	232.83 230.44	57.30	62.86
42	287.93 285.84	229.38 226.78	58.85	64.39
43	285.84 283.62	225.84 223.25	60.19	65.90
44	283.62 281.34	222.31 219.67	61.49	67.39
45	281.34 279.10	218.65 216.03	62.88	68.84
46	279.10 276.66	214.96 212.33	64.24	70.43
47	276.66 274.27	211.03 208.56	65.67	71.84
48	274.27 271.76	207.38 204.70	66.98	73.35
49	271.76 269.22	203.39 200.85	68.37	74.91
50	269.22 266.50*	199.34* 196.87	69.76	76.36

TABLE II.—(Contd.)

K	R	P	$\frac{\Delta_2 F''(K)}{= R(K) - P(K)}$	$\frac{\Delta_2 F''(K)}{= R(K-1) - P(K+1)}$
51	266.50* 263.70	195.42 192.84	70.97	77.84
52	263.70 261.33*	191.24 188.70	72.50	79.27
53	21261.33* 258.46	21187.17 184.50	74.06	80.94
54	258.46 255.65	182.95 180.20*	75.45	82.59
55	255.65 252.73	178.68 175.93	76.89	84.13
56	252.73 249.78	174.29 171.56	78.33	85.70
57	249.78 246.68	169.84 167.15	79.74	87.25
58	246.68 243.51	165.34 162.67	81.09	88.77
59	243.59 240.36	160.84 158.08	82.48	90.26
60	240.36 237.15	156.20 153.47	83.92	91.80
61	237.15 233.75	151.54 148.73	85.32	93.38
62	233.75 230.44	146.73 144.02	86.72	94.91
63	230.44 227.15	141.90 139.18	88.26	96.37
64	226.78 223.70	137.07 134.38*	89.70	98.11
65	223.25 220.14	132.06 129.32	91.23	99.59
66	219.67 216.58	127.04 124.27	92.47	101.15
67	216.03 212.89	121.91 119.18	93.92	102.60
68	212.33 209.10	116.79 114.27	95.19	104.03
69	208.56 205.39	111.91* 108.95	96.51	105.73
70	204.70 201.40	106.38* 103.60	98.06	107.38
71	200.85 197.51	100.80 98.31	99.63	108.85

TABLE II—(Contd.)

K	R	P	$\frac{\Delta_2 F''(K)}{= R(K) - P(K)}$	$\frac{\Delta_2 F''(K)}{= R(K-1) - P(K+1)}$
72	196.87 193.50	095.40 093.00	100.99	110.45
73	192.84 189.38*	089.88 087.58	102.38	112.02
74	21188.70 185.35	21084.35 081.99*	103.85	113.55
75	184.50 181.10	078.68 076.44	105.24	115.22
76	180.20 176.90	072.99 070.62	106.75	116.72
77	175.93 172.59	067.11 065.05	108.18	118.32
78	171.96 168.22	061.26 059.20	109.86	119.84
79	167.15 163.71	055.41 053.43	111.01	..
80	162.67 159.20	049.48	122.92
81	158.08 154.62	043.52 041.50	113.84	124.44
82	153.47 150.16	037.53 035.47	115.32	125.96
83	148.73 145.31	031.33 029.36	116.68	..
84	144.02 140.47	129.00
85	139.18 135.67	019.11 016.93	119.35	130.47
86	134.38 130.78	012.85 010.70	120.81	132.02
87	129.32 125.83	006.56 004.25	122.17	133.57
88	124.27 120.69	000.13 20997.89	123.47	135.06
89	119.18 115.74	993.67 991.36	124.95	136.47
90	113.70 110.67	987.21 984.81	126.18	138.14
91	108.45 105.36	980.52 978.12	127.59	139.53
92	103.14 100.05	973.87 971.45	128.94	140.99

TABLE II—(Contd.)

K	R	P	$\Delta_2 F'(K)$ $= R(K) - P(K)$	$\Delta_2 F''(K)$ $= R(K-1) - P(K+1)$
93	097.80 094.64	967.14 964.70	130.30	142.50
94	092.28 089.26	960.32 957.87	131.68	143.99
95	21086.79 083.71	20953.54 950.93	133.02	..
96	081.19 078.17	946.47	146.97
97	075.57 072.56	939.60 936.96	135.79	148.54
98	069.74 ..	932.44 929.84	..	150.09
99	063.97 061.26	925.25 922.71	138.64	..
100	058.13 055.41	918.20 915.68	139.80	152.97
101	052.17 049.48	911.05* 908.25	141.18	154.51
102	046.29 043.52	903.63 900.89	142.65	156.15
103	040.47 037.53	896.00 893.36	144.32	157.64
104	034.11 031.33	888.68 885.86	145.45	159.33
105	027.74 ..	881.12 878.22
106	021.57 019.11	873.61
107	015.20 012.85		
108	008.77 006.56			
109	002.29 000.13			
110	20995.77 993.67			
111	988.18 987.21			
112	982.46 980.52			
113	975.68 973.87			

TABLE II—(Concl'd.)

<i>K</i>	<i>R</i>	<i>P</i>	$\Delta_2 F'(K)$ $= R(K) - P(K)$	$\Delta_2 F''(K)$ $= R(K-1) - P(K+1)$
114	968.86 967.14			
115	961.92 960.32			
116	20954.98 953.54			
117	947.90 946.47			
118	940.81 939.60			
119	933.60 932.44			
120	926.41 925.25			
121	919.17 918.20			
122	911.76 911.05			
123	904.45 903.63			
124	896.67 896.00			
125	889.15 888.68			

TABLE III

(0, 1) Band

K	R	P	$\frac{\Delta_2 F''(K)}{= R(K) - P(K)}$	$\frac{\Delta_2 F''(K)}{= R(K-1) - P(K+1)}$
9		20327.33 323.53		
10		326.09 322.14		
11		324.50 320.72		
12		323.03 319.21		
13		321.52 317.87		
14		319.91 316.06		
15		318.25 314.29		
16		316.46* 312.54		
17		314.29 310.82		
18		312.93 308.92		
19		310.82 306.96		
20		308.92 304.96		
21		306.96 302.84		
22		304.96 300.66		
23		302.84 298.52		
24		300.66 296.27		
25		298.52 293.92		
26		296.27 291.81		
27		293.92 289.30		
28		291.20 286.55		

TABLE III—(Contd.)

K	R	P	$\Delta_z F''(K)$ $= R(K) - P(K)$	$\Delta_z F''(K)$ $= R(K-1) - P(K+1)$
29		288.76 284.03		
30		20286.19 281.55		
31	20327.33 323.53	283.54 279.30	44.01	
32	326.09 322.14	280.93 276.58	45.36	49.18
33	324.50 320.72	278.15 273.85	46.61	50.65
34	323.03 319.21	275.33 270.99	47.96	52.31
35	321.52 317.87	272.53 268.07	49.40	53.79
36	319.91 316.06	269.54* 265.12	50.66	55.33
37	318.25 314.29	266.62 262.11	51.91	56.69
38	316.46* 312.54	263.62 258.98	53.20	58.11
39	314.29 310.82	259.86 255.87	54.69	59.60
40	312.93 308.92	257.08 252.72	56.03	60.86
41	310.82 306.96	253.92 249.47	57.20	62.45
42	308.92 304.96	250.92 246.04	58.46	63.81
43	306.96 302.84	247.42 242.51	59.94	65.39
44	304.96 300.66	244.06 239.05	61.26	66.88
45	302.84 298.52	240.58 235.47	62.66	67.87
46	300.66 296.27	236.97 231.92	64.02	69.84
47	298.52 293.92	233.45 228.26	65.37	71.40
48	296.27 291.89	229.56* 224.57	66.98	72.74
49	293.92 289.30	226.16 220.80	68.13	74.35

TABLE III—(Contd.)

K	R	P	$\Delta_2 F''(K)$ $= R(K) - P(K)$	$\Delta_2 F''(K)$ $= R(K-1) - P(K+1)$
50	291.81 286.55	222.36 217.03	69.49	75.87
51	20289.30 284.03	20218.38 213.11	70.92	77.26
52	286.55 281.42	214.55 209.29	72.06	79.06
53	284.03 278.72	210.59 204.63	73.44	80.69
54	281.85 276.00	206.40 200.19	75.65	82.03
55	279.30 273.23	202.40 196.29	76.91	83.46
56	276.50 270.40	198.25 192.69	77.98	85.30
57	273.85 267.42	194.02 187.91	79.67	86.88
58	270.99 264.38	189.51 183.64*	81.11	88.37
59	268.07 261.23	185.38 179.16	82.49	89.86
60	265.12* 258.24	180.99 174.66	83.86	91.25
61	262.11 255.07	176.58 170.32	85.14	93.19
62	258.98 251.85	172.08 164.91*	86.92	94.43
63	255.87 248.55	167.38 160.94	88.05	95.98
64	252.72 245.24	162.95 155.92	89.55	97.75
65	249.74 241.81	158.22 150.71	91.18	99.41
66	246.04 238.29	153.27 145.87	92.60	100.80
67	242.90 234.74	148.88 140.81	93.98	102.21
68	239.45 231.06*	143.69* 136.23	95.30	103.75
69	235.99 227.50	139.24 130.91	96.67	105.08
70	232.44 223.88	134.40* 125.96	97.98	107.00

TABLE III—(Concl'd.)

K	R	P	$\frac{\Delta_2 F'(K)}{R(K) - P(K)}$	$\frac{\Delta_2 F''(K)}{R(K-1) - P(K+1)}$
71	228.81 219.97	128.90 120.59	99.65	108.67
72	20225.16 216.09	20123.63 115.35	101.14	109.93
73	221.47 212.23	118.71 110.22	102.39	..
74	217.64 208.21	112.75
75	213.91 204.20	108.78 099.43	104.95	114.65
76	209.87 200.19	102.51* 094.05	106.75	116.11
77	206.04 195.70	097.61 088.19	107.92	118.23
78	201.90 191.05	091.32 082.29	109.67	119.78
79	197.79 186.25	086.11 076.08	110.93	121.34
80	194.02 180.99*	080.94 069.34	112.37	123.03
81	189.51* 175.35	075.31 062.68	113.44	125.16
82	185.38 169.59*	069.34 055.36	115.14	126.73
83	180.99* 163.46	063.83 047.57	116.53	.
84	130.19
85	172.08 151.81	052.21 032.86	119.41	..
86	167.38 145.87	045.68
87	162.25 ..	038.79*
88	156.98 ..	032.86

TABLE IV

Constant	Upper State	Lower State
B_0 ..	0.3447 cm. ⁻¹	0.3776 cm. ⁻¹
B_1 ..	0.3420 "	0.3746 "
B_e ..	0.3461 "	0.3791 "
α_e ..	0.0027 "	0.0030 "
I_e .	80.852×10^{-40} gm. cm. ²	73.814×10^{-40} gm. cm. ²
r_e ..	1.889 Å	1.805 Å

The D values are of the order of 10^{-7} cm.⁻¹

ABSTRACT (SUMMARY)

The spectrum of columbium oxide has been photographed in the first and second orders of a 21-ft., 30,000 lines per inch concave grating with dispersions of 1.2 Å/mm. and 0.6 Å/mm. respectively. The rotational analysis of the (1, 0), (0, 0) and (0, 1) bands of system A at $\lambda\lambda$ 4510, 4689 and 4915 respectively in this spectrum has been carried out for the first time. As in vanadium oxide, four distinct branches, namely R_1 , R_2 , P_1 and P_2 , have been identified in each band. The rotational constants, calculated by the least squares method, are reported.

REFERENCES

- Christy, A. (1929). Quantum Analysis of the TiO bands. *Phy. Rev.*, **33**, 701.
 Lowater, F. (1929). The band systems of titanium oxide. *Proc. Phy. Soc. (Lond.)*, **41**, 557.
 ——— (1932). The band system of zirconium oxide. *Proc. Phy. Soc. (Lond.)*, **44**, 51.
 ——— (1935). Rotational analysis of band spectrum of zirconium oxide. *Roy. Soc. Phil. Trans.*, **234 A**, 355.
 Mahanti, P. C. (1935). The band spectrum of vanadium oxide. *Proc. Phy. Soc. (Lond.)*, **47**, 433.
 Ramakrishna Rao, V. (1950). The complex band spectrum of columbium oxide. (The diatomic molecule, CbO). *Ind. J. Phy.*, **24**, 35.
 Suryanarayana Rao, K. (1954). Rotational analysis of the columbium oxide bands. *Nature*, **173**, 1240.

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ENERGY LEVELS OF PXII

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The spectrum of phosphorus arising from its various stages of ionization has been observed and analysed by H. A. Robinson. His work is still unpublished for ionization stages higher than the fifth, but the results of his analysis from unpublished data are quoted by Charlotte E. Moore (1949). In course of a theoretical study of the atoms of the CI isoelectronic sequences, Naqvi (1953) found that Robinson's energy values of the level intervals of the configuration of PX are not in accordance with the trend of the isoelectronic sequence; they are in fact much lower. The present paper is a report on the energy levels belonging to PXII.

PXII belongs to the BeI isoelectronic sequence. The ground configuration for this sequence is $1s^2 2s^2$, and the electronic shells are completely filled giving rise to a single energy level, namely 1S_0 . The next excited configuration, $1s^2 2s 2p$ has four energy levels 1P_1 and $^3P_{2,1,0}$. We shall be chiefly concerned with the level intervals $^3P_2 - ^3P_1$ and $^3P_1 - ^3P_0$ of this configuration. Robinson's values for these intervals are

$$^3P_2 - ^3P_1 = 6600.0 \text{ cm.}^{-1}$$

$$^3P_1 - ^3P_0 = 3200.0 \text{ cm.}^{-1}$$

Naqvi (1951) made a theoretical study of this isoelectronic sequence in which he calculated the values of the mutual magnetic interaction integral M_0 and another parameter ζ' for the atoms of this sequence. The parameter ζ' which differs inappreciably from the spin-orbit interaction integral ζ , is given by

$$\zeta' = \zeta - 30 M_0 - 4 N^{-1} + 2 N^1 \quad \dots \quad (1)$$

where M_0 , N^{-1} , N^1 are mutual magnetic interaction integrals (for their definitions and that of ζ reference may be made to Condon and Shortley, 1935 and Marvin, 1947). Briefly the method is as follows:

The theoretical expressions for the energy of the levels, obtained by taking into account the complete spin-orbit interaction as well as the mutual magnetic interactions, are equated to the energy values obtained from spectroscopic analysis. These equations are

$$\left. \begin{aligned} ^3P_2 - ^3P_1 &= \zeta' + 12M_0 + \frac{\frac{1}{2}\zeta^2}{^1P_1 - ^3P_1} \\ ^3P_1 - ^3P_0 &= \frac{1}{2}\zeta' - 30M_0 - \frac{\frac{1}{2}\zeta^2}{^1P_1 - ^3P_1} \end{aligned} \right\} \dots \quad (2)$$

They are then solved for the unknown parameters, ζ' and M_0 occurring in these equations.

Table I gives the values of ζ' and M_0 for the BeI isoelectronic sequence. It is clear that the value of M_0 for PXII is erroneous. Further if we plot the fourth roots of ζ' against the atomic numbers, the points lie fairly closely on a straight line, except the point for PXII which is below the line, indicating that the value of ζ' for PXII is lower than the trend along this sequence.

TABLE I

Sequence	ζ'	M_0
BeI ..	2.2	0.0137
BII ..	15.8	0.0555
CIII ..	55.0	0.1486
NIV ..	141.1	0.2416
OV ..	300.4	0.4639
FVI ..	566.0	0.7083
NeVII
NaVIII ..	1575	1.5833
MgIX ..	2436	1.1388
AlX ..	3591	2.0833
SiXI ..	5140	2.2361
PXII ..	6505	2.388

As for the case of PX, we conclude that the discrepancy is on account of erroneous energy values for the levels of PXII. The extrapolated values of ζ' and M_0 are

$$\zeta' = 7130 \text{ cm.}^{-1}$$

$$M_0 = 2.45 \text{ cm.}^{-1}$$

Using these extrapolated values of ζ' and M_0 and equations (2) we obtain for the level intervals of PXII

$$^3P_2 - ^3P_1 = 7310 \text{ cm.}^{-1}$$

$$^3P_1 - ^3P_0 = 3340 \text{ cm.}^{-1}$$

These values are substantially different from those of Robinson quoted above.

In the 1953 paper, it was reported for a number of isoelectronic sequences that the level intervals belonging to a particular configuration fit remarkably well with a fourth degree polynomial. This conclusion has been further checked and extended. For the case of CI isoelectronic sequence it was found that the fit was very poor if PX was included, whereas it became almost perfect when PX was excluded. We have found the same result to hold for the BeI isoelectronic sequence. When PXII is included, the fit with a fourth degree polynomial comes out to be extremely poor as is shown in Table II. However, when PXII is excluded, the fit is

TABLE II

Sequence	$[^3P_2 - ^3P_1]_{\text{obs.}}$	$[^3P_2 - ^3P_1]_{\text{calc.}}$	$\Delta_{\text{calc.}}$ (obs. - calc.)	$[^3P_1 - ^3P_0]_{\text{obs.}}$	$[^3P_1 - ^3P_0]_{\text{calc.}}$	$\Delta_{\text{calc.}}$ (obs. - calc.)
BeI ..	2.35	-31.96	34.31	0.68	-7.38	8.06
BII ..	16.40	64.19	-47.79	6.40	17.78	-11.38
CIII ..	56.80	91.84	-35.04	23.00	31.73	-8.73
NIV ..	144.20	133.59	10.61	63.20	59.58	3.62
OV ..	306.20	259.18	47.02	136.70	124.70	12.00
FVI ..	576.00	525.65	50.35	260.00	248.69	11.31
NeVII
NaVIII ..	1604.00	1645.88	-41.88	730.00	750.82	-20.82
MgIX ..	2472.00	2550.14	-78.14	1162.00	1163.32	-1.32
AlX ..	3660.00	3696.32	-36.32	1690.00	1703.41	-13.41
SiXI ..	5250.00	5077.88	172.12	2420.00	2383.84	36.16
PXII ..	6600.00	6675.58	-75.58	3200.00	3215.62	-15.62

considerably improved as can be seen from Table III. The extrapolated values for the level intervals are

$$^3P_2 - ^3P_1 = 7316 \text{ cm.}^{-1}$$

$$^3P_1 - ^3P_0 = 3348 \text{ cm.}^{-1}$$

TABLE III

Sequence	$[^3P_2 - ^3P_1]$ calc.	Δ (obs. - calc.)	$[^3P_1 - ^3P_0]$ calc.	Δ (obs. - calc.)
BeI ..	3.23	-0.88	-0.10	0.78
BII ..	14.62	1.78	7.57	-1.17
CIII ..	56.44	0.36	24.43	-1.43
NIV ..	145.61	-1.41	62.06	1.14
OV ..	307.33	-1.13	134.64	2.06
FVI ..	575.00	1.00	258.86	1.14
NeVII
NaVIII ..	1603.07	0.93	741.96	-11.96
MgIX ..	2471.52	0.48	1147.06	14.94
AlX ..	3661.99	-1.99	1696.30	-6.30
SiXI ..	5249.12	0.88	2419.20	0.80

The extrapolated values of the level intervals obtained by these two methods give results in fairly good agreement with one another. The following mean values should be adopted for the energy intervals, rounded off in the last significant figure

$$^3P_2 - ^3P_1 = 7310 \text{ cm.}^{-1}$$

$$^3P_1 - ^3P_0 = 3350 \text{ cm.}^{-1}$$

The energy values of the 1P_1 level of the $2s2p$ configuration of PXII as given by Robinson appears to be allright.

SUMMARY

H. A. Robinson's values for the energies of the three $3P$ levels of the configuration $1s^2 2s2p$ of PXII appear to be in error. They have therefore been calculated by two different methods of extrapolation along the isoelectronic sequence, which give results in good agreement with one another.

REFERENCES

- Condon, E. U., and Shortley, G. H. (1935). *The Theory of Atomic Spectra*, Cambridge University Press.
- Marvin, H. H. (1947). Mutual Magnetic Interactions of Electrons. *Phy. Rev.*, **71**, 102.
- Moore, Charlotte E. (1949). *Atomic Energy Levels*, Vol. I, Circular No. 467, National Bureau of Standards, Washington, D.C.
- Naqvi, A. M. (1951). *Mutual Magnetic Interactions in p-Electron Configurations*, Thesis, Harvard University.
- Naqvi, A. M. (1953). Energy Levels of PX. *Proc. Nat. Inst. Sc.*, **19**, 461.

AN ESTIMATE OF THE OPTICAL THICKNESS OF A SPHERICALLY SYMMETRIC, NON-CONSERVATIVE SCATTERING ATMOSPHERE

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The equation of transfer in the case of spherically symmetric atmosphere in the non-conservative isotropic scattering case was solved by the author (1954), by Chandrasekhar's method of replacing the integrals by Gaussian sums. To solve the equation of transfer, two sets of boundary conditions were considered, namely (a) the existence of definite outward flux at the lower boundary and the vanishing of both inward and outward intensities at the upper boundary of the atmosphere, (b) the existence of an outward flux at the lower boundary and the equality of inward and outward intensities, both considered weak at the outward boundary. In the second case an estimate of the optical thickness (for a definite frequency) of the extensive atmosphere was made from the outward flux ratios in the inner and the outer boundary for different values of albedo for single scattering. In the present note, a method of estimating such optical thickness of an extensive atmosphere from the expression for flux obtained by the consideration of the first set of boundary conditions has been given. The results obtained show that the values of optical thicknesses found in this way for different percentages of scattering are nearly the same as those obtained from the second set of boundary conditions (*Proc. Nat. Inst. of Sciences of India*, Vol. XX, No. 1, 18, 1954).

The values of flux at any depth obtained by the solution of the equation of transfer for spherically symmetric, non-conservative isotropic scattering under the sets of boundary conditions (a) is given by (*Proc. Nat. Inst. of Sciences of India*, Vol. XX, No. 1, 15, 1954; equation (28))

$$F(z) = q \frac{F(z_1)z^\nu I_{\nu-1}(z)}{\sqrt{3}z_1^\nu I_\nu(z_1)} \quad \dots \quad (1)$$

where $z = q\tau$, τ being the optical thickness defined by $\tau = \int_r^\infty k\rho dr$, $q = \sqrt{3(1-\omega_0)}$,

ω_0 being the albedo for single scattering. A point in the deduction of this equation has to be noted. It has not been obtained as solution of an equation for $F(z)$, and $F(z_1)$ representing the value of the function at $z = z_1$. On the other hand $F(z_1)$ on the right-hand side appears through the intensity function I_{+1} at $z = z_1$, viz.

$F(z_1) = \frac{2}{\sqrt{3}}I_{+1}(z_1)$, whereas $F(z)$ on the left comes through K satisfying equation

(15) (*Proc. Nat. Inst. of Sciences of India*, Vol. XX, 13, 1954), K being obtained as solution of the second order differential equation (16) (*Ibid.*).

This being so, if we put $z = z_1$ in equation (1), we do not get a trivial identity, but an equation giving us z_1 in terms of q . It is to be remembered that, $z_1 = q\tau_1$, τ_1 being the optical thickness of the atmosphere.

As before we take $k\rho$ to vary as some inverse power of n , and let us put $n = 2$,

$$\text{Then } \nu = \frac{n+1}{2(n-1)}, \text{ gives } \nu = \frac{3}{2}.$$

Hence on putting $z = z_1$ in (1), we obtain

$$1 = \frac{q}{\sqrt{3}} \cdot \frac{I_{\frac{1}{2}}(z_1)}{I_{\frac{3}{2}}(z_1)} \quad \dots \quad (2)$$

which may also be written as

$$\coth z_1 = \frac{q}{\sqrt{3}} + \frac{1}{z_1} \quad \dots \quad (3)$$

If in this equation we put $\frac{1}{z_1} = x$, we get

$$\coth \frac{1}{x} = \frac{q}{\sqrt{3}} + x \quad \dots \quad (4)$$

as the relation connecting the optical thickness with q .

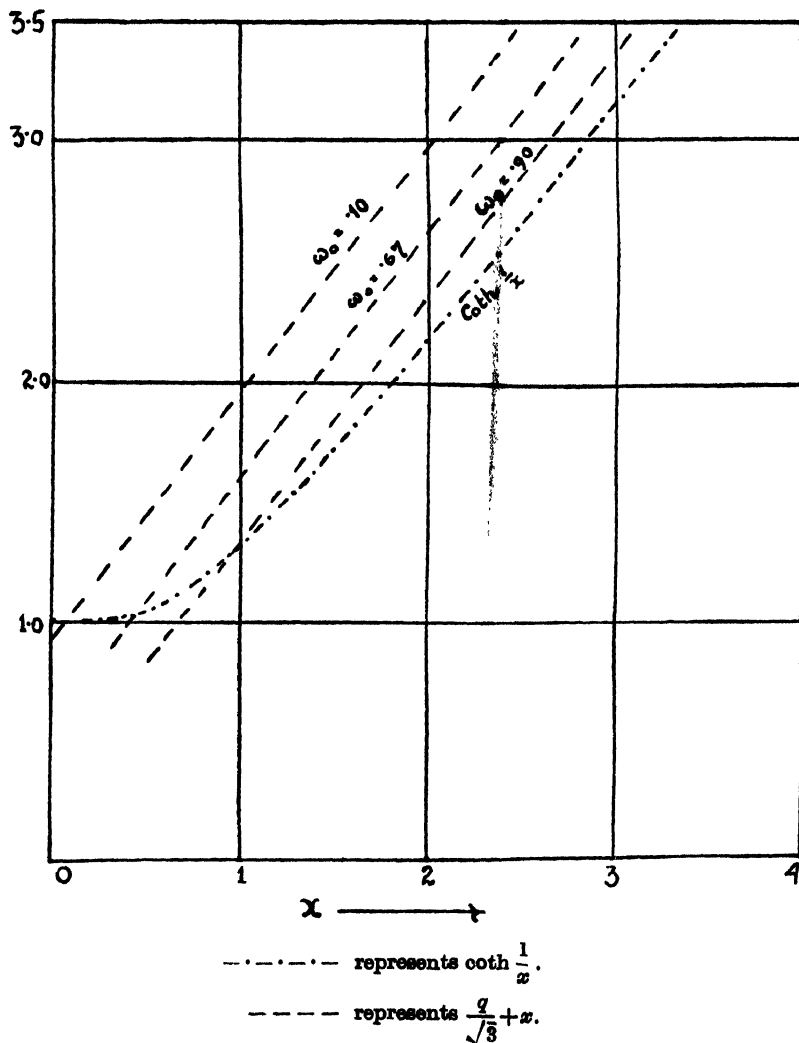
The values of x and hence of z_1 , and τ_1 were obtained from (4) by the graphical method, for different values of ω_0 , the albedo for scattering. The results are shown in the following table and are also plotted in the figure.

TABLE

x	$\frac{1}{x}$	$\coth \frac{1}{x}$	$\frac{q_1}{\sqrt{3}}$ (for $\omega_0 = .90$)	$x + \frac{q_1}{\sqrt{3}}$	$\frac{q_2}{\sqrt{3}}$ (for $\omega_0 = .67$)	$x + \frac{q_2}{\sqrt{3}}$	$\frac{q_3}{\sqrt{3}}$ (for $\omega_0 = .10$)	$x + \frac{q_3}{\sqrt{3}}$
0.25	4.00	1.001	0.316	0.566	0.577	0.827	0.949	1.199
0.50	2.00	1.037	"	0.816	"	1.077	"	1.449
1.00	1.00	1.313	"	1.316	"	1.577	"	1.949
1.50	0.67	1.709	"	1.816	"	2.077	"	2.449
2.00	0.50	2.164	"	2.316	"	2.577	"	2.949
2.50	0.40	2.632	"	2.816	"	3.077	"	3.449
3.00	0.33	3.140	"	3.316	"	3.577	"	3.949

From the figure, it appears that larger scattering is to be associated with smaller total optical depth for the frequency concerned. This is in accord with non-conservative scattering in a plane parallel atmosphere considered by Chandrasekhar under somewhat different boundary conditions. This result seems to suggest that the optical depth considered here is principally dependent on the amount of scattering which a pencil of rays starting from the photospheric surface suffers in course of its transmission through the radiation field, larger scattering implying more rapid weakening of the intensity in the ray. Thus for $\omega_0 = .90$, $x = 1.0$, $z_1 = 1.0$, $\tau_1 = 1.8$, and for $\omega_0 = .67$, $x = .466$, $z_1 = 2.15$, $\tau_1 = 2.15$, and for $\omega_0 = .10$, τ_1 will be extremely large. Thus we can study how the value of the optical thickness of the atmosphere for the boundary condition under contemplation varies with ω_0 , the albedo for scattering. This gives an estimate of the value of optical thickness of extensive atmospheres for different albedos.

In conclusion, it is a pleasure for me to express my gratitude to Prof. N. R. Sen for his kind advice during the preparation of this work.



SUMMARY

In the present note a method of estimating the optical thickness of extensive stellar atmosphere is considered for the case of spherically symmetric, non-conservative isotropic scattering. Different percentages of scattering have been taken into account, and it is shown how the thickness of the atmosphere is dependent on the value of the albedo for scattering.

REFERENCES

- Chandrasekhar, S. (1934). Radiative equilibrium of Extended Stellar Atmospheres. *Monthly Notices, Roy. Astro. Soc., London*, 94, 444-457.
- (1945). On the radiative equilibrium of a Stellar Atmosphere. V. *Astrophysical Journal*, 101, 95-107.
- (1950). Radiative Transfer, 364-370, 18-20, 344-346. Clarendon Press, Oxford.
- Kosirev, N. A. (1934). Radiative Equilibrium of Extended Stellar Photosphere. *Monthly Notices, Roy. Astro. Soc., London*, 94, 430-433.
- Sen, K. K. (1954). On the problem of transfer of radiation in a spherically symmetrical Stellar Atmosphere for the non-conservative case. *Proc. Nat. Inst. of Sci. Ind.*, 20, 12.

THE IDENTIFICATION OF RADIO STARS

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I. INTRODUCTION

The first discrete source of radio noise was observed by Hey, Parson and Phillips (1946), in the constellation of Cygnus. Since then nearly 300 of these objects have been reported by various observers in Australia and England (see Table I). Their identification with visual objects has met with little success. Only 16 of them have been identified so far, 5 with objects in our own galaxy and the rest with extra-galactic objects (Baade and Minkowski, 1954*a, b*).

The discrete sources are usually referred to as radio stars, although we know that several of them are unusual objects or even extra-galactic nebulae.

The major difficulty in their identification is the extremely poor resolution obtained at radio wavelengths. Thus a precise position of a radio star is not known. Ryle, Smith and Elsmore (1950) compared the positions of the radio stars observed by them with those of extra-galactic nebulae, stars brighter than the 4th magnitude, unusual but fainter visual stars (up to 11th magnitude only), nearby stars, large proper-motion stars, novae and supernovae and galactic nebulae and clusters but they failed to detect any significant coincidences.

Mills (1952) discussed the galactic distribution of 77 radio stars, and classified them into two classes. Class I objects are the brighter radio stars and show a galactic concentration. They are presumably at great galactic distances which, according to Mills' estimation, may be of the order of 1,000 parsecs. Class II objects are the fainter radio stars, which are more or less randomly distributed. From the random distribution of the Class II objects two possibilities arise: (1) they are of extra-galactic origin, or (2) they are nearby objects within our own galaxy. In the present paper we have considered the second possibility. The most common type of nearby stars are the *M* dwarf stars of the main sequence. Earlier, Unsöld (1949) had suggested that the general galactic radio noise may arise from the late type dwarf stars.

The coincidences in the position of the radio stars and the *M* dwarfs appear to us to be quite significant. At present the surveys of the radio stars as well as those of the *M* dwarfs are quite incomplete, so that even if the *M* dwarfs are the radio stars we would not expect to find perfect coincidences. When the surveys become more complete, this identification can be tested again.

II. COMPARISON OF POSITIONS OF THE RADIO STARS AND THE *M* DWARFS

Table I gives the sources of observed data on the radio stars. The total number of objects reported by various workers is 313, but several objects are common to two or more lists. The frequencies used by various workers are not much different from one another except the frequency of 18.3 Mc/s used by Shain

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and Higgins (1954a). The range of flux densities of radio stars as observed by different observers are comparable and lie between 5×10^{-25} and 2000×10^{-25} watts m^{-2} $(c/s)^{-1}$ except for the flux densities at 18.3 Mc/s, which lie between 250×10^{-25} and 66000×10^{-25} watts m^{-2} $(c/s)^{-1}$. A few radio stars are common between the lists of Shain and Higgins and that of Bolton, Stanley and Slee. A comparison of the flux densities for the two frequencies shows (also see Shain and Higgins, 1954b) that the reasons for the higher range of flux densities observed by Shain and Higgins is due to the different frequency used by them rather than that their radio stars have intrinsically higher brightnesses.

TABLE I

No.	Observer	Fre- quency	Region covered	Flux density 10^{-25} watts $m^{-2}(c/s)^{-1}$	No. of stars
		Mc/sec.	Dec.		
1	Stanley and Slee (1950)	100	+50° to -50°	10 to 1250	22
2	Ryle, Smith and Elsmore (1950)	81	+90° to +10°	2.5 to 2200	50
3	Mills (1952)	100	+50° to -90°	5 to 1300	77
4	Brown and Hazard (1953)	158	+70° to +40°	0.5 to 925	23
5	Shain and Higgins (1954a)	18.3	+10° to -90°	250 to 66000	37
6	Bolton, Stanley and Slee (1954)	100	+50° to -50°	4 to 1200	104

Various authors have given the probable errors in right ascension and declination, except Bolton, Stanley and Slee (1954), who have given the probable errors in rising time and azimuth. When the latter are converted into probable errors in right ascension and declination a number of them turn out to be very large. The maximum limits of probable error considered by us are $\Delta \alpha = 20$ min. and $\Delta \delta = 5^\circ$. All probable errors larger than the above limits have been considered equal to these values.

The positions of M dwarfs are taken from the three lists by Vyssotsky and his co-workers (1943, 1946, 1952). These lists give the right ascension and declination, the spectral class, etc. for all the stars and trigonometric parallaxes for about 46% of the stars. The declinations lie between -30° and $+90^\circ$.

In comparing the positions of the radio stars and the M dwarfs, we have restricted ourselves to the limit of declination from -25° to $+60^\circ$. In this range there are 156 radio stars and about 300 M dwarfs. Twelve out of the sixteen radio stars already identified lie within our range of declination and are also included here. We find that 41 M dwarfs lie within the range of probable error of 28 radio stars. Table IIA gives the positions, etc. for these coincidences.

Further 20 M dwarfs lie within the range of probable error of either the right ascension or the declination and within twice the range of probable error of the other co-ordinate of 17 radio stars. The relevant data on the second set of coincidences is given in Table IIB. The total number of radio stars listed in Tables IIA and IIB is 45 or about 30% of the total number of radio stars considered.

In order to obtain an idea of the significance we can attach to the above-mentioned coincidences, we selected two areas as follows:

Selected area I α : 0 to 4 hours $\delta = -14^\circ$ to $+60^\circ$

Selected area II α : 8 to 12 hours $\delta = -14^\circ$ to $+60^\circ$

TABLE IIA

S. No.	Radio Source				Optical Source								
	Catalogue No.	Position (1950)			Flux density	Catalogue No.	Position (1950)		Spectral class	Distance			
		R.A.	Dec.				R.A.	Dec.					
			h	m							s	°	'
1	R. 00-01	00 42	6	38	5	4	BD or AC	h	m	°	'	Pc	
2	B.S.S. 105	00 27	15	45	4	5	+34-106	00 40	35	16		dM0	19-0
3	B.S.S. 92	01 38	17	40	32	6	-19-111	00 42	-19	13		dM0	19-0
4	B.S.S. 97	02 36	14	40	3	45	+29-4398	01 55	29	34		dM0	19-0
5	B.S.S. 40	03 09	18	45	41	10	+3-339	02 25	04	12		dM2	14-0
6	B.S.S. 100	04 20	13	15	2	40	+39-710	03 04	40	11		dM0	34-5
7	B.S.S. 84	08 00	20	48	3	3	-15-728	04 07	-14	48		dM0	33-3
8	B.S.S. 86	09 16	20	46	5	2	+44-1710	08 01	44	07		dM0	33-3
9	R. 09-02	09 32	6	39	2	20	+47-2368-79	09 08	46	49		dM0	33-3
10	S. 09-1	09 23		-12		3-5	+40-2208	09 28	39	44		dK8	10-1
						400	-8-2689	09 26	-9	02		dM0	10-1
11	H.B. 13	10 30	2	57	2		-12-2918	09 29	-13	17		dM2	10-1
12	S. 10+1	10 45		16		>0-5	+57-1274	10 29	57	22		dM0	10-1
13	B.S.S. 98	10 42	7	00	2	1300	..	11 00	16	38		dM0	10-1
14	B.S.S. 87	11 42	20	31	5	12	+1-2013-63	10 50	00	45		dM0	10-1
						10	-1-2457	10 50	-01	48		dK8	10-1
							+33-238	11 42	32	50		dM0	10-1
							+31-2290	11 42	31	15		dK8	10-1
							+30-2201	11 52	29	50		dK8	10-1
15	H.B. 15	12 15	3	47	1	0-5	+29-2228	11 52	29	01		dM0	10-1
							..	12 15	46	52		dM	10-1
16	B.S.S. 30	12 19	6	05	1	14	+6-2572	12 13	05	56		dK8	10-1
							+4-2612	12 22	04	12		dM0	10-1
17	S. 12-0	12 53		-01		4100	-1-2764	12 59	-02	26		dM0	10-1
18	R. 13-02	13 40	2	38	1	3	+39-2675	13 41	39	30		dK8	10-1
19	B.S.S. 78	13 26	16	10	4	4	+11-2576	13 27	10	41		dM2	10-1
							+10-2540	13 18	10	11		dM0	10-1

* 1 Unit = 10^{-25} watts m^{-2} $(c/s)^{-1}$.

TABLE IIA—(Contd.)

S. No.	Radio Source				Optical Source			
	Catalogue No.	Position (1950)		Flux density	Catalogue No.	Position (1950)		Spectral class
		R.A.	Dec.			R.A.	Dec.	
		h m s	° ' ±	Units*		h m	° '	
20	B.S.S. 81	16 36 20	41	8	+39-3048	16 48	39 22	dM0+
21	B.S.S. 55	17 23 13	20	7	+23-3151	17 36	22 59	dM0
					+19-997-273	17 36	18 35	dM0
22	B.S.S. 36	18 47 14	12	10	+10-3724	18 53	10 55	dM0
					+10-3665	18 46	10 42	dK8
23	B.S.S. 23	18 45 17	08	12	+10-3665	18 46	10 42	dK8
					+8-142-349	18 53	08 21	dM2
24	B.S.S. 11	18 39 8	02	27	+6-3993	18 55	05 53	dM2
25	S.S. 18	18 11	-15	20	+3-2528-176	18 48	03 01	dM2
26	S. 19+0	19 30	02	1100	-12-4935	18 07	-11 31	dM2
					+4-4157	19 32	04 27	dM0
27	S. 21-1	21 20	-15	1400	+0-4241	19 30	00 29	dM0
28	B.S.S. 77	23 07 13	05	4	+3-5945	21 27	-15 14	dK8
					+4-4988	23 14	05 25	dM0
					+3-2781-116	23 06	03 03	dM0

 * 1 Unit = 10^{-25} watts m^{-2} $(c/s)^{-1}$.

TABLE IIB

S. No.	Radio Source			Optical Source			
	Catalogue No.	Position (1950)		Flux density	Catalogue No.	Position (1950)	
		R.A.	Dec.			R.A.	Dec.
		h m ± m s	° ' ± "	Units*		h m	° ' "
1	M. 01-2	01 50 8	-22 20	8	BD or AC	01 51	-22 39
2	B.S.S. 40	03 09 18 45	41 2 40	8	+37-693	03 15	38 04
3	B.S.S. 53	03 03 8 45	03 2 30	9	+1-543	02 50	01 50
4	S.S. 8	04 38	28 5	30	+25-733	04 45	26 06
5	B.S.S. 32	07 20	15 30	6	+14-1684	07 28	14 49
6	R. 08-03	08 22 3	36 1 40	4	+36-28826	08 25	35 13
7	M. 08+1	08 50 4	13 1	16	+12-1944	08 54	11 51
8	R. 10-03	10 33 4	56 37	3-3	+56-1458	10 27	56 15
9	M. 11+3A	11 45 8	31 40	10	+36-2219	11 49	35 31
10	M. 11+3B	11 35 4	31 40	10	+31-2290	11 42	31 15
11	B.S.S. 65	13 08 10 20	30 2 15	10	...	13 16	33 37
					+35-2439	13 18	34 33
					+35-2406	13 05	34 39
					+29-2405	13 21	29 29
					+30-2512	14 20	29 53
					-4-3873	15 19	-04 36
					+24-4329	21 06	24 59
					+8-4887	22 30	09 16
					+0-5017	23 33	01 19
					-13-6464	23 46	-13 16
12	B.S.S. 44	14 43 20	30 5	10			
13	B.S.S. 74	15 20 8 30	-2 1 40	5			
14	B.S.S. 31	21 13 20	20 4 50	10			
15	B.S.S. 45	22 20 12	04 3 20	5			
16	B.S.S. 77	23 07 13 20	05 4 10	4			
17	B.S.S. 48	23 17 14 15	-17 4 4	5			

* 1 Unit = 10^{-25} watts $m^{-2} (c/s)^{-1}$.

The selected area I is represented in figure 1. The position of a radio star is indicated by a rectangle, which represents the range of probable error in right ascension and declination. The maximum probable errors used are 20 minutes in right ascension and 5° in declination. The position of *M* dwarfs are indicated by dots. The number of *M* dwarfs in the selected areas is 28 and 70 respectively. The fraction of the total area covered by all the rectangles is 12.3% for selected area I and 17% for selected area II. The number of chance coincidences to be expected is 3.4 and 11.9 for the two areas, whereas the actual number of coincidences is 9 and 17 respectively.

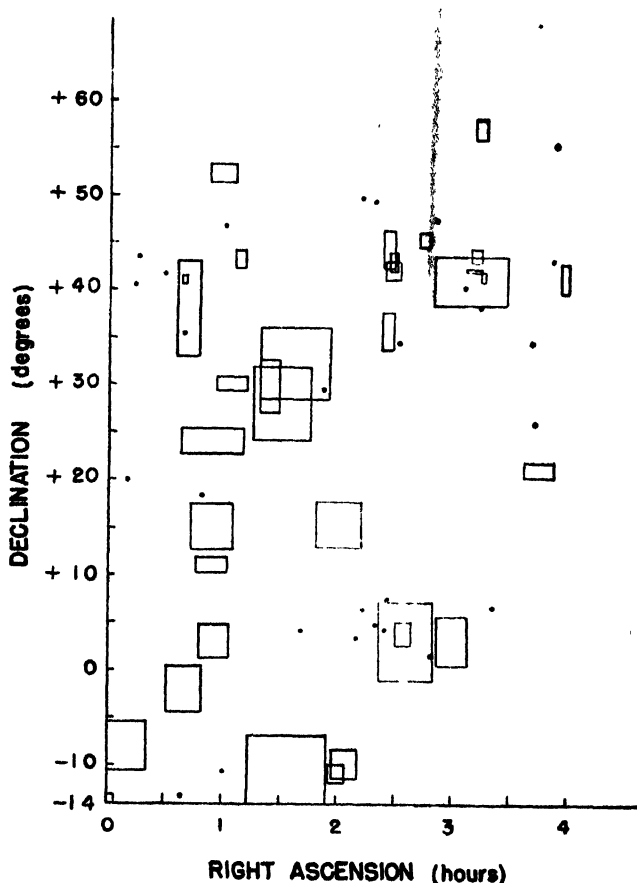


FIG. 1. The positions of the radio stars and *M* dwarf stars are indicated by the rectangles and the dots (for the selected area I) respectively. A rectangle represents the range of probable error in right ascension and declination.

We also considered the possibility of the identification of the radio stars with certain other types of stars such as the *T*-Tauri stars, the metallic line stars, the flare stars, the bright *H* and *K* emission stars, but could not get any significant coincidences. The best coincidences were obtained for the *H* and *K* emission stars (list of Joy and Wilson, 1949). The comparison of the positions of 445 of these stars with those of the radio stars gave only 20 coincidences, of which 12 were dwarf stars (7 *M* dwarfs, others *G* or *K* dwarfs) and 8 were giants or super-giants. For the case of *H* and *K* emission stars the number of actual coincidences was less than the number of chance coincidences.

FERMI'S THEORY OF MULTIPLE PION PRODUCTION AND THE ZERO-POINT ENERGY

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INTRODUCTION

I. Fermi's statistical theory of multiple pion production in high energy nucleon-nucleon collisions (1950) has received considerable attention in recent years. The earlier calculations have been improved upon (Fermi, 1953) by taking into account the conservation of charge and isotopic spin (this has been done up to a maximum number of pions produced ≤ 3). As has been stressed by Fermi himself, and others (Bhabha, 1953), the theory is not free from 'conceptional' difficulties and, because of this and other simplifying assumptions introduced in the treatment, the theory can at best provide a rough picture of the phenomena concerned. Nevertheless, the results are on the whole in fair agreement with experiment, though this may be partly due to accident (this view is supported by considerations of section III).

In the next section we extend (following Cocconni, 1954) Fermi's theory to the case of a nucleon-nucleus collision. The incident high energy nucleon is supposed to interact with the nucleons, say n in number, which lie in its path as it 'tunnels' through the target nucleus (for heavy nuclei, $A \sim 100$, a reasonable value for n is 5). Further, we assume that for a pion produced in the collision process, the lowest permissible energy, ϵ_0 say, is equal to the ground state energy of a free particle of mass μ (the pion rest mass) enclosed in a volume equal to the volume of the 'reaction-space'. In the case of a nucleon-nucleon collision, as Auluck and Kothari (1953) have pointed out, because of the large contraction of the reaction-space in the centre of mass system (henceforth called C -system), the zero-point energy becomes comparable with the total available energy, resulting in a substantial decrease in the number of pions produced. For very high primary energies, the number of particles (pions) produced becomes independent of energy. In the case of a nucleon-nucleus collision, because of the length of the 'tunnel' being n times the transverse direction, for relatively small bombarding energies (\sim a few Bev), there is no appreciable increase in the zero-point energy due to Lorentz contraction, but it is still fairly effective in cutting down the number of pions produced. For very high energies the contraction becomes large, and as in the case of the nucleon-nucleon collision, we obtain an upper limit for the number of pions produced. The introduction of the zero-point energy implies a serious departure from Fermi's original theory and it is consequently of considerable importance to determine which of the two versions of the theory is in reasonable accord with observation.

In section III, we return to Fermi's original thermodynamic theory and show that his estimate of the total number of pions generated is in error by a factor of about 2 (Kothari, 1954). It is seen that as an indirect consequence of the contraction of the reaction-space, one obtains, in addition to the number of pions estimated by Fermi, an approximately equal number N_0 , which, in the C -system, have all the same energy μc^2 . The existence of this number N_0 is inherent in Fermi's theory.

II. We consider a collision between a high energy nucleon and a nucleus at rest in the laboratory system. We assume that the nucleon in going through the

nucleus hits n nucleons, the collision being regarded as taking place between the primary nucleon and the 'lump' of these n nucleons. For a heavy nucleus ($A \sim 100$), $n \simeq 5$. The velocity of the centre of mass of the total system composed of the primary nucleon and the lump of n nucleons is (in units of c)

$$\beta_c = \frac{(\gamma^2 - 1)^{\frac{1}{2}}}{(\gamma + n)}, \quad \dots \dots \dots (1)$$

where γMc^2 is the energy of the primary nucleon in the laboratory system. The total available energy in the C -system is

$$W = Mc^2(2n\gamma + n^2 + 1)^{\frac{1}{2}}, \quad \dots \dots \dots (2)$$

and the Lorentz contraction factor is

$$\frac{1}{\gamma_c} = \frac{(2n\gamma + n^2 + 1)^{\frac{1}{2}}}{(\gamma + n)}. \quad \dots \dots \dots (3)$$

The volume (in the C -system) in which the pions are produced may be taken as

$$\Omega = \frac{n\Omega_0}{\gamma_c}, \quad \dots \dots \dots (4)$$

where

$$\Omega_0 = \frac{4\pi}{3} \left(\frac{\xi \hbar}{\mu c} \right)^3, \quad \xi \sim 1.$$

If the final state consists of N particles, the probability $P(N)$ of this state being realized according to Fermi's theory is

$$P(N) = k \left(\frac{\Omega}{\hbar^3} \right)^{N-1} \frac{dQ_{N-1}}{dW}, \quad \dots \dots \dots (5)$$

where k is a constant and $Q_{N-1}(W)$ is the volume of the momentum space containing all those states for which the energy is less than or equal to W . Assuming that the final state consists of N relativistic pions and S non-relativistic nucleons, we write

$$W = \frac{1}{2M} \left[\sum_{i=1}^{S-1} P_i^2 + \left(- \sum_{i=1}^{S-1} P_i \right)^2 \right] + SMc^2 + \sum_{i=1}^N cp_i \quad \dots \quad (6)$$

In writing down this expression we have assumed momentum conservation for the heavy particles. Let ϵ_0 be the least permissible energy for a pion produced in the collision. Then the volume of the momentum space bounded by the surface defined by (6) and

$$cp_i > \epsilon_0, \quad (i = 1, 2, \dots, N)$$

is given by

$$Q_{N-1}(W) = \frac{(4\pi)^N (2\pi M)^{\frac{3(S-1)}{2}} (W - SMc^2 - N\epsilon_0)^{3N + \frac{3}{2}(S-1)}}{c^{3N} S^{\frac{3}{2}}} \left(\sum_{r=0}^{2N} \frac{a_r \alpha^{2N-r}}{\Gamma\left(r + N + \frac{3S}{2} - \frac{1}{2}\right)} \right), \quad \dots \dots (7)$$

where

$$\alpha = \frac{\epsilon_0}{W - SMc^2 - N\epsilon_0},$$

and the coefficients a_r are given by

$$(2 + 2k + k^2)^N = \sum_{r=0}^{2N} a_r k^{2N-r} \quad \dots \dots \dots (8)$$

Substituting in (5) for Ω and $Q_{N-1}(W)$ from (4) and (7), we obtain after some simple calculation

$$P(S, N) = kS^{-3/2}(\pi/2)^{(S-1)/2} \left(\frac{4\xi^3}{3\pi} \frac{M^3}{\mu^3} \right)^{N+S-1} (Mc^2)^{-3N-\frac{3}{2}(S-1)} \frac{n^2 w}{(n^2 + w^2 - 1)^{N+S-1}} \times \\ \times \sum_{r=0}^{2N} \left[\frac{a_r \alpha^{2N-r} W_0^{3N+\frac{3S}{2}-\frac{5}{2}}}{\Gamma\left(r+N+\frac{3S}{2}-\frac{1}{2}\right)} \left\{ 3\left(N+\frac{S-1}{2}\right) \frac{W_0}{\epsilon_0} \frac{d\epsilon_0}{dW} + \left(N+r+\frac{3S}{2}-\frac{3}{2}\right) \times \right. \right. \\ \left. \left. \times \left(1-N \frac{d\epsilon_0}{dW} - \frac{W_0}{\epsilon_0} \frac{d\epsilon_0}{dW}\right) \right\} \right] \dots \quad (9)$$

$$\left. \begin{array}{l} \text{Here} \\ \text{and} \end{array} \right\} \begin{array}{l} w = \frac{W}{Mc^2}, \\ W_0 = (W - SMc^2 - N\epsilon_0). \end{array} \dots \dots \dots (10)$$

If the 'tunnel' be considered as a cylinder with area of cross-section πr_0^2 and length $\frac{2nr_0}{\gamma_c} \left(r_0 = \frac{\eta\hbar}{\mu c}, \eta \sim 1 \right)$, the zero-point energy of a pion will be given by

$$\epsilon_0 = \frac{c\hbar}{r_0} \left[5.784 + \eta^2 + \frac{\pi^2 \gamma_c^2}{4n^2} \right]^{\frac{1}{2}} \\ = \frac{c\hbar}{r_0} \left[5.784 + \eta^2 + \frac{\pi^2}{4n^2} \frac{(\gamma+n)^2}{(2n\gamma+n^2+1)} \right]^{\frac{1}{2}} \dots \dots (11)$$

The maximum number, N_{\max} , of pions produced is

$$N_{\max} < \frac{W - SMc^2}{\epsilon_0} \\ = \frac{Mc^2 r_0}{c\hbar} \left[1 - \frac{S}{(2n\gamma+n^2+1)^{\frac{1}{2}}} \right] \times \left[\frac{5.784 + \eta^2}{(2n\gamma+n^2+1)} + \frac{\pi^2}{4n^2} \frac{(\gamma+n)^2}{(2n\gamma+n^2+1)} \right]^{-\frac{1}{2}} \dots (12)$$

For very high primary energies

$$N_{\max} < \frac{4}{\pi} \frac{M}{\mu} \eta n^2 = 8.49 \eta n^2, \dots \dots (13)$$

$$\text{taking} \quad \frac{\mu}{M} = 0.15.$$

Explicit calculations of the relative probabilities have been made for $n = 1$ and $n = 5$. $n = 1$ corresponds to the case of a collision between two nucleons, while $n = 5$ corresponds to the collision between a nucleon and a heavy nucleus.

(i) Nucleon-Nucleon Collision :

In this case

$$\frac{\epsilon_0}{Mc^2} = \frac{\mu}{M} \frac{1}{\eta} \left(5.784 + \eta^2 + \frac{\pi^2 w^2}{16} \right)^{\frac{1}{2}} \dots \dots (14)$$

Calculating $\left(\frac{d\epsilon_0}{dW}\right)$ and substituting in (9), we obtain (neglecting nucleon pair creation)

$$P(2, N) = \frac{k}{Mc^2} \left(\frac{4\xi^3}{3\pi} \frac{M^3}{\mu^3}\right)^{N+1} w^{-(N+1)} \left(\frac{\epsilon_0}{Mc^2}\right)^{8N+\frac{1}{2}} \times \\ \times \left[\frac{3\pi^2}{16} \left(\frac{\mu}{\eta M}\right)^2 \frac{wMc^2}{\epsilon_0} (N+\frac{1}{2}) \left(\sum_{r=0}^{2N} \frac{a_r z^{N+r+\frac{1}{2}}}{\Gamma(N+r+\frac{5}{2})} \right) + \right. \\ \left. + \left\{ 1 - \frac{\pi^2}{16} \left(\frac{\mu}{\eta M}\right)^2 \frac{w(w-2)}{(\epsilon_0/Mc^2)^2} \right\} \left\{ \sum_{r=0}^{2N} \frac{a_r z^{N+r+\frac{1}{2}}}{\Gamma(N+r+\frac{3}{2})} \right\} \right], \quad \dots \quad (15)$$

where

$$z = \frac{W_0}{\epsilon_0} = \frac{W-2Mc^2-N\epsilon_0}{\epsilon_0} = \frac{w-2}{(\epsilon_0/Mc^2)} - N,$$

and

$$w = \frac{W}{Mc^2} = (2\gamma+2)^{\frac{1}{2}}.$$

γMc^2 denotes the energy of the bombarding nucleon in the laboratory system. Table I shows the relative probabilities for the production of various numbers N of pions calculated from equation (15) for $\xi = 1$ and $w = 3, 4$ and 5 . \bar{N} is the average number of pions produced. For comparison we reproduce in Table II Fermi's results, obtained without introducing the zero-point energy.

TABLE I

γ	w	ϵ_0/Mc^2	$N = 0$	1	2	3	\bar{N}	$N_{\max.}$
3.5	3	0.597	20	80			0.80	1.67
7.0	4	0.696	2	55	43		1.42	2.88
11.5	5	0.805	0	12	78	10	1.98	3.73

Relative probabilities for the production of N pions in a nucleon-nucleon collision when the zero-point energy is taken into account. $N_{\max.}$ is the maximum number of pions produced, while \bar{N} is the average number of pions produced during the collision.

TABLE II

γ	w	$N = 0$	1	2	3	4	5	6	7	\bar{N}
3.5	3	9	59	30	2					1.2
7.0	4		13	40	33	11	2			2.5
11.5	5		2	15	34	31	14	3	1	3.5

Relative probabilities for the production of N pions in a nucleon-nucleon collision, disregarding the zero-point energy.

(ii) Nucleon-Nucleus Collision :

In this case we take $n = 5$. ϵ_0 then changes inappreciably in the range $\gamma < 10$, and hence in differentiating Q_{N-1} we may regard it as constant. Carrying out the

calculation, we finally obtain for the probability of a state consisting of six nucleus and N pions,

$$P(6, N) = \frac{k}{Mc^2} \left(\frac{\epsilon_0}{Mc^2} \right)^{8N+\frac{1}{2}} \left(\frac{4\xi^3}{3\pi} \frac{M^3}{\mu^3} \right)^{N+5} \left(\frac{25w}{24+w^2} \right)^{N+5} \left(\sum_{r=0}^{2N} \frac{a_r z^{N+r+\frac{1}{2}}}{\Gamma(N+r+\frac{1}{2})} \right), \quad (16)$$

where $z = \frac{w-6}{(\epsilon_0/Mc^2)} - N$.

$\left(\frac{\epsilon_0}{Mc^2} \right)$ may be taken as

$$\frac{\epsilon_0}{Mc^2} = \frac{c\hbar}{2(\xi\hbar/\mu c)} \frac{1}{Mc^2} = \frac{\pi}{\xi} \frac{\mu}{M} = \frac{0.471}{\xi} \quad \dots \quad (17)$$

Table III shows the relative probabilities for the production of various numbers N of pions calculated from eqn. (16) for $w = 8$ and $w = 10$ and $\xi = 1$. In Table IV are given the relative probabilities obtained by letting $\epsilon_0 \rightarrow 0$. The last column in each table gives the average number of pions emitted.

TABLE III

γ	w	$N = 0$	1	2	3	4	5	6	\bar{N}
3.8 7.4	8 10	12	54	33 5	1 26	46	21	2	1.22 3.89

Relative probabilities for the production of N pions in a nucleon-nucleus collision, taking into account the zero-point energy; γMc^2 is the energy of the bombarding nucleon and \bar{N} is the average number of pions produced.

TABLE IV

γ	w	$N = 0$	1	2	3	4	5	6	7	8	9	10	11	12	\bar{N}
3.5 7	7.8 9.8	2	12	27	30	19 2	8 6	2 13	20	22	18	11	5	2	2.8 8.0

Relative probabilities for the production of N pions in a nucleon-nucleus collision, disregarding zero-point energy.

III. In this section we return to Fermi's thermodynamic theory and show that his estimate of the total number of pions produced has appreciable error (Kothari, 1954). According to Bose-Einstein statistics, the number of particles n_i in the energy state ϵ_i is

$$n_i = \frac{1}{\frac{1}{A} e^{\epsilon_i/kT} - 1},$$

where A is the degeneracy parameter. Fermi assumes that the number of pions produced in an extremely high energy collision is given by

$$N = \sum_i n_i = \sum_i \frac{1}{e^{\epsilon_i/kT} - 1}, \quad \dots \quad (18)$$

where T is the temperature of the pion assembly. He replaces the sum by the integral

$$\frac{4\pi g\Omega}{(ch)^3} \int_0^\infty \frac{\epsilon^2 d\epsilon}{e^{\epsilon/kT} - 1} = \frac{4\pi g\Omega}{(2\pi c\hbar)^3} \cdot 2! \zeta(3) (kT)^3 = N_1, \text{ say } \dots \quad (19)$$

g is the weight factor of the pions (equal to 3) and Ω is the volume in which the pions are assumed to be generated. It is given by

$$\Omega = \frac{4\pi}{3} \left(\frac{\hbar}{\mu c} \right)^3 \frac{2Mc^2}{W} = \Omega_0 \frac{2Mc^2}{W}, \quad \dots \quad (20)$$

W being the total available energy in the C -system. If we write

$$N_0 = N - N_1,$$

then

$$N_0 = \alpha g (kT/\mu c^2), \quad \dots \quad (21)$$

α being a numerical factor of the order of unity. $(kT/\mu c^2)$ is the first term of the sum (18) for $(\mu c^2/kT)$ small. The temperature of the pion gas is given by Stefan's Law, viz.,

$$\frac{W - 2Mc^2}{\Omega} = \frac{g}{2} \frac{6\zeta(4)}{\pi^2 c^3 \hbar^3} (kT)^4 \quad \dots \quad (22)$$

Substituting for kT from this equation, we obtain from (19) and (21)

$$\frac{N_0}{N_1} = \alpha \frac{[9\pi\zeta(4)]^{\frac{1}{4}}}{2\zeta(3)} \left(\frac{g\mu}{M} \right)^{\frac{1}{4}} (wf)^{-\frac{1}{4}}, \quad \dots \quad (23)$$

where $w = \frac{W}{Mc^2}$, and $f = \frac{\Omega}{\Omega_0}$.

When $f \sim 1$, $\frac{N_0}{N_1} \rightarrow 0$ as $w \rightarrow \infty$.

However, the actual value of Ω assumed by Fermi is (20); hence we have

$$\begin{aligned} \frac{N_0}{N_1} &= \alpha \frac{[9\pi\zeta(4)]^{\frac{1}{4}}}{2\zeta(3)} \left(\frac{g\mu}{2M} \right)^{\frac{1}{4}} \\ &\simeq 0.54 \text{ (for } \alpha = \tfrac{1}{2} \text{).} \end{aligned}$$

Thus we see that the number N_0 is of the same order as N_1 and hence Fermi's expression (equation 19) seriously underestimates the total number of pions produced. In the laboratory system, the N_0 pions will form a narrow jet moving along the direction of the incident nucleon, each pion having energy $\frac{w}{2} \mu c^2$.

In conclusion, I wish to express my thanks to Dr. D. S. Kothari and Dr. F. C. Auluck for help in completing this work. I am also grateful to the Atomic Energy Commission, India, for the award of a research fellowship.

SUMMARY

Fermi's statistical theory of multiple pion production is extended to the case of a nucleon-nucleus collision, taking into account the zero-point energy of the pions. Numerical calculations of the relative probabilities are made. It is seen that the introduction of the zero-point energy substantially reduces the number of pions produced.

Fermi's estimate of the number of pions produced on the basis of statistical thermodynamics is shown to have appreciable error.

REFERENCES

- Auluck, F. C., and Kothari, D. S. (1953). Fermi's Theory of Nucleon Collisions and the Zero-point Energy of Pions. *Phys. Rev.*, **90**, 1002.
- Bhabha, H. J. (1953). Production of Mesons and Localisation of Field Energy. *Proc. Roy. Soc.*, **219**, 293.
- Cocconni, G. (1954). Fermi's Theory of Collisions of High Energy Particles. *Phys. Rev.*, **93**, 1107.
- Fermi, E. (1950). High Energy Nuclear Events. *Prog. Theor. Phys.*, **5**, 570.
- (1953). Multiple Production of Pions in Nucleon-Nucleon Collisions at Cosmotron Energies. *Phys. Rev.*, **92**, 452.
- Kothari, D. S. (1954). Fermi's Thermodynamic Theory of the Production of Pions. *Nature*, **173**, 590.

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MO CALCULATION ON THE ABSORPTION SPECTRA OF TETRAPHENYL PORPHINE

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INTRODUCTION

It has been shown by Basu (1954) that the MO calculation on as complex a molecule as phthalocyanine can be used in predicting the spectral transition for this compound. In view of the insolubility of this compound in common organic solvents, an experimental verification of the calculated results have not been made. Recently Calvin *et al.* (1946) have reported the results of their experimental investigation on the absorption spectra of tetraphenyl porphine (Fig. 1) in alcohol and in chloroform. Since MO calculation can be extended to this compound with equal ease, computation has been made on tetraphenyl porphine and the results have been found to agree fairly closely with the absorption spectra of this compound, as reported in the present paper.

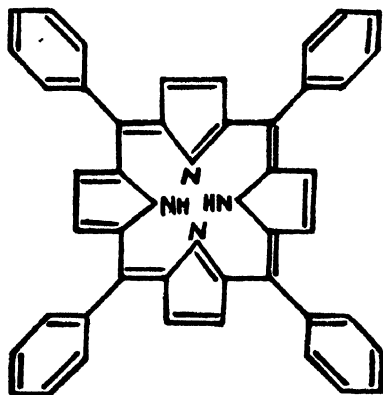


FIG. 1. Tetraphenyl Porphine.

METHOD OF CALCULATION

The method has already been described in detail (Basu, 1954). Only a short account will be given here.

In the LCAO MO method each electron is considered to be in a self-consistent field due to other electrons and atomic nuclei, and the single electron wave function ψ_j is approximated by a linear combination of atomic orbitals ϕ_k , each belonging to one of the nuclei. We treat the problem as a π -electron system, excluding the σ -electrons involved in the formation of single bonds.

$$\psi_j = \sum a_{jk} \phi_k$$

The electronic energy E_j of the molecular orbital ψ_j can be obtained as roots $E = E_j$ ($j = 1, 2, 3, \dots, l$) of a secular equation

$$D = |H_{mk} - S_{mk}E| = 0$$

where l denotes the number of π -electron atomic orbitals and H_{mk} and S_{mk} are given by the following equations respectively

$$H_{mk} = \int \phi_m^* H \phi_k d\tau, S_{mk} = \int \phi_m^* \phi_k d\tau$$

For simplicity it is assumed that the atomic orbitals are orthogonal and normalised, viz.,

$$\int \phi_m^* \phi_k d\tau = 1, \text{ if } m = k \\ = 0, \text{ otherwise.}$$

On these assumptions the energy of an electron in ψ is given by

$$(H_{mm} - E)c_m = \sum_{k \neq m} H_{mk}c_k = 0$$

We make further assumption that H_{mm} (E_0) is same for all carbon atoms, $E_0 + 2\beta$ for nitrogen and $H_{mk} = \beta$ for any two neighbouring atoms bounded to each other.

With these assumptions the secular equation takes the form

$$(E - E_0)c_r + \Sigma \beta c_s = 0 \\ \text{or } Wc_r + \Sigma \beta c_s = 0 \\ \text{or } xc_r + \Sigma c_s = 0$$

where $x = (E - E_0)/\beta$ an energy parameter.

There are in all 48 MOs and 50 π -electrons in tetraphenyl porphine. With restrictions that two electrons of opposite spin can be adopted by each orbital, 25 out of 48 orbitals will be occupied. The calculation of absorption frequencies due to electronic transition between the occupied and vacant levels is made with the following prescription:—

- (i) Compute the roots (x) of the simple secular equation.
- (ii) Set $\beta = -23,000 \text{ cm.}^{-1}$ for aromatic C—C bonds.
- (iii) The transition frequencies are given by x (empty) $-x$ (highest filled) and is governed by the symmetry of the corresponding MOs.

RESULTS

Like phthalocyanine the group symmetry in tetraphenyl porphine is D_{4h} and π -electron MOs belong to group species A_{2u} , B_{2u} , A_{1u} , B_{1u} , and E_g . The last species exist in doubly degenerate pairs. The energy equations for these species are as follows:—

$$A_{2u}: x^8 + x^7 - 10x^6 - 9x^5 + 27x^4 + 22x^3 - 16x^2 - 12x = 0$$

$$B_{2u}: x^7 - x^6 - 8x^5 + 7x^4 + 18x^3 - 13x^2 - 9x + 4 = 0$$

$$A_{1u}: x^4 - x^3 - 2x^2 + x + 1 = 0$$

$$B_{1u}: x^5 - 4x^3 + 5x - 2 = 0$$

$$E_g: x^{10} - 12x^8 + x^7 + 36x^6 + 10x^5 - 41x^4 - 44x^3 + 32x^2 + 23x - 4 = 0$$

All these equations are solved by method of successive approximation correct up to first decimal place. The values of x (the energy parameter) for different species are listed in Table I in increasing order.

TABLE I

Energy parameter for tetraphenyl porphine

No.	Group species	x	No.	Group species	x
1	E_g	-3.25	23	B_{1u}	0.48
2	E_g	-3.25	24	A_{2u}	0.90
3	E_g	-2.70	25	E_g	0.95
4	E_g	-2.70	26	E_g	0.95
5	A_{2u}	-2.33	27	B_{2u}	0.98
6	B_{2u}	-2.00	28	E_g	1.00
7	B_{1u}	-1.98	29	E_g	1.00
8	B_{2u}	-1.80	30	E_g	1.00
9	B_{1u}	-1.18	31	E_g	1.00
10	E_g	-1.00	32	A_{1u}	1.00
11	E_g	-1.00	33	B_{1u}	1.00
12	A_{2u}	-1.00	34	E_g	1.10
13	E_g	-0.75	35	E_g	1.10
14	E_g	-0.75	36	E_g	1.20
15	A_{2u}	-0.61	37	E_g	1.20
16	B_{2u}	-0.60	38	A_{1u}	1.61
17	E_g	-0.55	39	B_{1u}	1.65
18	E_g	-0.55	40	B_{2u}	1.71
19	A_{2u}	0.00	41	A_{2u}	2.00
20	E_g	0.31	42	A_{2u}	2.13
21	E_g	0.31	43	B_{2u}	2.25
22	B_{2u}	0.40			

Only 43 energy levels have been listed in the table since calculated transition extends up to that level.

Lowest 25 of these levels are filled. The electron transition between the highest filled level and the vacant levels are controlled by the requirements that there can be no transition between levels belonging to the same group species, i.e. there can be no transition between E_g and E_g or between B_{1u} and B_{1u} and so on. With this restriction the first allowed transition is between the levels 25 (E_g) and 27 (B_{2u}), the second between 25 and 32 (A_{1u}), and so on. But it may be noted that the energy difference between these states are extremely small, so there will be strong configurational interaction and the levels will mix, as discussed in the previous communication (Basu, 1954). These transitions will give absorption in the infra-red region, the exact position of which will be rather uncertain. Transition accompanied with appreciable energy difference so as to give absorption in the visible and near ultraviolet region are listed in Table II along with the calculated and observed wavelength.

TABLE II

Calculated and observed absorption maxima for tetraphenyl porphine

Levels	Group species	Degeneracy	Wavelength (Å)	
			Calculated	Observed
25-38	$A_{1u}-E_g$	Double	6521	6500
25-39	$B_{1u}-E_g$	"	6210	6100
25-40	$B_{2u}-E_g$	"	5434	5500
25-41	$A_{2u}-E_g$	"	4223	5200
25-43	$B_{2u}-E_g$	"	3344	..

It may be observed that the agreement between the calculated and observed wavelength at maximum absorption is fairly close. A better agreement is not expected in view of the approximations used in solving the secular determinant and the neglect of configurational interactions for these high energy transitions.

SUMMARY

Calculation of energy levels in tetraphenyl porphine has been made by LCAO MO method and the result has been correlated with the observed spectral transition.

REFERENCE

- Basu, S. (1954). Molecular Orbital calculation on phthalocyanine. *Indian Jour. Phys.*, 28, 511.
Calvin, M., Ball, R. H., and Dorrough, G. D. (1946). A further study of the Porphine-like products of the reaction of benzaldehyde and pyrrole. *J. Amer. Chem. Soc.*, 68, 2278.

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AN X-RAY STUDY OF FLUORANTHENE CRYSTAL

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INTRODUCTION

Fluoranthene, a crystalline principle isolated from high boiling fraction of coal-tar by the normal process of distillation and repeated fractional crystallization, has the molecular formula $C_{16}H_{10}$ as determined from a study of chemical reactions and formation of various derivatives. Formerly, its molecular formula was given as $C_{15}H_{10}$ by Fittig and Gebhard, but later on Braun and Anton (1929) established uniquely the molecular formula of Fluoranthene as $C_{16}H_{10}$. The present paper reports some new morphological data, differing from those given by Groth and an X-ray investigation of single crystal of Fluoranthene.

EXPERIMENTAL RESULTS AND CONCLUSIONS

Single crystals of Fluoranthene of suitable size were obtained by the slow evaporation of the concentrated solution of the substance in ethyl alcohol. On a morphological examination with the help of a two-circle goniometer, the crystal was found to be monoclinic with (100), (001), (10 $\bar{2}$) and (111) faces developing moderately well, but $b(010)$ face being absent. The crystal growth has a tendency to be elongated along b axis. Rotation photographs taken about various possible crystallographic axes were used in determining the fundamental axes. The following Table I gives the results of the complete goniometric study of single crystal of Fluoranthene.

TABLE I

Inter-facial angle of Fluoranthene crystal according to the present author.

(111) : ($\bar{1}\bar{1}\bar{1}$) = 68°
(111) : (001) = 86°
(102) : (001) = 36° 40'
(102) : (101) = 33° 26'
(101) : (100) = 51° 39'
(100) : (001) = 58° 15'
β = 121° 45'

Thus it can be seen that the indices of the so-called m and r faces referred by Groth should be modified to (111) and (102). The axis which has been referred by Groth as c axis is really a diagonal axis, i.e. [111], and this is also corroborated by the X-ray rotation photograph taken about this diagonal axis and, consequently, the new monoclinic angle should be 121° 45'. From rotation photographs axial lengths and axial angle were obtained as follows:

$$\begin{aligned}a &= 18.46 \text{ \AA} \\b &= 6.205 \text{ \AA} \quad \beta = 121^\circ 45' \\c &= 22.11 \text{ \AA}\end{aligned}$$

whence $a : b : c = 2.975 : 1 : 3.563$.

TABLE II

*Indexing of spots and their estimated intensities.
b axis zero-layer*

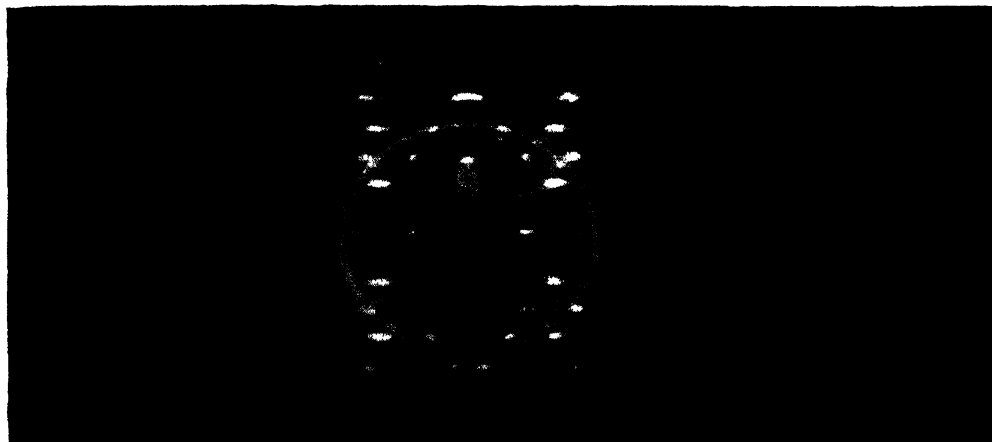
Index of spot	Intensity	Index of spot	Intensity	Index of spot	Intensity
200	m.	20(12)	v.w.	506	m.w.
300	v.s.	20(14)	w.	508	v.w.
500	m.w.	20(16)	w.	50(10)	v.w.
600	m.	302	s.	602	m.
700	m.	304	m.	604	m.
800	m.	306	w.	608	v.w.
900	m.w.	308	m.	702	m.
(10)00	v.w.	30(10)	w.	704	v.w.
(11)00	w.	30(12)	v.w.	802	m.w.
(12)00	v.w.			804	m.
(13)00	v.w.	30(14)	v.w.	902	m.w.
(15)00	v.w.	30(16)	v.w.	906	v.w.
102	m.	402	v.w.	908	w.
104	m.w.	404	m.	90(10)	v.w.
106	m.w.	406	m.	(10)04	v.w.
202	s.	408	m.	(11)02	w.
204	m.w.	40(10)	v.w.	(11)04	v.w.
				(13)02	w.
206	m.	40(12)	v.w.		
208	m.	502	v.w.	(13)04	w.
20(10)	w.	504	m.		

TABLE III

*Indexing of spots and their estimated intensities.
a axis zero-layer*

Index of spot	Intensity	Index of spot	Intensity	Index of spot	Intensity
002	v.s.	020	s.	035	v.w.
004	v.s.	040	v.w.	039	v.w.
006	s.	06-0	w.	03(10)	v.w.
008	s.	021	s.	03(11)	v.w.
00(10)	m.	022	m.	041	m.
00(12)	w.	023	w.	042	m.
00(14)	v.w.	024	s.	043	w.
00(16)	w.	026	s.	044	m.w.
012	w.	027	w.	046	v.w.
013	m.	028	m.	047	v.w.
014	w.	029	w.	048	m.w.
015	v.w.	02(10)	m.w.	049	w.
016	v.w.	02(11)	w.	04(11)	m.w.
017	v.w.	02(12)	m.w.	04(12)	w.
018	v.w.	02(13)	m.w.	04(13)	v.w.
019	m.	02(14)	m.w.	04(14)	v.w.
01(11)	v.w.	02(15)	m.w.	062	v.w.
01(12)	v.w.	02(17)	m.w.	063	v.w.
01(13)	v.w.	031	v.w.	065	v.w.
01(15)	w.	032	v.w.	066	v.w.
01(16)	w.	033	v.w.		
		034	v.w.		

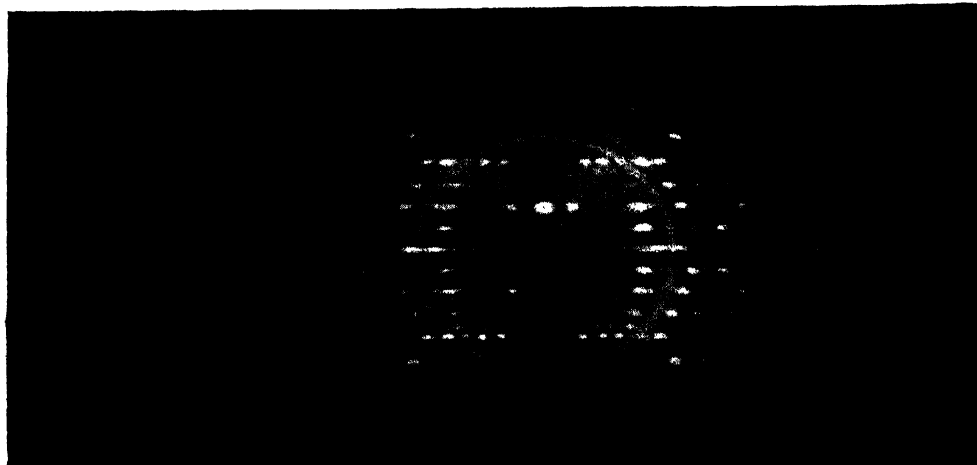
* (001) planes observed in *a* and *b* axes zero-layer Weissenberg photographs.



(a)



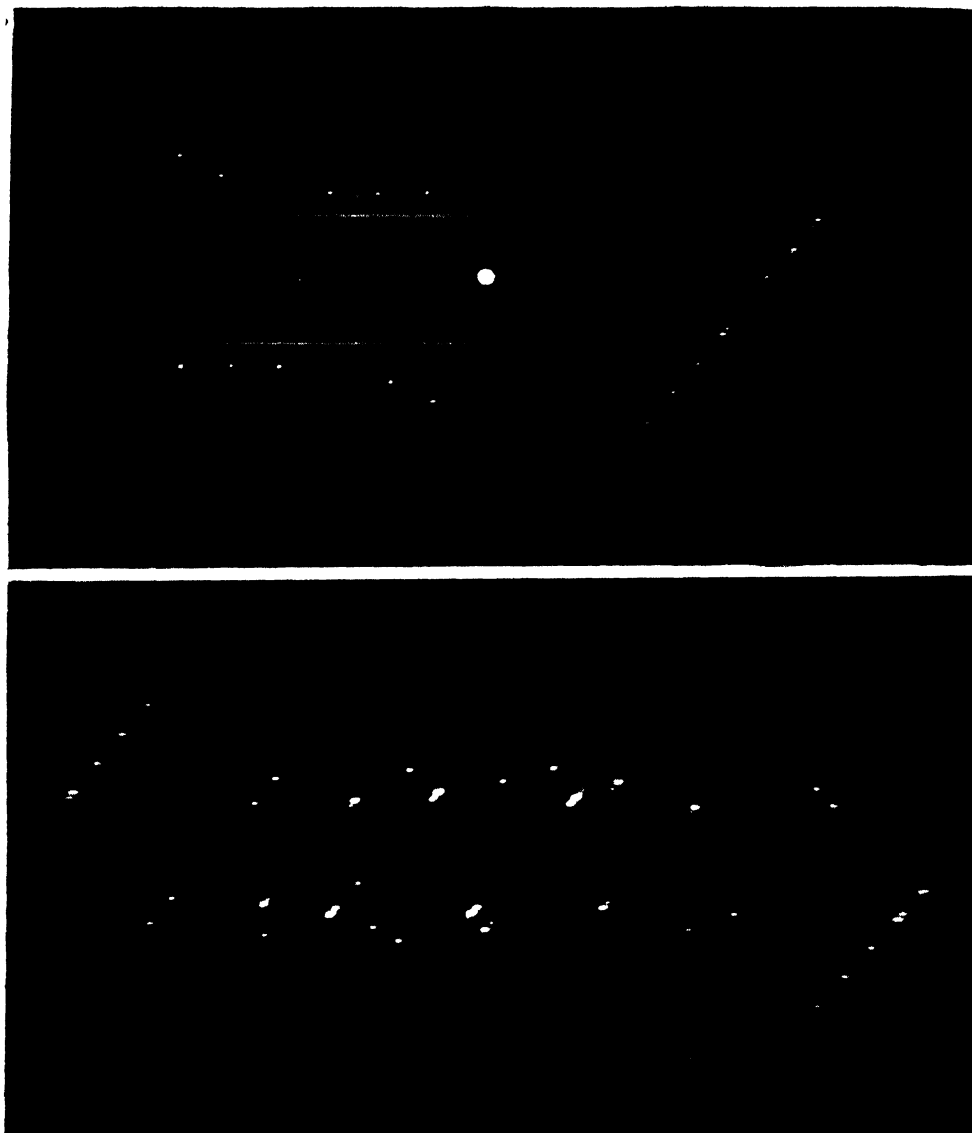
(b)



(c)

- (a) Rotating Crystal Photograph taken about a axis.
(b) Rotating Crystal Photograph taken about b axis.
(c) Rotating Crystal Photograph taken about c axis.

Camera : Cylindrical ; radius 4.501 cm.



Zero-level Weissenberg Oscillation Photograph taken about a axis (top fig) and b axis (bottom fig.).

The density of the crystal determined by floatation method was found to be 1.238 ± 0.01 gm. cm.⁻³. On the basis of this value and those already obtained from values of the axial parameters the number of molecules in the unit cell was calculated and found to be 8.

Over-exposed zero-level Weissenberg photographs were taken about b and a axes by the normal-beam method, giving ($h0l$) and ($0kl$) reflections respectively. The spots were identified by drawing requisite Weissenberg charts by the method of Schneider (1928). The relative intensities, as determined by the eye-estimation, were noted down against all the spots in Tables II and III.

First and second-level Weissenberg photographs about b axis were also taken by the equi-inclination method, giving ($h1l$) and ($h2l$) reflections respectively and the indexing of the spots showed that there were no systematic absence in (hkl) planes.

From Tables II and III it is observed that the conditions of extinction of the spots were

($h0l$) planes absent when l is odd.

($0k0$) planes absent when k is odd.

The extinction of $h0l$ when l is odd implies the presence of a glide plane parallel to ($0k0$) with glide component $c/2$. Then the space-group of the crystal definitely contains a glide plane. Again the extinction of ($0k0$) when k is odd implies that b axis is a two-fold screw axis. So the space-group of the crystal may be written as $P2_1/c$ or C_2^5 . Now the number of equivalent point in this space-group is 4, i.e. if one molecule occupies any general position in the unit cell of the crystal, by symmetry operations there should be altogether 4 molecules in the unit cell. But experimentally it has been found that the number of formula weights $C_{16}H_{10}$ in the unit cell is 8. Thus there are two sets of 4 molecules in the unit cell of the crystal, the orientation of the two sets being independent of each other.

ABSTRACT

Single crystal of Fluoranthene with the molecular formula $C_{16}H_{10}$ has been studied both morphologically and by means of X-rays. Goniometric study, together with rotation photographs and density measurement, showed that the crystal was monoclinic with the following axial lengths and axial angle:

$$a = 18.46 \text{ \AA}, b = 6.205 \text{ \AA}, c = 22.11 \text{ \AA}$$

$$\beta = 121^\circ 45'$$

$$(\text{number of molecules per unit cell} = 8)$$

Weissenberg photographs about the crystallographic axes showed that ($h0l$) planes are present when l is even and ($0k0$) plane is present when k is even. The crystal is found to belong to the space-group $C_2^5 - P2_1/c$, and so there are four asymmetric pairs of Fluoranthene molecules in the unit cell.

ACKNOWLEDGEMENTS

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REFERENCES

- Braun, J. Von, and Anton, E. (1929). Benzopolymethylene Compounds. XV. Composition, constitution and synthesis of Fluoranthene. *Ber. Dtsch. Chem. Ges.*, **62** (B), 145-151.
Groth, P. (1906-19). *Chemische Kristallographie*, **5**, 430, Leipzig, Engelmann.
Schneider, W. (1928). Über die graphische Auswertung von Aufnahmen mit dem Weissenbergschen Röntgengoniometer. *Zeits. Krist. (A)*, **69**, 41-48.

PAPER CHROMATOGRAPHIC STUDY OF JUTE HOLOCELLULOSE

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INTRODUCTION

Jute contains about 10.5–14% lignin, 24–30% hemicellulose and 55–63% α -cellulose. From the carbondioxide values of jute, the total hemicellulose and the yield of furfural, its hemicellulose components have been subdivided into polyuronides, pentosans and hexosans. Little attempt has been made in the past to isolate and identify these hemicellulose components even by selective extraction and precipitation. The idea that polyuronide is polygalacturonic acid and pentosan is xylan, is based more on assumption than on experimental evidence.

With the development of partition chromatography the isolation and characterisation of the components of polysaccharides have become easier. The success of the preliminary partition chromatography (Sarkar, Mazumdar and Pal, 1950; Bose and Burma, 1952; Das, Roy Chaudhuri and Wareham, 1952) suggested that useful information might be obtained in applying the methods of paper chromatography (Consden, Gordon and Martin, 1944) as applied to sugars (Partridge, 1946) to a detailed study of sugar components in jute holocellulose and this communication describes the separation and identification of the different sugars present in jute holocellulose.

In selecting the acid for hydrolysing jute holocellulose, it was thought desirable to try a weak acid so as to prevent decomposition of sugar components as far as possible. In this connection the work of Spoehr (1947) on the use of formic acid as a hydrolysing agent for algenic acid seemed to be encouraging as he showed that uronic acid was not decarboxylated with boiling formic acid in contrast to boiling mineral acid solution. In addition the normal method of using mineral acids is rather cumbersome whereas in the formic acid method no neutralisation is required and therefore avoids the possible loss entailed through adsorption of the inorganic salts.

In view of the observations (Das, Mitra and Wareham, 1952, 1953, 1954) that some pentosans remain associated with the so-called α -cellulose, it is desirable to hydrolyse the holocellulose completely. The main difficulty encountered in this study is the drastic treatment required for complete hydrolysis of the holocellulose. The usual process of dissolving holocellulose is the 72% H_2SO_4 method of Norman and Jenkins (1933) which seemed to be too drastic for the detection of sugars which may be present only in traces. Hence it was thought desirable to try repeated treatments of formic acid, which being weak is unlikely to cause any serious decomposition of sugars. Another advantage of this procedure is to obtain several fractions and thus to study homogeneity of these fractions with respect to their composition. This procedure was also undertaken to establish the observation of Das, Mitra and Wareham (1952, 1953, 1954) that cellulosan is present in jute.

72% sulphuric acid has been used by Sarkar, Mazumdar and Pal (1950) for hydrolysing jute α -cellulose, 72% sulphuric acid and also hydrochloric acid of varying concentrations have been used here as hydrolysing agent for jute holocellulose in order to compare the results obtained by the mineral acids with those found by formic acid method.

METHODS AND MATERIALS

Jute.—The sample was white jute (*Corchorus Capsularis*) obtained from 24-Parganas District of West Bengal and was of good and soft quality. The whole fibre stem was purified by extraction with alcohol-benzene mixture (1:1) in a Soxhlet apparatus followed by washing in water. After drying in air, it was freed from adhering bark and specks by hand, cut into short length (1 cm.), any traces of grease absorbed in the process being afterwards removed by further extraction with alcohol-benzene mixture. The jute was then rinsed in distilled water, filtered and dried in air. This was then mixed thoroughly, and stored over 72% R.H. acid at 35°C. in a desiccator. The ash content of jute was determined by igniting samples of known dry weight (1 g.) till constant in weight. In order to characterise jute, the lignin content was determined by the method of Norman and Jenkins (1933), α -cellulose was estimated by treating the holocellulose obtained after chlorite treatment (Chattopadhyay and Sarkar, 1946) with 17.5% caustic soda solution according to A.C.S. method slightly modified by Sarkar, Chatterjee, Mazumdar and Pal (1948). The hemicellulose content was calculated by difference. Furfural was determined by the method of Hibbert *et al.* (1923) as modified by Chattopadhyay and Sarkar (1946). The formula given by Doreè (1947a) was used in the calculation. The carbondioxide value of the fibre was estimated by the method of Nanji, Paton and Link (cf. Doreè, 1947b). The results are given in Table 1.

TABLE 1
Chemical composition of jute

	(% on dry weight)			
α -Cellulose	61.40
Hemicellulose	25.12
Lignin	11.78
Ash	0.81
Furfural	9.38
CO ₂ value	1.26

A. Preparation of holocellulose.—

The holocellulose was prepared by sodium chlorite method (Taylor *et al.*, 1940) as used for jute by Chattopadhyay and Sarkar (1946), which involved the treatment of cut defatted jute with 0.7% sodium chlorite (liquor ratio 1:50) for 2 hours at the boiling water bath at pH between 4.5 to 5 using acetic acid and sodium acetate buffer. It was then filtered, washed thoroughly with distilled water and then dried in air. The loss in weight recorded was 11.85% which approximately corresponds to the value of lignin content (11.78%). The lignin content of holocellulose was 0.33%.

B. Conditions of hydrolysis.—

Three methods of hydrolysis have been tried.

METHOD (i): Sulphuric acid.—Holocellulose was treated with 72% sulphuric acid (liquor ratio 1:20) at temperature below 10°C. for 16 hours. The product was then diluted to a strength of 3% sulphuric acid, followed by heating in a water bath under reflux for 2 hours and filtering through a sintered crucible. The filtrate was then neutralised with barium carbonate, centrifuged and then passed through a charcoal filter. The clear solution was then evaporated under reduced pressure to a syrup when the distillate was treated for furfural yielding products with aniline reagent and phloroglucinol.

METHOD (ii): Formic acid.—0.5 g. jute holocellulose and 30 c.c. of 85% formic acid were taken in a test tube with ground joint (B24). The mixture was

thoroughly shaken to obtain as complete a suspension as possible and to avoid formation of lumps. The tube was then connected with an air condenser with glass joint, the upper end provided with a calcium chloride tube and the mixture boiled for 30 minutes in an oil bath maintained at 130–135°C. The tube was then cooled rapidly under a tap, the contents were filtered through a sintered crucible and washed with a few changes of water. The filtrate was termed as fraction 1 and the precipitate fraction 2.

Fraction 1.—The filtrate with its washings was then evaporated to dryness under reduced pressure in a water bath and the distillate tested for furfural yielding substances as usual. The residue was dissolved with a small quantity of water and again evaporated to dryness under reduced pressure. This was then refluxed with 30 c.c. of water in boiling water bath for 3 hours and again evaporated to dryness under reduced pressure. This process of dilution and distillation was repeated until the distillate was only faintly acidic to litmus. The residue was then taken with water and treated with activated charcoal and then filtered through a charcoal bed. The filtrate with washings was then found to be neutral which was subsequently evaporated under reduced pressure to a syrup.

Subfractions.—The residue (fraction 2 mentioned above) was estimated by drying at room temperature in a vacuum desiccator over P_2O_5 until constant in weight, further hydrolysed with 85% boiling formic acid (liquor ratio 1:60) for 24 hours, and the filtrate termed as subfraction 2, and the residue again hydrolysed in the same way when the filtrate was denoted as subfraction 3. The process was repeated until subfraction 4, subfraction 5 and subfraction 6 were obtained. The residue left was termed as subfraction 7.

Subfractions 2, 3, 4, 5 and 6 were purified, decolourised and made to syrups in the same way as that of fraction 1, while for subfraction 7 sulphuric acid method of hydrolysis and purification, as explained above, was used.

Rate of hydrolysis.—In order to study the rate of hydrolysis of holocellulose and also to evaluate the percentage of these fractions and subfractions on the total holocellulose, 0.5 gm. holocellulose was hydrolysed with 30 c.c. of 95% boiling formic acid for the required length of time and at the end of this period, the contents were filtered through a tared sintered crucible, washed until free from acid. The filtrate was purified, decolourised and evaporated to dryness in the same way as before and then taken up with 1 c.c. water while the residue with its container (sintered crucible) was dried in an oven at 105°C. to constant weight. In all cases the first distillate was noted for furfural yielding substances as usual.

METHOD (iii): Hydrochloric acid of different concentrations (1N–6N) was tried for hydrolysing holocellulose, the procedure adopted being the same as that of formic acid, the duration of boiling in all cases was however 24 hours. The syrup was always slightly acidic.

C. Chromatographic Techniques: Apparatus.—

Round drums made up with galvanised sheets were used as chromatographic chambers while the solvent troughs were of glass. A vacuum desiccator top with grease was fitted on the chamber in order to make it air-tight. For inserting solvent during chromatography without disturbing the equilibrium a separating funnel was fitted on the lid of the desiccator, end of the former reaching to the solvent trough.

A conical flask with a fine glass jet connected with an electric air blower was found quite satisfactory for fine and uniform spray. Generally No. 1 Whatman filter paper was used whereas No. 3 occasionally, especially for isolation of sugars when larger quantity was desirable. Goodbrand air oven with arrangement for constant draught of pre-heated air ($105^\circ \pm 2^\circ\text{C.}$) was used for drying and developing the chromatograms.

Solvents.—General solvents used are phenol saturated with water (9:1) and *n*-butanol saturated with water (4:1); the upper layers of the mixtures were only used as running liquids whereas the lower layers were placed at the bottom of the chambers.

The major advantage of selecting moist butanol and moist phenol as solvents is the fact that while with the former galactose runs slower than glucose and arabinose slower than xylose; these cases are reversed in the case of latter solvent and thus helping detection of these four sugars all of which are present in jute holocellulose. For certain specific purpose especially for separating galactose from glucose the upper layer of the mixture of ethyl acetate-water-pyridine (2:2:1) as suggested by Jermyn and Isherwood (1949) was used. The lower layer as usual was used at the bottom of the chamber.

Spray liquids.—In general a saturated water solution of aniline oxalate (Partridge, 1948a) was used. Ammoniacal silver nitrate solution was also tried occasionally but in view of the fact that the aniline oxalate develops a beautiful pink colour for pentose, yellow for rhamnose, brown to yellow for hexoses and a pinkish brown colour for uronic acid, aniline oxalate was preferred to silver nitrate.

General procedure.—As one dimensional chromatography has been used in all cases, it was sometimes essential to run the chromatograms as long as 72 hours in moist phenol and 128–168 hours in moist *n*-butanol. The latter procedure has been found to be quite suitable to separate glucose from galactose. In the case of ethyl acetate-water-pyridine mixture (Jermyn and Isherwood, 1949), it has been found that 32 hours are quite suitable for this purpose. 3 μ l solution was used for each spot.

D. Identification of sugars.—

The colour developed by different sugars in the chromatogram after spraying with aniline oxalate together with their relative positions were the primary guides for their identification. But as there is the possibility of the presence of some of the sugars either only in traces or of similar R.F. value, both of which prevent separation, misleading information is likely to obtain by the normal chromatography alone. This is especially found to be true in the case of identification of galactose in the hydrolysate in presence of glucose, the former being present in jute holocellulose only in small proportion to that of the latter. For this purpose it is desirable to separate the individual sugars by paper chromatography followed by the characterisation by their physical properties (e.g. optical rotation) and compound formation. While this is being in progress and the results will be communicated later on, in the present investigations the sugars were separated by paper chromatography and provisionally identified in the following way.

No. 3 Whatman filter paper was generally used for this purpose. Consecutive spots of same dimension of the syrups of different fractions were spotted on the starting line of a 20 cm. wide paper (length 55 cm.) and the chromatogram was run in moist phenol for about 72 hours. It was then dried at 105°C. and a 2 cm. strip from side across the length was cut off and sprayed with aniline oxalate reagent and then the strips were developed by baking as usual. The different strips from the under-developed portion corresponding to the different bands of sugars were then cut off, across the width of the paper. These strips were then extracted separately with water and evaporated to a syrup as usual. The purity of different sugars obtained and their identifications were tested by further chromatography and whenever possible by specific colour tests as explained below:

(i) **By repeated chromatography.**—About a 7 cm. wide filter paper was spotted with three different spots, one with the test syrup, the next with the corresponding pure reference sugars and the last with a mixture of test solution and corresponding pure sugar. The chromatograph was then run in two different solvents for sufficient

length of time to effect separation of sugars of close R.F. values, if present. In most of the cases single non-diffused spot was observed indicating pure sugar but occasionally some mixtures of two sugars were also obtained. In the latter case the process of chromatographic separation (in these cases by No. 1 Whatman filter paper) as described above was repeated when pure sugars were obtained. This was especially experienced in separating and identifying galactose in presence of glucose.

From the rate of travel, the spot No. 2 was considered to be a di- or tri-saccharide. This isolated sugar when heated with naphthoresorcinol and hydrochloric acid and then extracted with ether, the extracted ether solution gave purple colour whereas the aqueous liquid was green indicating the presence of uronic acid and other non-extractable sugar. Spot No. 2 when further hydrolysed with 95% formic acid for 1 hour gave spots for methyl glucuronic acid and xylose only.

(ii) *By colour tests.*—Several colour reactions have been used to identify the sugars. These are naphthoresorcinol and hydrochloric acid test for Forsyth (1948); naphthoresorcinol and trichloroacetic acid test of Partridge (1948a); aniline phthalate in butanol as suggested by Partridge (1949) and further extended by Hough, Jones and Wadman (1950); aniline oxalate in water (Partridge, 1948b) and lastly the usual basic lead acetate test. The test results of the different syrups with these coloured reagents are recorded in Table 5.

TABLE 2

Degree of hydrolysis and degradation of holocellulose by different acids

Acid used for hydrolysis	Concentration of the acid	Duration of hydrolysis (hrs.)	% Total holocellulose	Test for furfural on the first distillate (intensity of colour reaction)
Formic ..	95%	$\frac{1}{2}$	31.9	Faint
" ..	"	3	33.9	+
" ..	"	6	37.4	++
" ..	"	12	49.3	+++
" ..	"	24	73.0	++++
" ..	"	48	88.8	+++++
" ..	"	72	96.0	+++++
Hydrochloric ..	1N	24	38.0	++++
" ..	2N	"	62.0	++++
" ..	4N	"	84.4	++++
" ..	6N	"	85.0	++++
Sulphuric acid ..	72%	18	100.0 (approx.)	++++

Above results show that when the duration of hydrolysis with formic acid is $\frac{1}{2}$ hour, decomposition of the pentoses and uronic acid is negligible or very little although about one-third of the total holocellulose is dissolved under this condition. Considering hemicellulose is likely to be attacked first than the α -cellulose, it should more or less represent the hemicellulose part of the holocellulose. As the jute holocellulose contains about 28.5% hemicellulose it seemed possible that the major part of hemicellulose, if not all, plus a small portion of α -cellulose are dissolved during this treatment.

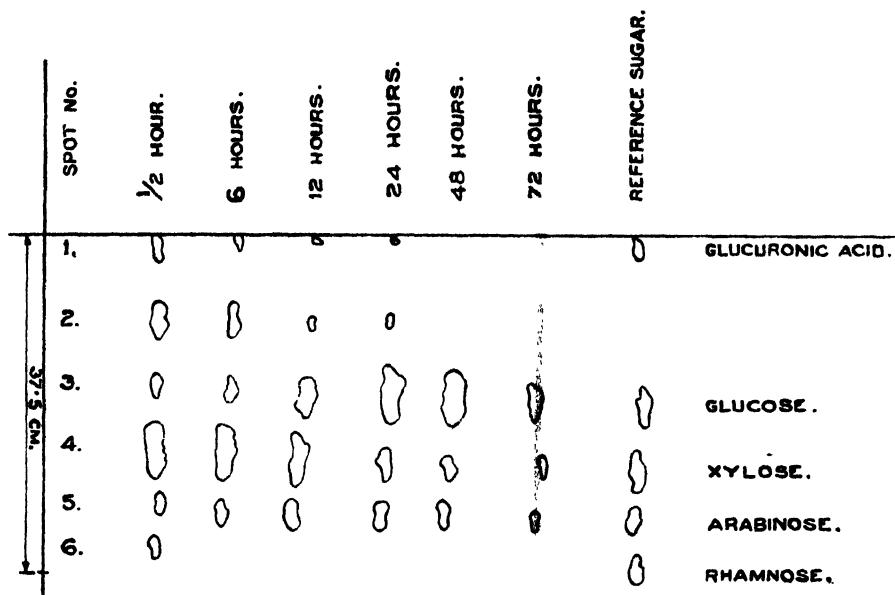


FIG. 1

Chromatogram in moist phenol. (72 hrs.) (Holocellulose hydrolysed by formic acid.)

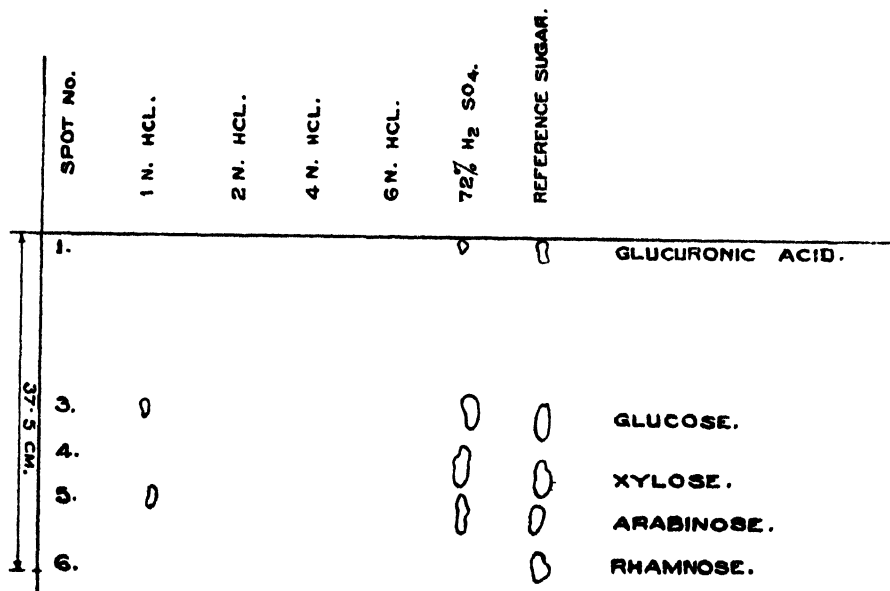


FIG. 2

Chromatogram in moist phenol. (72 hrs.) (Holocellulose hydrolysed by mineral acid.)

In the case of hydrolysis with hydrochloric acid, however, although the loss in weight with 1N acid is 38.0%, a value similar to that of formic acid when the duration is 6 hours, the high intensity of the furfural yield with the former method indicates that formic acid method is superior to hydrochloric acid method.

TABLE 3

Colour intensity of the spots shown in Figs. 1 and 2 after spraying with aniline oxalate solution

Ref. Sugar arranged in descending line	Glucuronic acid		Glucose and galactose	Xylose	Arabinose	Rhamnose
Spot No. (Figs. 1 and 2)	1	2	3	4	5	6
(1) Hydrolysate by formic acid; duration—						
(a) $\frac{1}{2}$ hr. ..	+++	+++	+	++++	++	++
(b) 6 hrs. ..	++	+++	+	++++	+++	(+)
(c) 12 " ..	+	++	++	++++	+++	—
(d) 24 " ..	+	+(+)	+++	+++	+++	—
(e) 48 " ..	Faint	(+)	++++	+(+)	++	—
(f) 72 " ..	—	—	++++	+	+(+)	—
(2) Hydrolysed by HCl for 24 hrs.; strength of acid being—						
(a) 1N ..	—	—	+	—	+	—
(b) 2N ..	—	—	—	—	—	—
(c) 4N ..	—	—	—	—	—	—
(d) 6N ..	—	—	—	—	—	—
(3) Hydrolysed by 72% sulphuric acid ..	+	—	+++ (no galactose)	++	+	—
Reaction with aniline oxalate after spraying and roasting ..	Pinkish yellow	Yellowish pink	Yellow brown	Pink	Pink	Yellow

This is clearly evident from the results of chromatography that while six spots could be identified with the formic acid method, only two have been noticed with HCl acid method and four by 72% H_2SO_4 method. Galactose and rhamnose have been completely missed by both the mineral acids hydrolysis whereas the uronic acid could be found with H_2SO_4 hydrolysate only in a small quantity but none with HCl hydrolysate. Spot No. 2, while very strong in the formic acid hydrolysate, is absent in the mineral acid hydrolysate. All these, therefore, suggest in favour of using formic acid method of hydrolysis. To ensure less degradation and also for convenience it was, however, thought desirable to use more a diluted form of formic acid available in the market (85% Merck) and this has been used in all the subsequent experiments.

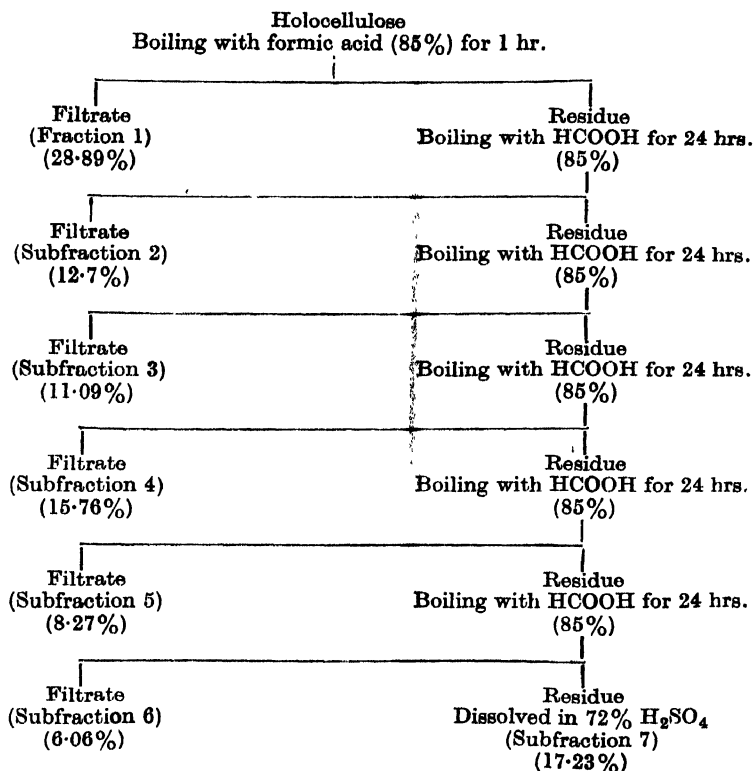
Fig. 3. *Distribution of holocellulose in different fractions*

TABLE 4

Colour intensity of the spots shown in Fig. 4 after spraying with aniline oxalate

Ref. Sugar arranged in descending line	Glucuronic acid		Glucose and galactose	Xylose	Arabinose	Rhamnose
Spot No. (Fig. 4)	1	2	3	4	5	6
Fraction 1 ..	++++	+++	+	+++++	++	+
Subfraction 2 ..	+	+	+++	+++++	+++	—
„ 3 ..	—	—	+++	++	++	—
„ 4 ..	—	—	+++++	+	+	—
„ 5 ..	—	—	+++++	+	+	—
„ 6 ..	—	—	+++++	(+)	+	—
„ 7 ..	—	—	+++++	—	—	—
Reaction with aniline oxalate after spray- ing and roasting ..	Pinkish yellow	Yellowish pink	Yellow brown	Pink	Pink	Yellow

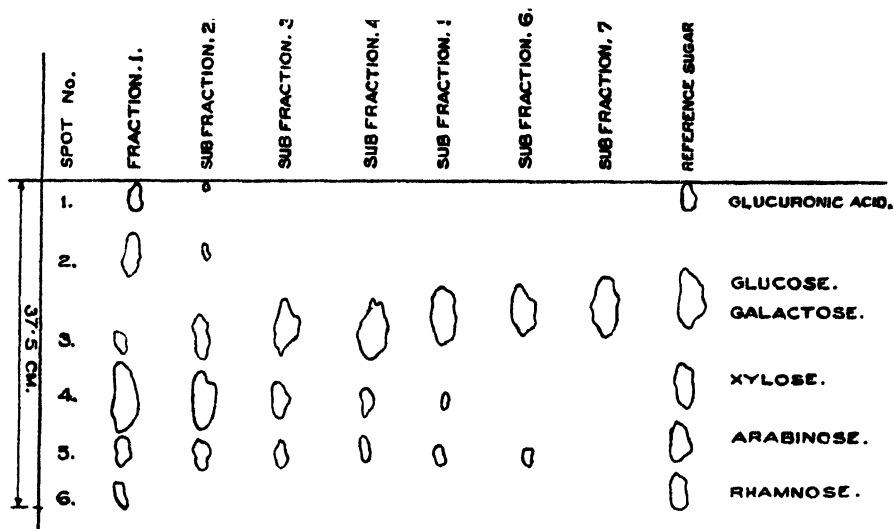


FIG. 4

Chromatogram of different fractions of jute holocellulose in moist phenol. (72 hrs.)

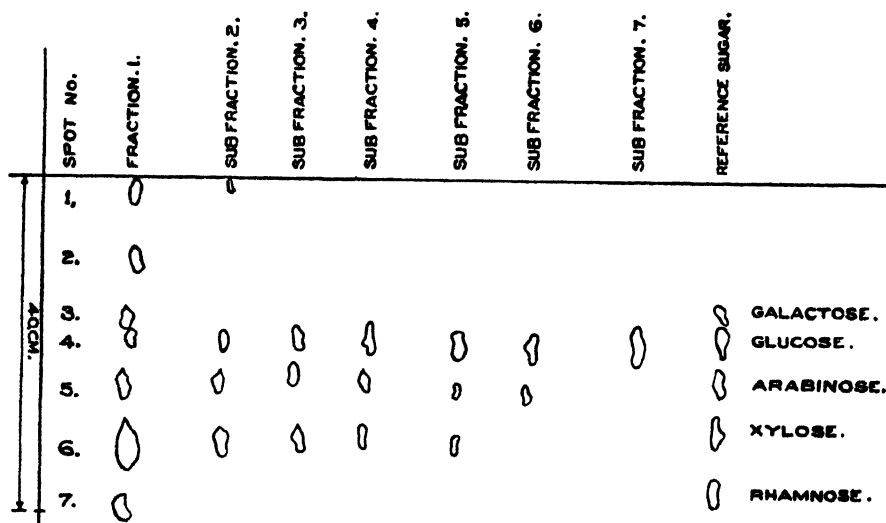


FIG. 5

Chromatogram of different fractions of jute holocellulose in moist n-butanol. (128 hrs.)

TABLE 5
Colour intensity of the spots shown in Fig. 5 after spraying with aniline oxalate

Ref. Sugars arranged in descending line	Glucuronic acid		Galactose	Glucose	Arabinose	Xylose	Rhamnose
Spot No. (Fig. 5)	1	2	3	4	5	6	7
Fraction 1	+++	- + +	++	+	++	+++	+
Subfraction 2	+	(+)	-	+	++	+++	-
" 3	-	-	-	+	++	+++	-
" 4	-	-	-	+	++	+++	-
" 5	-	-	-	+	++	+++	-
" 6	-	-	-	+	++	+++	-
" 7	-	-	-	+	++	+++	-
Reaction with aniline oxalate after spraying and roasting	Pinkish yellow	Yellowish pink	Yellow to brown	Brown	Pink	Pink	Yellow

TABLE 6
Identification by colour tests

Colour reagents	SPOT NO. (from Tables 3 and 4)						
	in moist <i>n</i> -butanol (Table 4)						
	1	2	3	4	5	6	7
	in moist phenol (Table 3)						
	1	2	3	3	5	4	6
	1	2	3	3	5	4	6
Aniline oxalate	Pinkish brown	Pinkish brown	Brown	Brown	Pink	Pink	Yellow
Heated with naphthoresorcinol and HCl	Violet blue extractable with ether	Green partly extractable with ether	Green not extractable with ether	Green not extractable with ether	Green not extractable with ether	Green not extractable with ether	Red not extractable with ether
Naphthoresorcinol and tri-chloroacetic acid ..	Green	Green	Green	Green	Bluish green	Bluish green	Green
Aniline phthalate in <i>n</i> -butanol	Pink	Reddish brown	Yellowish brown	Yellowish brown	Red	Red	Yellowish brown
<i>p</i> -anisidine hydrochloride in <i>n</i> -butanol	Cherry red	Yellow	Yellow	Yellow	Yellow	Yellow	Yellow
Heated with basic lead acetate	Light brown ppt.	Brown ppt.	Brick red ppt.	Brown ppt.	Brick red ppt.	Brown ppt.	No change in colour
Inference	Methyl gluconic acid	Methyl gluconic acid/xylose complex	Galactose	Glucose	Arabinose	Xylose	Rhamnose

DISCUSSION

The results presented in Tables 2 and 3 and Figs. 1 and 2 show that under the conditions of our experiments hydrolysis by formic acid ensures less degradation of the different sugars present in jute holocellulose than in the case with either hydrochloric or sulphuric acid. Thus formic acid method has been found to be especially suitable in detecting rhamnose which is apparently present in jute holocellulose only in small quantities and also uronic acid.

Fig. 5 shows that glucose can be completely separated from galactose by means of moist *n*-butanol when the duration of chromatogram is as high as 128 hours. Using this solvent as well as that of Jermyn and Isherwood (1949) no galactose has been detected in the hydrolysate of jute holocellulose when the hydrolysis is carried out by 72% sulphuric acid method, whereas the same is found in fraction 1 (Fig. 5) obtained by the formic acid method. These observations together with the fact that formic acid can hydrolyse jute holocellulose, a part at least, to simple sugars suggest that this acid should preferably be used in hydrolysing different carbohydrates especially when identification of different sugar constituents and preparation of some oligosaccharides (Das, Mitra and Wareham, 1953) are aimed at. Practically all jute holocellulose (96%) is dissolved with formic acid when the duration of hydrolysis is 72 hours. In this respect the values recorded in Fig. 3, however, are much lower than those of the corresponding ones in Table 2. This is mainly due to the fact that in obtaining the values recorded in Fig. 3, the process of hydrolysis was not continuous as in the other case but was interrupted at different stages followed by washing and drying with the repeated formation of hard masses which are less prone to attack by the acid. The lower value is also due to the more dilute (85%) acid being used for obtaining the results recorded in Fig. 3 than that (95%) for the results of Table 2.

When however the duration of hydrolysis is high as is required for dissolving higher percentages of holocellulose, degradation of sugars occurs as is evident from the presence of furfural (Table 1). From Figs. 1 and 2 and Table 3 the absence of rhamnose, uronic acid and with only a trace of xylose is especially remarkable. Spot 2 is further hydrolysed and finally disappears when duration of hydrolysis is over 48 hours. Arabinose and glucose seem to be less prone to attack by formic acid. All these inferences support the recent study of Das, Mitra and Wareham (unpublished investigation) on the rate of decomposition of different sugars with formic acid obtained with Somogyi's (1945) reagent.

From the chromatographic results recorded in Figs. 4 and 5, Tables 4 and 5 and also the colour tests recorded in Table 6, methyl glucuronic acid, glucose, galactose, xylose, arabinose and rhamnose have been provisionally identified in jute holocellulose. This has been supported by separating these sugars and identifying them by further chromatography as explained previously. That the uronic acid component is gluco—and not galacto—has been concluded from its failure to give any mucic acid test but forms saccharic acid which was identified in the form of *k*-salt.

Spot No. 2 has been separated and identified as a compound of methyl glucuronic acid and xylose from the observation that on further hydrolysis of this compound with formic acid it splits up into these two sugars. This proves the existence of methyl uronic acid-xylose linkages in the jute hemicellulose chains.

The above results also show that the seven fractions or subfractions obtained by the formic acid method from the jute holocellulose differ in chemical composition. First two fractions represent most of the hemicellulose and only a small amount of glucose, coming presumably from α -cellulose. None of the other subfractions contain any uronic acid, oligosaccharide or rhamnose but all of them except the last one more or less contain glucose, arabinose and probably xylose thus supporting the view of Das, Mitra and Wareham (1952, 1953, 1954) that cellulosan exists in

jute. The absence of xylose or its presence only in traces in the last two fractions is due to the possibility of this sugar being seriously decomposed as the duration of hydrolysis is very great.

That arabinose is present in every fraction or subfraction of holocellulose and also that the rate of hydrolysis of arabinose in jute is slow supports the suggestion of Das, Mitra and Wareham (1953) that arabinose might be present in jute in pyranose form.

SUMMARY

It has been shown that under the conditions of the experiments hydrolysis by formic acid causes much less destruction of jute holocellulose than is the case with mineral acids.

From the rate of hydrolysis of jute holocellulose it seems that the degree of decomposition of xylose, glucuronic acid and rhamnose by formic acid is quite marked whereas in the case of arabinose and glucose it is less pronounced especially in the latter.

The formic acid method has been found suitable for the detection of uronic acid and also rhamnose, the latter being present in jute only in a small amount.

Methyl glucuronic acid, galactose, glucose, xylose, arabinose and rhamnose have been chromatographically separated and provisionally identified from jute holocellulose.

A complex (spot No. 2) has been detected which indicates the presence of a linkage between methyl uronic acid and xylose.

It has been shown that galactose can easily be separated from glucose by using *n*-butanol as the solvent if the time of chromatograph is as high as 168 hours.

By fractional hydrolysis of jute holocellulose, seven fractions or subfractions were obtained. Some of these have been shown to differ in chemical composition.

The presence of pentose with glucose in almost all the subfractions supports the cellulosan theory.

It supports the suggestion that arabinose in jute, some of it at least, exists in pyranose form.

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REFERENCES

- Bose, R. K., and Burma, D. P. (1952). Paper chromatography in the study of sugar constituents of jute hemicellulose. *Science and Culture*, 18, 39-40.
- Chattopadhyay, H., and Sarkar, P. B. (1946). A new method for the estimation of cellulose, with special reference to jute. *Proc. Nat. Inst. Sci. Ind.*, 12, 23-46.
- Consden, R., Gordon, A. H., and Martin, A. J. P. (1944). Qualitative analysis of proteins; a partition chromatographic method using paper. *Biochem. J.*, 38, 224-232.
- Das, D. B., Mitra, M. K., and Wareham, J. F. (1952). A study on the association of xylan with α -cellulose in jute by partition chromatography. *Science and Culture*, 18, 249-250.
- (1953). Association of glucose, arabinose and xylose in jute. *Nature*, 171, 613-614.
- (1954). Pentoses associated with jute α -cellulose. *Nature*, 174, 228-229.
- Das, D. B., Roy Chaudhuri, P. K., and Wareham, J. F. (1952). A note on the constitution of jute by paper chromatography. *Science and Culture*, 18, 197.
- Doreé, C. (1947a). Method of cellulose chemistry, London, Chapman & Hall, 377.
- (1947b). *Ibid.*, 393.
- Forsyth, W. G. C. (1948). Colour reagents for paper chromatography of sugar. *Nature*, 161, 239-240.
- Hibbert, H., Henderson, W. F., Johnsen, B., Mitscherling, W. O., and Wise, L. E. (1923). The preparation of standard cotton cellulose and the proposed method of analysis. *J. Indus. Eng. Chem.*, 15, 748-751.
- Hough, L., Jones, J. K. N., and Wadman, W. H. (1950). Quantitative analysis of mixtures of sugars by the method of partition chromatography. *J.C.S.*, 345, 1702-1706.
- Jermyn, M. A., and Isherwood, F. A. (1949). Improved separation of sugars on the paper partition chromatogram. *Biochem. J.*, 44, 402-407.
- Norman, A. G., and Jenkins, S. H. (1933). A new method for the determination of cellulose, based upon observation on the removal of lignin and other encrusting materials. *Biochem. J.*, 27, 818-831.

- Partridge, S. M. (1946). Application of paper partition chromatogram to the qualitative analysis of reducing sugars. *Nature*, **158**, 270-271.
- (1948a). Filter paper partition chromatography of sugars. *Biochem. J.*, **42**, 238-250.
- (1948b). Application to carbohydrate studies. *Biochem. J.*, **43**, Proc. XLVIII.
- (1949). Aniline hydrogen phthalate as a spraying reagent for chromatography of sugars. *Nature*, **164**, 443.
- Sarkar, P. B., Mazumdar, A. K., and Pal, K. B. (1950). Association of xylan with α -cellulose in jute. *Nature*, **165**, 897-898.
- Sarkar, P. B., Chatterjee, H., Mazumdar, A. K., and Pal, K. B. (1948). The combination between lignin and polyuronic acid in jute fibre. *J. Textile Institute*, **39**, T1-T7.
- Somogyi, M. (1945). A new reagent for determination of sugars. *J. Biol. Chem.*, **160**, 61-68.
- Spoehr, H. A. (1947). The hydrolysis of alginic acid with formic acid. *Archives of Biochemistry*, **14**, 153.
- Taylor, M. C., White, J. F., Vincent, G. P., and Cunningham, G. L. (1940). Sodium chlorite—Properties and reactions. *J. Indus. Eng. Chem.*, **32**, 899-903.

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EFFECT OF E.M. RADIATION ON LAMB SHIFT—II

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INTRODUCTION

The ultra-violet catastrophe in the line-shift problem is associated with the electromagnetic mass of the electron. Bethe (1947) postulated that the observed electronic mass already contains this part. The electromagnetic mass therefore is a non-observable quantity. This idea is sufficient to separate out at once the infinity, in the expression for the line-shift: A field-theory must allow for the free-electron contribution in its theoretical apparatus for calculating the observable effects of the transverse self-energy on the quantum states of a bound electron. This contribution also diverges, but since the two divergences are of the same order, the residue got through the process of mass-renormalization converges to a small finite value. The value so obtained (Salpeter, 1953) is in excellent agreement with the Lamb shift in the fine structure of hydrogen (Triebwasser, Dayhoff and Lamb, 1953). Hence the observed $2S_{\frac{1}{2}}-2P_{\frac{1}{2}}$ displacement is of electrodynamic origin. It is due to the interaction of the electron with the vacuum. Keeping this shift in view, there is perfect agreement between the observed hydrogen spectrum and that predicted on Dirac's theory, according to which the $2S_{\frac{1}{2}}$ - and $2P_{\frac{1}{2}}$ -levels should be degenerate.

Another very encouraging conclusion which emerges from Bethe's work is that, it showed for the first time that the higher orders involved in perturbation theory also correspond to reality and thus there is no 'deep-seated limitation' in the perturbation technique itself, as was formerly supposed (Heitler, 1944).

The Lamb shift is due to the perturbing effect of the zero-point energy of the radiation field. If, however, an external transverse electromagnetic field is switched on, the higher modes of the oscillating field are also excited, which, too, interact with the electron. Thus there is an additional displacement of the energy-states of the atom from the Dirac energy-level scheme. This depends essentially on the energy density and spectral composition of the radiation field perturbing the bound electron. It is the purpose of the present paper to give a theoretical treatment of this problem and to evaluate numerically the expressions for the line-shift so obtained. The case of a free electron was considered in a previous paper (Inderjit Singh, 1954), which is hereafter referred to as paper I.

I. Consider a system consisting of a hydrogen atom immersed in a radiation bath. It is the case of a bound electron in interaction with the surrounding field. The Hamiltonian for such a system is

$$H = H_{\text{Rad}} + H_{\text{Mat}} + H_{\text{Int}} \quad \dots \quad (1)$$

$$\left. \begin{aligned} H_{\text{Rad}} &= \hbar \int d\vec{k} |\vec{k}| \sum_{\lambda=1}^2 N_{\lambda}(k) \\ H_{\text{Mat}} &= \vec{\alpha} \cdot \vec{p} + \beta \mu + V(\vec{r}) \\ H_{\text{Int}} &= H' = - \sum e (\vec{\alpha} \cdot \vec{A}(\vec{r})) \end{aligned} \right\} \quad \dots \quad (2)$$

Here e is the unit electronic charge and $\vec{\alpha}, \beta$ are the usual Dirac matrices

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \dots \quad \dots \quad \dots \quad (3)$$

Here $\vec{\sigma}$ is the two-component Pauli matrix. \vec{A} is the vector-potential of the external transverse field, given by

$$\vec{A} = \sum_{\lambda} \vec{e}_{\lambda} (A_{\lambda}/2) (\vec{C}_{\lambda} \cdot \exp(i \vec{k}_{\lambda} \cdot \vec{r}) + \text{conj.}) \quad \dots \quad \dots \quad (4)$$

where the constants and the operators are the same as in paper I. $\lambda = 1, 2$ describe the types of polarization of a photon.

The first term in the Hamiltonian given by Eq. (1) gives the total energy of the radiation field. The second term leads to the Dirac energy-levels, while the last term represents the electrodynamic shift in these levels. We are interested only in this term, which arises due to the interaction between the radiation field and the charged particle. It is not possible to deal with this interaction exactly. The usual technique is to consider \hat{H}_{int} as a small perturbation and then solve the Schrödinger equation describing the system by the method of perturbation theory due to Dirac. It consists essentially in finding the time-dependent amplitude of the system as a series-expansion by the method of successive approximations.

In the present problem, the first significant interaction term is the one which is quadratic in the perturbation energy. Here transition is possible only if it takes place *via* the intermediate states $|i\rangle$. The second-order matrix element for the interaction Hamiltonian is

$$W = \sum_i \frac{(n | H' | i)(i | H' | n)}{E_n - E_i} \quad \dots \quad \dots \quad \dots \quad (5)$$

The intermediate states can arise in two ways:

- (1) A photon of momentum \vec{k} is absorbed by the electron in state $|n\rangle$. So there are $[N(k)-1]$ quanta in the intermediate state $|i\rangle$. In going to the original state, the electron emits the absorbed photon.
- (2) The electron emits a quantum of momentum \vec{k} to go to the intermediate state. Hence there are $[N(k)+1]$ quanta of radiation present in the state $|i\rangle$. The transition to the final state is made by absorbing a quantum of the same energy, so that the total energy in the state $|n\rangle$ remains conserved. However, in the intermediate state there need be no conservation of energy. Hence all intermediate states are possible. Since these transitions may involve a violation of the law of conservation of energy, they are essentially virtual, though in the process 'real quanta may be absorbed and emitted', as is the case in the present problem.

Taking account of these processes in Exp. (5) we have

$$W = 2\pi e^2 \hbar^2 \sum_{i, \lambda} \left\{ \frac{(N(k)+1)}{k} \frac{(n | \vec{\alpha} | i_1)(i_1 | \vec{\alpha} | n)}{E_n - E_{i_1}} + \frac{N(k)}{k} \frac{(n | \vec{\alpha} | i_2)(i_2 | \vec{\alpha} | n)}{E_n - E_{i_2}} \right\} \dots \quad (6)$$

where the sum is over all intermediate states and also over the directions of polarization of quanta. The present problem was first considered by Auluck and Kothari (1952), but they overlooked transitions of type (1), which arise only when the atom

is situated in a real photon field, but are absent in the problem of the Lamb shift. We are, however, not taking the line-width of the levels into account in the present treatment, as it will not materially affect our conclusions.

In the non-relativistic approximation, the Dirac operator $\vec{\alpha}$ should be replaced by the velocity vector \vec{v} . Also the oscillator strength $f(n, i)$ for the transition $n \rightarrow i$ is defined by the relation

$$\left| (n, l | \vec{p}_e | i) \right|^2 = \frac{\mu}{2} (E_i - E_n) f(n, i) \quad \dots \quad (7)$$

where μ is the mechanical mass of the electron and n and l are the usual quantum numbers. Using Eq. (7), the non-relativistic expression for the interaction energy becomes

$$W = -\frac{2\alpha}{\mu} \int \left[N(k) \sum_i \frac{f(n, i) (E_i - E_n)^2}{(E_i - E_n)^2 - k^2} + \frac{f(n, i)}{(E_i - E_n + k)} \right] k dk \quad \dots \quad (8)$$

Here α is the Sommerfeld fine structure constant.

The second term arises even in the absence of all quanta of radiation and it is this term which is responsible for the Lamb shift. The change in Lamb shift is therefore given by

$$W' = -\frac{2\alpha}{\mu} \sum_i \int dk \cdot N(k) \cdot k \cdot \frac{(E_i - E_n)^2 \cdot f(n, i)}{(E_i - E_n)^2 - k^2} \quad \dots \quad (9)$$

II. We have now to evaluate expression (9). For this we have first to replace $N(k)$ by a suitable distribution function for the photons, determined by the quality of the radiation field in interaction with the atom.

Following Auluck and Kothari (1952) we consider three distinct cases:

- (1) The atom is entrapped in a Hohlraum, surrounded by impenetrable walls at a temperature T . In this case the energy density and spectral composition of the radiation field is governed by the relation

$$N(k) = \frac{1}{\exp(k/RT) - 1} \quad \dots \quad (10)$$

This gives the average number of photons of energy k per oscillator.

- (2) The atom is under the influence of a transverse radiation field as in case (1), but the effective spectrum is cut off beyond a certain frequency, which is taken to be less than the ionizing frequency of the atom.
- (3) The atom is surrounded by isotropic radiation of frequency $|k|$ contained in a bath of volume G . If $I(k)$ is the intensity of the radiation in the bath,

$$N(k) = G \cdot \frac{I(k)}{k} \quad \dots \quad (11)$$

For case (1), exp. (9) reduces to

$$W'(T) = -\frac{2\alpha}{\pi\mu} \int_0^\infty \frac{x dx}{\exp(2\pi x) - 1} \sum_i \frac{(E_i - E_n)^2 \cdot f(n, i)}{q_i^2 - x^2} \quad \dots \quad (12)$$

where

$$q_i = \frac{E_i - E_n}{2\pi RT} \quad \dots \quad (13)$$

It can also be expressed in the alternate form

$$W'(T) = \left(\frac{4\pi\alpha}{\mu} \right) (RT)^2 \sum_p \sum_q \{ \exp(-2\pi pq) \cdot E_i(pq) + \exp(2\pi pq) E_i(-pq) \} \cdot q^2 f(q) \quad \dots (14)$$

where

$$E_i(-x) = - \int_x^\infty \exp(-t) \cdot \frac{dt}{t} \quad \dots \dots \dots (15)$$

and tables for it are available.

The only interesting case is when $RT \ll 1$.

The first term in exp. (12) is capable of being expressed as a power series containing the well-known Bernoulli numbers, B_{2m+1} , and integrating term by term we get

$$W'(T) = \frac{\alpha}{2\pi\mu} \sum_i \sum_m \frac{(-1)^{m+1}}{(m+1)} B_{2m+1} \frac{(2\pi RT)^{2m+2}}{(E_i - E_n)^{2m}} \cdot f(n, i) \quad \dots (16)$$

where i denotes the summation over the discrete and continuous intermediate states and m takes all integral values starting from zero onwards.

In the first approximation

$$W'(T) \cong - \frac{\pi\alpha}{3\mu} (RT)^2 \quad \dots \dots \dots (17)$$

Since

$$\sum_i f(n, i) = 1 \quad \dots \dots \dots (18)$$

and

$$B_1 = \frac{1}{6} \quad \dots \dots \dots (19)$$

Hence the shift increases quadratically with temperature and in the first approximation is the same for all levels. Note that the Lamb shift is pronounced only for S -levels. If the higher order terms are considered, we have

$$W'(T) = - \frac{\pi\alpha}{3\mu} (RT)^2 \left[1 - \frac{2\pi^2}{7} \sum_i \frac{f(n, i)}{(E_i - E_n)^2} (RT)^2 + \dots \right] \quad \dots (20)$$

since

$$B_3 = \frac{1}{42}.$$

Expressions $\sum_i \frac{f(n, i)}{(E_i - E_n)^2}$, etc. are to be evaluated numerically.

The case when $RT \gg 1$ has no physical significance, for the atom has little chance to survive in such a radiation bath. It becomes of interest only if the radiation spectrum is cut off at $|\vec{k}_0|$, where $|\vec{k}_0|$ is always less than the ionizing frequency of the atom. When this is the case, the exponential term in Eq. (12) is capable of being evolved in a power series, giving after integration over x ,

$$\begin{aligned}
 W'(T) = & -\frac{\alpha}{\pi\mu} \sum_i \left[RT(E_i - E_n) \cdot f(n, i) \cdot \log \left| \frac{E_i - E_n + k_0}{E_i - E_n - k_0} \right| \right. \\
 & \left. + \frac{1}{2}(E_i - E_n)^2 \cdot f(n, i) \cdot \log \left| \frac{(E_i - E_n)^2 - k_0^2}{(E_i - E_n)^2} \right| \right] \\
 & + \text{terms containing powers of } (1/RT).
 \end{aligned}
 \quad (RT \gg 1) \quad \dots (21)$$

Finally, consider the case when $N(k)$ represents a monochromatic radiation field of frequency $|\vec{k}|$ and intensity $I(|\vec{k}|)$ contained in a volume G , which also contained the electron under consideration. Then

$$W'(k) = -\frac{8\alpha G}{\pi\mu} \cdot \frac{I(k)}{k^2} \sum_i \frac{(E_i - E_n)^2 \cdot f(n, i)}{(E_i - E_n)^2 - k^2} \quad \dots \quad (22)$$

Expressions (20), (21) and (22) give the shift in the three cases when the summations over the intermediate states consisting of the discrete spectrum and the continuum are carried out.

III. In order to perform the summations \sum_i , we need the values of $f(n, i)$ for the discrete and continuous spectrum of hydrogen. Tables for the oscillator strength for transitions to the discrete levels are available (Bethe, *Handbuch der Physik*, 24/1), but it has been pointed out by various authors (Bethe, Brown and Stehen, 1950) that Bethe's Table 16 contains many numerical mistakes. The strengths for a few transitions to the $2s$ - and $2p$ -levels have been re-evaluated by Bethe, Brown and Stehen (1950). We give here a much larger revised table (Table I). Since present-day experimental precision demands increased accuracy, we give values correct to four significance places.

The oscillator strength for the transition $|n > \rightarrow |i >$ is given by the relation

$$f(n, i) = (2\mu/\hbar)\nu_{ni} \left| \int (n | r | i) d\tau \right|^2 \quad \dots \quad (23)$$

with

$$\nu_{ni} = \frac{E_n - E_i}{h} \quad \dots \quad (24)$$

Clearly

$$\sum_i f(n, i) = 1. \quad \dots \quad (25)$$

For transitions to the discrete spectrum Eq. (23) reduces to (Bethe, *Handbuch der Physik*, 24/1)—

$$f(n, m) = \frac{1}{3} \frac{\max(l, l')}{2l+1} \frac{\nu_{ni}^{nl}}{R_y} \frac{(R_{ni}^{ml'})^2}{a^2} \quad \dots \quad (26)$$

$R_{ni}^{ml'}$ is the dipole moment of the hydrogen atom of radius a and is given by the relation

$$R_{ni}^{ml'} = \int_0^a R_{ni} R_{mi'} r^2 dr \quad \dots \quad (27)$$

TABLE I
Oscillator strengths for hydrogen line spectrum

Transition involving	Oscillator strength					
	1s	2s	2p		3s	3p
Initial level			ns	nd	np	ns
Final level	np	np				nd
$n = 1$	-0.1387	-0.02637
2	0.4162	-0.04077	-0.1449
3	0.07910	0.4349	0.01359	0.6958
4	0.02899	0.1028	0.003044	0.1218	0.4847	0.03225
5	0.01394	0.04193	0.001210	0.04437	0.1210	0.007428
6	0.007799	0.02163	0.0006180	0.02163	0.05139	0.003032
7	0.004814	0.01274	0.0003613	0.01233	0.02737	0.001579
8	0.003183	0.008182	0.0002308	0.007756	0.01655	0.0009412
9	0.002216	0.005583	0.0001570	0.005220	0.01086	0.0006110
10	0.001605	0.003988	0.0001119	0.003693	0.007554	0.0004227
Asymptotic	$a_1 n^{-3} \left(1 + \frac{8}{3} n^{-2}\right)$ with	$a_2 n^{-3} \left(1 + \frac{25}{3} n^{-2}\right)$ with	$a_3 n^{-3} \left(1 + \frac{28}{3} n^{-2}\right)$	$a_4 n^{-3} \left(1 + \frac{37}{3} n^{-2}\right)$	$a_5 n^{-3} (1 + 26n^{-2})$	$a_6 n^{-3} (1 + 21n^{-2})$ $a_7 n^{-3} (1 + 22n^{-2})$
Formulae	$a_1 = 1.56293$	$a_2 = 3.66414$	$a_3 = 0.10178$	$a_4 = 3.25702$	$a_5 = 6.24291$	$a_6 = 0.33975$ $a_7 = 6.11551$

where R_{nl} and $R_{m'l'}$ are the renormalized radial wave-functions for the hydrogen atom and has been calculated by Gordon, such that

$$R_{n,l}^{m,l-1} = \frac{(-1)^{m-l}}{4(2l-1)!} \cdot \left[\frac{(n+l)! (m+l-1)!}{(n-l-1)! (m-l)!} \right]^{\frac{1}{2}} \cdot \frac{(4nm)^{l+1} (n-m)^{n+m-2l-2}}{(n+m)^{n+m}} \\ \cdot \left\{ F\left(-n_r, -m_r, 2l, \frac{-4nm}{(n-m)^2}\right) - \frac{(n-m)^2}{(n+m)^2} \cdot F\left(-n_r-2, -m_r, 2l, \frac{-4nm}{(n-m)^2}\right) \right\} \quad (28)$$

where, F is a hypergeometric series

$$F(\alpha, \beta, \gamma, x) = \sum \frac{\alpha(\alpha+1) \dots (\alpha+\nu-1) \beta \dots (\beta+\nu-1) x^\nu}{\gamma \dots (\gamma+\nu-1) \nu!} \quad (29)$$

with the radial quantum numbers—

$$\left. \begin{array}{l} n_r : n-l-1 \\ m_r : m-l \end{array} \right\} \quad (30)$$

The expressions for the oscillator strength for transitions from a given shell to the continuum are obtained by making use of theoretical relations for the photo-electric absorption coefficients.

In this way we get

$$df(1s, n) = \frac{128}{3} \nu^{-4} d\nu \frac{\exp(-4n \operatorname{arc} \cot n)}{1 - \exp(-2\pi n)} \quad (31)$$

$$df(2s, n) = 4\nu^{-4} d\nu \left(\frac{4}{3} + \nu^{-1} \right) \frac{\exp\left(-4n \operatorname{arc} \cot \frac{n}{2}\right)}{1 - \exp(-2\pi n)} \quad (32)$$

$$df(2p, n) = \frac{8}{9} \nu^{-5} d\nu \left(\frac{3}{2} + \nu^{-1} \right) \frac{\exp(-4n \operatorname{arc} \cot)}{1 - \exp(-2\pi n)} \quad (33)$$

and so on. Here we have taken

$$E_n = R_y/n^2 \quad (34)$$

The contributions from the continuum and discrete states to the oscillator strengths for the $2s$ - and $2p$ -states have been evaluated in a manner outlined in the next section. The results are tabulated below (Table III).

TABLE II
Oscillator strengths for hydrogen atom in $n = 2$ state

State	Oscillator strength			
	Discrete	Continuous	Total	Actual
$2s$	0.64895	0.35109	1.00004	1
$2p$	0.80915	0.19078	0.99993	1

IV. The oscillator strength for transitions into a small frequency interval $d\nu$ of the continuum, the initial level being the $2s$ -state, is given by

$$df(2s, n) = 4\nu^{-4}d\nu \left[\frac{4}{3} + \nu^{-1} \right] \frac{\exp \left(-4n \arccot \frac{n}{2} \right)}{1 - \exp(-2\pi n)} \quad \dots (32)$$

The contribution from such transitions to the oscillator strength $f(2s, n)$ is obtained by integrating (32) with respect to the variable ν . It is more convenient to use the variable n while performing numerical work, where

$$\nu = \frac{1}{4} + \frac{1}{n^2} \quad \dots \dots \dots (35)$$

When n is small, (32) can be expanded and integrated analytically. In this way we find that contribution from the region $n = 0$ to $n = 0.04$ is negligible.

$$\begin{array}{l} \text{From } n = 0.04 \text{ to } n = 0.1, \text{ interval} = 0.01 \\ 0.1 \text{ to } n = 1 \text{ interval} = 0.05 \end{array} \quad \dots \dots (36)$$

From $n = 1$ to $n \rightarrow \infty$ we introduce again, following Bethe, Brown and Stehen, a new variable

$$y = \frac{1}{n} \quad \dots \dots \dots (37)$$

instead of n itself.

$$\text{From } y = 0 \text{ to } y = 1, \text{ interval} = 0.05 \quad \dots \dots (38)$$

The numerical integration is then performed, using rules like Simpson's and Waddle's.

$df(2p, n)$ is to be tackled in a similar manner. The contributions from the discrete spectrum are evaluated using Table I. For $n > 10$, it is sufficiently accurate to use the asymptotic expressions given therein. The error in Simpson's rule is greater than that in Waddle's and varies very approximately as the fourth power of the interval used.

To evaluate Eq. (21) for RT very large, we need the value of

$$A = \sum (E_i - E_n) \cdot f(n, i) \log \frac{E_i - E_n + k_0}{E_i - E_n - k_0} \quad (39)$$

It consists of two parts, such that

$$A = A_D + A_C \quad \dots \dots \dots (40)$$

where A_D is the contribution from the discrete transitions and A_C arises because of the continuum. We take the case when k_0 is equal to one Rydberg. Consider the second term in Eq. (4). This contribution is assessed in a manner outlined above. However, the term

$$T_1 = (E_i - E_n) \cdot f(n, i) \cdot \log |E_i - E_n - k_0| \quad \dots \dots (41)$$

tends to infinity as $(E_i - E_n) \rightarrow k_0$. Starting from $\nu = 0$, therefore we sum up to $\nu = 1 - \epsilon$, and then from $\nu = 1 + \epsilon$ to $\nu = \infty$. There is little contribution from the range $\nu = 1 - \epsilon$ to $\nu = 1 + \epsilon$, to Eq. (41). In our case

$$\epsilon = 0.0275 \quad \dots \dots \dots (42)$$

The contribution to

$$T_2 = (E_i - E_n) \cdot f(n, i) \cdot \log |E_i - E_n + k_0| \quad \dots \quad (43)$$

from the region $\nu = 1 - \epsilon$ to $\nu = 1 + \epsilon$ is however finite. Hence

from $n = 0$ to $n = 1$, intervals as in (36).

But (38) is modified :

$$\left. \begin{array}{lll} y = 0 & \text{to } y = 0.85 & \text{interval} = 0.05 \\ = 0.85 & = 0.8817 & = 0.0317 \\ = 0.8817 & = 1 & = 0.0394 \end{array} \right\} \quad \dots \quad (44)$$

To find the contribution from the discrete spectrum to Exp. (40), we note that

$$A_D = \sum_i 2\nu_i^2 \cdot f(n, i) \left[1 + \frac{\nu_i^2}{3} + \frac{\nu_i^4}{5} + \dots \right] \quad (45)$$

which is easily evaluated using Table I. In this way we get Table III.

TABLE III

$$A = \sum (E_i - E_n) \cdot f(n, i) \cdot \log \left| \frac{E_i - E_n + 1}{E_i - E_n - 1} \right|$$

State	Contribution from		
	Discrete (A_D)	Continuum (A_C)	Total (A)
2s	0.0356	0.2782	0.3138
2p	-0.1085	0.1356	0.0271

V. Using the values given in Table III, in Eq. (21), we find

$$\begin{aligned} W'(T, 2p) - W'(T, 2s) &\cong 58 (RT/R_y) \text{ megacycles} \\ &\left\{ \begin{array}{l} RT \gg 1R_y \\ k_0 = 1. \end{array} \right. \end{aligned}$$

The additional shift is in a direction opposite to that for the Lamb shift. As in the case of the Lamb shift, the shift for the 2p-level is much smaller than that for the 2s-level. The ratio

$$\frac{W'_2(T, 2s)}{W'_2(T, 2p)} \sim 10.$$

However, the 2p-level shift is not so insignificant here as in the case of the Lamb shift, where the ratio

$$\frac{\text{Lamb shift } (2s)}{\text{Lamb shift } (2p)} \sim 300.$$

The Lamb shift is due to 'photon-vacuum', while the additional shift is due to photons, and increases as the temperature is raised.

The Lamb shift and the influence of e.m. field on it, for the case of a harmonic oscillator and other allied problems, will be the subject of a subsequent paper.

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SUMMARY

The effect of interaction with the electromagnetic field in modifying the spectral terms of a Dirac electron in an external field of force is investigated, using the conventional form of perturbation theory. The magnitude of the shift is evaluated numerically. It is shown that this displacement of the energy-levels differs in many respects from the Lamb shift, which is due to the effect of the scalar radiation field as it exists in the vacuum.

REFERENCES

- Auluck, F. C., and Kothari, D. S. (1952). Effect of E.M. Radiation on the Lamb shift. *Proc. Roy. Soc., A*, **214**, 137.
Bethe, H. A. (1947). The E.M. Shift of Energy Levels. *Phys. Rev.*, **72**, 339.
——— (1933). *Handbuch der Physik*, 24/1.
Bethe, H. A., Brown, L. M., and Stehen, J. R. (1950). Numerical Value of the Lamb shift. *Phys. Rev.*, **77**, 370.
Heitler, W. (1944). The Quantum Theory of Radiation, 87.
Salpeter, E. E. (1953). The Lamb shift for Hydrogen and Deuterium. *Phys. Rev.*, **89**, 92.
Singh, Inderjit (1953). Effect of E.M. Radiation on Lamb shift. *Prog. Theor. Phys.*, **10**, 476.
——— (1954). Effect of E.M. Radiation on the Self-Energy of Free-electrons. *Proc. N.I.S.*, **20**, No. 5, 557.
Triebwasser, S., Dayhoff, E. S., and Lamb, W. E. (1953). Fine Structure of the Hydrogen Atom. *Phys. Rev.*, **89**, 98.

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ON THE LAMB SHIFT AND OTHER RADIATIVE EFFECTS—III

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INTRODUCTION

The technique of mass and charge renormalization provides a useful procedure for calculating the reactions of the field within the present 'faulty' structure of quantum electrodynamics. It involves the tacit assumption that although the mass m and charge e of an 'isolated' electron (isolated even from vacuum fields) appear in the original description of the atom without radiation field, all the final results of the theory should depend on the experimentally observable mass ($m + \Delta m$) and the experimentally observable charge ($e + \Delta e$) where the Δ s arise due to the perturbing effect of the field which is present in every physical situation, being a property of the vacuum. It is further supposed ad hoc, that these corrections are to be small. Thus, quantum electrodynamics in its present form is good enough to handle observable radiative effects, although its weakness becomes apparent when the electron itself is to be dealt with.

Welton (1948) provided a physical foothold to these quantum mechanical calculations. He showed that the Lamb shift is due to a mean extension of the position of the electron, because of its coupling to the vacuum. These position fluctuations are the result of the infinite energy of zero-point field oscillations. Though itself unobservable, it makes the electron perform a sort of Brownian movement. Because of this Welton-Bewegung, the charge is no more a point singularity but behaves as distributed over a finite area. In a state devoid of orbital angular momentum, the electron is under strong influence of the nucleus. The spherical charge due to the fluctuating electric vector is not so well bound to the nucleus, as is a point charge. This results in a slight upward shift of the S -levels. It is significant to note that in the actual calculation, we assume of a quantum mechanical force behaving in a classical manner.

Clearly the phenomenon of Lamb shift depends essentially on the electron behaviour at distances from the nucleus which are of the order of the Bohr radius, rather than of the electron Compton wavelength. It is because of this very fact, that the Lamb shift cannot be treated by a straightforward expansion of self-energy effects in powers of the external field, even though the break-down of the expansion is a mild one. It is also clear that the non-relativistic effects depend, in an essential way, on the structure of the atom, and so should be carefully separated from the relativistic effects.

In this paper we have calculated the shift in the energy levels of a Harmonic Oscillator, due to its coupling with the vacuum. The problem is investigated in two ways, using perturbation technique supplemented with mass-renormalization idea and on Welton's phenomenological theory. It is shown that both methods lead to the same result. The presence of transverse radiation field will be the source of additional perturbation. This effect can be handled by both theories, but the two approaches lead to different results. The reason for this divergence is indicated. Koba's improvement of Welton's model (1949) through the introduction of Schrödinger Bewegung is also discussed, and has been used to calculate the spin-coupling energy contribution to Lamb shift. Some scattering problems

involving electrons are reviewed with special reference to electro-magnetic inertial effects, on Welton-Koba theory. In the last section, we have calculated the additional magnetic moment due to black-body radiation at a temperature T , in which the electron is supposed embedded, and also the problem of the anomalous magnetic moment of the nucleon.

2. RADIATIVE CORRECTION TO OSCILLATOR-SPECTRUM

Let us apply Bethe's renormalization procedure to the case of a Spatial Oscillator. For simplicity, we confine ourselves to a simple type of oscillatory system, one which is one-dimensional, linear and conservative. It consists of a charged particle of mass M and charge e executing forced vibrations, under the influence of an 'elastic' restoring force which varies as the first power of the displacement. Such a system has an energy spectrum given by

$$W_n = (2n+1) \frac{\hbar\omega}{2} \quad \dots \quad (1)$$

assuming that the particle is not influenced by the vacuum, which is not so. Thus, in our problem is involved simultaneous interaction of the particle with the restoring force as well as with the electro-magnetic field. We consider the field in its lowest energy state (complete absence of photons); and calculate the correction it produces, treating it as a small perturbation.

Following Bethe, the displacement in energy levels is given by

$$W'_n = \frac{2e^2}{3\pi\hbar c^3} \sum_i \left| \langle n | \vec{v} | i \rangle \right|^2 (E_i - E_n) \ln \frac{K}{|E_i - E_n|} \quad \dots \quad (2)$$

for the oscillator-states E_n , where E_i are the intermediate virtual states in which the particle finds itself, because of the perturbing effect of the field. Here $\langle n | \vec{v} | i \rangle$ is the velocity-matrix, and

$$K \sim Mc^2 \quad \dots \quad (3)$$

The Hamiltonian of the system (without radiation) is

$$H(\vec{Q}, \vec{P}) = \frac{P^2}{2M} + \frac{AQ^2}{2} \quad \dots \quad (4)$$

which gives

$$\frac{\hbar}{jM} \vec{P} = \vec{Q} \vec{H} - \vec{H} \vec{Q} \quad [j = \sqrt{-1}] \quad \dots \quad (5)$$

Or

$$\langle n | \vec{v} | i \rangle = \frac{j}{\hbar} \langle n | \vec{q} | i \rangle (E_n - E_i) \quad \dots \quad (6)$$

Making use of (9) in (4), we get

$$\begin{aligned} W'_n &= \frac{2e^2 \cdot [E_i - E_n]^3}{3\pi\hbar c^3} \sum_i \left| \langle n | \vec{q} | i \rangle \right|^2 \ln \frac{K}{|E_i - E_n|} \\ &= \frac{1}{3\pi} \left(\frac{e^2}{\hbar c} \right) \frac{\hbar^2 \omega^2}{Mc^2} \ln \frac{K}{\hbar\omega} \quad \dots \quad (7) \end{aligned}$$

The same expression for the shift is obtained on Welton's theory. The effect of fluctuations on the potential energy is:

$$\delta V = \frac{1}{3\pi} \left(\frac{\epsilon^2}{\hbar c} \right) \left(\frac{\hbar}{Mc} \right)^2 \cdot \nabla^2 V \cdot \ln \frac{K}{k_1} \quad \dots \quad (8)$$

When this is averaged over the quantum mechanical state,

$$U_n(q) = \left(\frac{\lambda}{\pi^{\frac{1}{2}} 2^n n!} \right)^{\frac{1}{2}} \cdot H_n(\lambda q) e^{-\frac{1}{2} \lambda^2 q^2} \quad \dots \quad (9)$$

with
$$\lambda = \left(\frac{M \cdot A}{\hbar^2} \right)^{\frac{1}{2}} \quad \dots \quad (10)$$

We get for the electrodynamic shift

$$W'_n = \frac{1}{3\pi} \left(\frac{\epsilon^2}{\hbar c} \right) \left(\frac{\hbar^2 \omega^2}{Mc^3} \right) \ln \frac{K}{k_1} \quad \dots \quad (11)$$

which agrees with (10), if

$$k_1 \sim \hbar \omega \quad \dots \quad (12)$$

Thus, the whole oscillator-spectrum is shifted upwards by a slight amount, given by (8), though the difference between any two levels remains unaffected.

The agreement between perturbation theory and semi-classical theory of radiative reactions leads us to a significant conclusion. It shows that the particle is behaving like a wave-packet, within which the potential energy is sensibly constant, so that the position and momentum vectors of the packet can be very closely represented by their expectation values. Thus we may conclude that perturbation theory will give sensible results so long as the electro-magnetic field does not change appreciably over the dimensions of the particle, or over distances of the order of Compton wavelength. At present, it is very difficult to say beyond which frequency precisely, present-day quantum electrodynamics fails. However, because of the almost non-relativistic nature of reactive effects, this lack of information does not involve much error. It is assumed that beyond the Compton frequency, the contribution from each wavelength falls off rapidly and is not much affected by the external potential. During renormalization, therefore, such contributions almost cancel each other.

3. EFFECT OF ELECTRO-MAGNETIC RADIATION ON LAMB SHIFT

In this section we consider the cases of a Hydrogen Atom and a Harmonic Oscillator immersed in black-body radiation at Temperature T .

A free electron executes steady forced vibrations under the action of an incident light wave and emits a scattered light wave of the same frequency. The effect of these position fluctuations will also be to make the electron behave as if its charge is effectively spread over a small region. Hence a bound electron in such a field will give rise to additional effects similar to those caused by zero-point oscillations of the electro-magnetic field.

We initiate our study by calculating the additional shift in the Hydrogen energy states due to coupling of the atom with the transverse radiation field surrounding it. It is a straightforward extension of Welton's idea. The electrodynamic shift due to additional perturbation is

$$W''_n(x) = \frac{8}{3} \alpha \cdot e^2 \left(\frac{\hbar}{mc} \right)^2 |\psi_n(0)|^2 \int_{x_0}^{\infty} \frac{dx}{x(e^x - 1)} \quad \dots \quad (13)$$

where
$$x = \hbar \omega / RT, \quad \dots \quad (14)$$

and $\alpha = (e^2/\hbar c)$ is Sommerfield Fine-structure constant. $x_0 RT$ is the average excitation potential of the Hydrogen atom and has a value of 16.64 Ry for the 2*S*-state, according to the numerical calculations of Bethe, Brown and Stehen. Using the familiar exponential integral

$$-E_i(-x) = \int_x^\infty \frac{e^{-t}}{t} dt \quad \dots \quad (15)$$

expression (13) is transformed into the equivalent form

$$\begin{aligned} W''(x) &= -\frac{8}{3} \alpha e^2 \left(\frac{\hbar}{mc}\right)^2 |\psi_n(0)|^2 \sum_x E_i(-x) \\ &= -\frac{8}{3\pi} \alpha^3 \cdot \frac{1}{n^3} \sum_x E_i(-x) \text{ Rydbergs.} \quad \dots \quad (16) \end{aligned}$$

Expression (18) is obtained by using the relation

$$\psi_n^2(0) = \left(\frac{Z}{na}\right)^3 / \pi \quad \dots \quad (17)$$

where *Z* is the nuclear charge number and *a* is the Bohr radius. For the 2*S*-state,

$$W''(x, 2s) = -136 \sum_x E_i(-x) \text{ Mc/sec.} \quad \dots \quad (18)$$

A few values of $-\sum E_i(-x)$ for various values of *x* are given below:—

x_0	$T^\circ\text{K.}$	$\sum_{x=x_0}^\infty -E_i(-x)$
0		∞
0.01	3×10^8	97.0662
0.1	10^7	8.2100
1	10^6	0.2868
10	10^5	0.0000
∞	0	0

The radiation shift is zero at the absolute zero of temperature and increases rapidly with increase in temperature. This steep rise is due to the exponential character of the integrand, and the shift tends to infinity as the lower limit tends to zero. Very high temperatures however have no physical meaning, because the atom will not survive in such a radiation bath.

Equation (18) predicts in general lower values than are given by the non-relativistic theory, which has been discussed by Auluck and Kothari (1952) and by Inderjit Singh (1955). It is shown by the later author that the shift is given by the expression

$$W''_n(T) = -\frac{2\alpha}{\pi\mu} \int_0^\infty \frac{x dx}{e^{2\pi x} - 1} \sum_i \frac{(E_i - E_n)^2 f(n, i)}{q_i^2 - x^2} \quad \dots \quad (19)$$

where

$$q_i = \frac{E_i - E_n}{2\pi RT} \quad \dots \quad (20)$$

$$\text{and} \quad f(n, i) = (2m/\hbar) \nu_{ni} \left| \int \psi_n^* r \psi_i d\tau \right| \quad (21)$$

It is interesting that while Lamb shift is due to the emission and absorption of virtual photons, this additional shift is brought about by transitions involving real photons. For $RT \ll 1$

$$W_n''(T) = \frac{\alpha}{2\pi\mu} \sum_i \sum_m \frac{(-1)^{m+1}}{m+1} \cdot B_{2m+1} \cdot \frac{(2\pi RT)^{2m+2}}{(E_i - E_n)^{2m}} \cdot f(n, i)$$

$$\text{with} \quad m = 0, 1, 2, \dots, \infty. \quad \dots \quad (22)$$

Here B_{2m+1} are Bernoulli numbers. Quantum mechanical calculations predict a shift of negative sign. On the other hand Welton's theory gives a positive shift. It predicts further that only S -levels should undergo a change, which is again contradicted by (16). Also, the fluctuations should contribute to the kinetic energy of the electron which shifts all levels by an equal amount. However, in Lamb shift one is concerned with the $2S-2P$ separation, and so the kinetic energy effect does not show itself. Nevertheless, this shift is there and is another weak point of the semi-classical theory.

It may be remarked here that it is the renormalization procedure which makes the Lamb shift positive. The self-energy of a bound electron is more than that of free-electron of the same energy. Since what we observe already includes this additional mass Δm of the electron, our experiments should reveal only the difference which is positive, i.e. the S -levels appear raised from the positions expected on Dirac's theory.

Consider now the case of a Harmonic Oscillator. Proceeding on perturbation theory, as for the Hydrogen atom, we get

$$W_n''(x) = -\frac{2}{3\pi} \left(\frac{e^2}{\hbar c} \right) \frac{\hbar^2 \omega^2}{Mc^2} \int_0^\infty \frac{x}{q^2 - x^2} \cdot \frac{dx}{e^x - 1} \quad \dots \quad (23)$$

$$\text{where} \quad q = \hbar\omega/RT \dots \quad (24)$$

On Welton's theory, we obtain

$$W_n''(x) = \frac{2}{3\pi} \left(\frac{e^2}{\hbar c} \right) \frac{\hbar^2 \omega^2}{Mc^2} \int_{x_0}^\infty \frac{dx}{x(e^x - 1)} \quad \dots \quad (25)$$

Thus, here again, there is disagreement. Equations (23) and (25) become identical for q , tending to zero. The same is true for (13) and (19). This leads us to another conclusion. Welton's theory is correct only in the ideal limit when the force of binding is small. The lower cut-off is arbitrary and the agreement with perturbation theory is not of much interest. The infra-red catastrophe in Welton theory is due to improper handling of the infra-red spectrum. These low energy transitions are playing a more important part here than for the case of the Lamb shift, and the lower cut-off is too high and washes out their contributions.

Gunther (1949) has calculated by Welton's method the radiation correction to the ionization energy of the Helium atom. The most accurate theoretical value is $198,319 \text{ cm}^{-1}$ while the experiment value is $198,305 \text{ cm}^{-1}$. It has not been possible to account for the whole discrepancy as being due to the effect of the vacuum. However, the effect of fluctuations on the mutual potential energy was neglected by him and so it is very difficult to say whether Welton's theory is capable of explaining this difference or not. As for the observed Lamb shift for He, it cannot be correctly assessed on the theories of Bethe and Welton.

4. Koba's Treatment of Reactive Corrections

Welton got a correction $\Delta\mu$ to the static magnetic moment μ associated with the spin of the electron of wrong sign, though of correct order of magnitude.

A magnetic dipole with a static moment μ placed in a homogeneous magnetic field \vec{H} , has an interaction energy $-\mu H$. So far we have neglected the influence of zero-point field. Its magnetic vector is undergoing incessant fluctuations. Due to this the dipole is tossed to and fro and makes on the average an angle θ with the magnetic field H , reducing the magnetic moment to $-\mu \cos \theta$, i.e. we should expect a correction of negative sign, and Welton's more refined calculation gives

$$\delta_1 \approx -e^2/2\pi\hbar c, \quad \dots \quad (26)$$

While on Tomonaga-Schwinger formalism, we get

$$\delta \approx e^2/2\pi\hbar c, \quad \dots \quad (27)$$

for the correction $\Delta\mu/\mu$ as has been independently shown by Schwinger (1948) and Luttinger (1948). Weisskopf (1949) concludes that while the line-shift problem is amenable to a simple pictorial understanding, the interaction between spin and magnetic field is not, because of the pure quantum mechanical character of spin.

Koba (1949) has shown that this failure is rather due to the neglect of virtual pair creation and annihilation, which plays an essential rôle in the magnetic moment problem. The vacuum polarization effect will be understood by paying due attention to Schrödinger's Zitterbewegung of the Dirac electron. This produces a correction

$$\delta_2 \approx e^2/\pi\hbar c \quad \dots \quad (28)$$

giving

$$\delta_1 + \delta_2 \approx e^2/2\pi\hbar c \quad \dots \quad (29)$$

which is in agreement with (27).

The Schrödinger bewegung will also produce a mean-square change in the position of the electron. The position vector q , according to Schrödinger, consists of two parts

$$q_j = \bar{q}_j + \tilde{q}_j \quad (j = 1, 2, 3) \quad \dots \quad (30)$$

where the first part commutes with the Hamiltonian and the other anticommutes with it. Welton bewegung affects the first part, while the second is affected only by Schrödinger bewegung. Considering both, we get for the mean-square extension

$$\langle (\Delta q_j)^2 \rangle_{Av} = \sum_k \left| \bar{q}_j(k) + \tilde{q}_j(k) \right|^2 \quad \dots \quad (31)$$

with

$$\bar{q}_j(k) = -\frac{eE_j(k)}{mk^2} \quad \dots \quad (32)$$

$$\tilde{q}_j(k) = +\frac{eE_j(k)}{mk(2m+k)} \quad \dots \quad (33)$$

giving for the correction, to Welton extension,

$$\delta_2 \langle (\Delta \tilde{q}_j)^2 \rangle_{Av} = \frac{\alpha}{2\pi} \left(\frac{\hbar}{mc} \right)^2 \quad \dots \quad (34)$$

This produces an additional shift—

$$W'_2 = \frac{\alpha}{4\pi} \left(\frac{\hbar}{mc} \right)^2 \langle \nabla^2 V \rangle_{Av} \quad \dots \quad (25)$$

For the $2S$ -level, we get

$$W'_2(2s) \approx 100 \text{ Mc/sec.} \quad \dots \quad (36)$$

However calculations on positron theory give for the spin-coupling energy a value of 68 Mc/Sec.

The contribution to line-shift due to E.M. radiation is given by

$$W'_2 = \frac{4\alpha}{3\pi} \left(\frac{\hbar}{mc} \right)^2 < \nabla^2 V >_{Av} \cdot RT \int_{x_0}^{\infty} \frac{dx}{(2\mu + xRT)(e^x - 1)} \quad \dots \quad (37)$$

and is therefore quite negligible.

5. RADIATIVE EFFECTS IN SCATTERING PROBLEMS

Consider the case of an electron of momentum \vec{p} , which undergoes elastic scattering due to interaction with an electrostatic field V . Like the line-shift problem, it involves simultaneous interaction of an electron with the fixed external potential and with the electro-magnetic field. The only difference is that now the scattering process is real. So the problem can be handled using Bethe's approach, and has been tackled in this manner by Lewis (1948), giving for the change in cross-section

$$\frac{\delta(d\sigma)}{d\sigma} = -\frac{2\alpha}{3\pi} \left(\frac{\hbar}{mc} \right)^2 (\vec{P} - \vec{K})^2 \int_T^{\infty} \frac{dk}{k} \quad \dots \quad (38)$$

Here \vec{K} is the momentum of the scattered electron and k is the frequency of the photon whose emission and re-absorption produces this effect. The lower limit T in the integral is the kinetic energy of the electron of mechanical mass m .

The same expression can be obtained by using Welton's idea that the position vector \vec{q} changes to $\vec{q} + \Delta\vec{q}$. We have merely to consider the change in the phase-factor introduced by the mean-square fluctuations. We get

$$\begin{aligned} & < \exp. i (\vec{P} - \vec{K}) (\vec{q} + \Delta\vec{q}) >_{Av} \\ &= \exp. i (\vec{P} - \vec{K}) \vec{q} \cdot \left[1 - \frac{1}{6} (\vec{P} - \vec{K})^2 < (\Delta\vec{q})^2 >_{Av} \right] + \dots \quad (39) \end{aligned}$$

Therefore

$$\frac{\delta(d\sigma)}{d\sigma} = -\frac{2\alpha}{3\pi} \left(\frac{\hbar}{mc} \right)^2 (\vec{P} - \vec{K})^2 \int_T^{\infty} \frac{dk}{k} \quad \dots \quad (40)$$

which is the same as (38). The same result is got using Epstein's (1948) method which consists essentially in calculating

$$\delta(d\sigma) = \frac{\delta(d\sigma)}{\delta M} \cdot \delta M \quad \dots \quad (41)$$

where δM is the difference between the experimental and the mechanical rest-masses. This illustrates in a very striking manner that the correction is the result of a change in the particle's rest-mass.

Again, the scattering cross-section is

$$d\sigma \approx \left| \int \exp \left[-i (\vec{P} - \vec{K}) \cdot \vec{q} \right] \cdot V \cdot \exp \left[i (\vec{P} - \vec{K}) \cdot \vec{q} \right] d\vec{q} \right|^2 \quad \dots \quad (42)$$

for a particle of zero spin. For a Dirac electron, the above expression is changed by a factor of

$$\delta' = 1 - \frac{p^2}{2m^2} (1 - \cos \theta) \quad \dots \quad (43)$$

where θ is the angle between \vec{p} and \vec{k} . Koba has got this factor as a consequence of the Zitterbewegung by using relation (31), so that

$$\exp \left[-i (\vec{k} \cdot \vec{q}) \right] \cdot \exp \left[i (\vec{p} \cdot \vec{q}) \right]^2 \approx 1 - \frac{p^2}{2m^2} (1 - \cos \theta) \quad \dots \quad (44)$$

which is the same as (43). This will also affect relation (40), and we get

$$\frac{\delta_1(d\sigma)}{d\sigma} = -\frac{\alpha}{2\pi} \left(\frac{\hbar}{mc} \right)^2 (\vec{P} - \vec{K})^2 \quad \dots \quad (45)$$

which is very small compared to the correction (40) due to Welton bewegung. A similar correction will arise while considering non-relativistic Compton scattering. Welton gets for the radiative correction to low-energy Compton scattering.

$$\frac{\delta(d\sigma)}{d\sigma} = -\frac{4\alpha}{3\pi} \left(\frac{\hbar k}{mc} \right)^2 \ln \frac{mc}{\hbar k_0} \quad \dots \quad (46)$$

Now we must also take into account the Zitterbewegung of the Dirac electron. The correction arising due to these fluctuations can be calculated in the same manner as for the elastic scattering for an electron, and which has been considered above. In this way we get

$$\frac{\delta_1(d\sigma)}{d\sigma} = -\frac{\alpha}{\pi} \left(\frac{\hbar k}{mc} \right)^2 \quad \dots \quad (47)$$

which is again small compared to (46).

We can again introduce the effect of external transverse radiation field when a charged particle of momentum \vec{p} is elastically scattered by a fixed external potential V , so that the scattered electron has a momentum \vec{K} . This problem is a dynamic analogue to the corresponding line-shift problem. The effect of the electro-magnetic field can be considered in a similar manner. The details of the calculation for this reason are not being reproduced here. We get

$$\frac{\delta'(d\sigma)}{d\sigma} = -\frac{4\alpha}{3\pi} \left(\frac{\hbar}{mc} \right)^2 (\vec{P} - \vec{K})^2 \int_{x_1}^{\infty} \frac{dx}{x(e^x - 1)} \quad \dots \quad (48)$$

for the radiation correction to the differential cross-section when the scattering process is taking place in space containing black-body radiation at a temperature T .

6. ANOMALOUS MAGNETIC MOMENT

Magnetic moment of a Dirac electron is due to circular currents of radius \hbar/mc . The zero-point oscillation of the field make this current role and pitch, thus reducing the magnetic moment by a cosine factor. In addition to this, there is the effect of the Zitterbewegung of the Dirac electron which makes the ring-current vibrate. The net effect is a correction of the order of $\alpha/2\pi$. If the electron is situated in an electro-magnetic field, the coupling between the two gives rise to

additional disturbance, thereby introducing a slight change in the magnetic moment which will be a function of the nature of the surrounding radiation field.

An electron having a spin vector, $\vec{\sigma}$, under the influence of field \vec{H} , will be governed by the relation

$$\frac{\partial \vec{\sigma}}{\partial t} = \frac{e}{mc} \vec{H} \times \vec{\sigma}$$

giving

$$\langle (\Delta \sigma)^2 \rangle_{Av} = \frac{4}{\pi} \left(\frac{e^2}{\hbar^2 c^3} \right) \left(\frac{\hbar}{mc} \right)^2 \sigma^2 \cdot (RT)^2 \int_0^\infty \frac{x dx}{e^x - 1} \dots \dots (49)$$

$$\text{Now} \quad \frac{e\hbar}{2mc} \sigma_2 \cdot H = \frac{e\hbar}{2mc} H |\vec{\sigma}| \cos \theta \dots \dots (50)$$

$$\text{and} \quad \langle \cos \theta \rangle_{Av} \cong \cos \theta \left[1 - \frac{1}{2} \langle (\Delta \theta)^2 \rangle_{Av} \right] \dots \dots (51)$$

$$\text{where} \quad \langle (\Delta \theta)^2 \rangle_{Av} = \frac{\langle (\Delta \sigma)^2 \rangle_{Av}}{\sigma^2} \dots \dots (52)$$

Hence, we get for the correction $\Delta \mu$ due to radiation field at temperature T to the magnetic moment μ of the electron,

$$\frac{\Delta \mu}{\mu} = -\frac{\pi}{3} \cdot \alpha \cdot \left(\frac{RT}{mc^2} \right)^2 \dots \dots (53)$$

Let us now consider the case of a nucleon. Again, according to Dirac's theory, the magnetic moment should be one nuclear magneton, if the particle is charged and none at all, if it is neutral. Experiments, however, show that

$$\begin{aligned} \mu_P &= 2.79353 \text{ nuclear magneton} \\ \mu_N &= -1.91354 \dots \dots (54) \end{aligned}$$

This means that quite unlike the electron, the proton and the neutron have pronounced anomalous magnetic moments. In fact, the neutron is behaving much like a spinning negative charge.

The cause of this discrepancy is to be sought for in the unique property of the nucleons to undergo transmutations which are the essence of nuclear forces. A proton, for example, can transform into a neutron and the converse is equally true. In fact, it is such transformations which keep a nucleus intact. But they also change the observable magnetic moment of the nucleon. This is because the magnetic moment of a proton, for example at any instant, is either unity or M/m , depending upon whether the proton is still a proton or has changed over into a neutron with the emission of a meson of mass m . M is the mass of the proton. A measurement cannot distinguish between the two and it represents therefore the probable value (March, 1951). A neutron has a negative magnetic moment, because of the emission of a negative meson, and also because the resulting proton will have a spin $-\frac{1}{2}$, because the spin vector of the meson must be in the direction of the field (Frohlich, Heitler and Kemmer, 1938). If the cause of the anomaly in the physically observable magnetic moment is the emitted meson field, then it is reasonable to expect that

$$\mu_P + \mu_N \sim | \dots \dots (55)$$

because

$$\Delta \mu_P + \Delta \mu_N \sim 0, \dots \dots (56)$$

if the surplus magnetic moment is to be ascribed to the meson-nucleon interaction. The values of μ_P and μ_N given in (54) amply corroborate Eq. (55).

The interaction is given by the second order matrix element

$$W = \sum_i \frac{\langle n | H_{\text{Int}} | i \rangle \langle i | H_{\text{Int}} | n \rangle}{E_n - E_i} \quad \dots \quad (57)$$

Hence, the self-energy of a proton surrounded by a positive vector meson field, due to interaction between the two sub-systems, is given by the expression

$$W_{\text{self}} = - \frac{|f|^2 \cdot \chi}{3\pi^2} \int_0^\infty \frac{(N(k)+1) \cdot dk \cdot k^4}{\chi^2 + k^2} \quad \dots \quad (58)$$

This consists of two parts, self-energy due to interaction with the vacuum meson field, and that due to interaction with the real field. The vacuum self-energy W_{vac} comes out divergent. If, however, the integral is cut off at $k \sim \chi$ we get

$$W_{\text{vac}} \approx \frac{2 \cdot |f|^2}{9\pi^2} \left[1 - \frac{3\pi}{8} \right] \cdot \chi^4 \quad \dots \quad (59)$$

where f is the interaction constant and χ gives the rest-energy of the meson. The second part of self-energy, W_2 , when the meson field is at temperature T , is

$$\begin{aligned} W_2 &= - \frac{|f|^2 \cdot \chi}{3\pi^2 \hbar^3} (RT)^3 \int_0^\infty \frac{x^4 dx}{q^2 + x^2} \cdot \frac{1}{e^x - 1} \\ &= - \frac{|f|^2 \cdot \chi}{3\pi^2 \hbar^3} (RT)^3 \cdot q^3 \sum_{p=0}^\infty \left\{ \left[C_i(pq) \sin pq \right. \right. \\ &\quad \left. \left. - S_i(pq) \cdot \cos pq + \frac{\pi}{2} \cos pq \right] + \frac{1}{p^3} (1 - p^2 q^2) \right\} \end{aligned}$$

where
$$q = \frac{\chi}{RT} \quad \dots \quad (60)$$

But since x is very large, we have for Exp. (60)—

$$W_2 \approx - \frac{|f|^2 \cdot \chi}{3\pi^2 q^2 \hbar^3} (RT)^3 \cdot \xi(5) \cdot \Gamma(5) \quad \dots \quad (61)$$

where
$$\xi(5) = 1.0369 \quad \dots \quad (62)$$

To calculate the surplus magnetic moment, we have to consider the system as placed in a weak magnetic field, then

$$W = W_{\text{self}} - \frac{|f|^2 \cdot \chi}{3\pi^2} \cdot \frac{eH}{\hbar} \int_0^\infty \frac{k^4 (N_k + 1) dk}{(\chi^2 + k^2)^2} \quad \dots \quad (63)$$

The correction $\Delta\mu_P$ due to the vacuum meson field is

$$\Delta\mu_P \approx \frac{5|f|^2}{12\pi^2} \left(1 - \frac{3\pi}{10} \right) \cdot \frac{e\chi^2}{\hbar} \quad \dots \quad (64)$$

and, due to the surrounding field, is

$$\Delta_{2\mu_P} \approx \frac{|f|^2}{3\pi^2\hbar^2} (RT) \int_0^\infty \frac{x^4 dx}{(q^2 + x^2)^2} \cdot \frac{1}{e^x - 1} \quad \dots \quad (65)$$

Eq. (66) is easy to integrate, but it leads to a cumbersome expression.

For the sake of a neater result, we again apply the approximation used before and get

$$\Delta_{2\mu_P} \approx \frac{|f|^2 \cdot e}{3\pi^2\hbar^2} \frac{(RT)^5}{\chi^3} \cdot \Gamma(5) \cdot \xi(5) \quad \dots \quad (66)$$

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SUMMARY

The influence of radiative forces on several processes, involving the interaction between a charged particle and a fixed potential, is investigated. The effect of fields as they exist in the vacuum, as well as external fields, is considered.

REFERENCES

- Auluck, F. C., and Kothari, D. S. (1952). Effect of E.M. Radiation on the Lamb Shift. *Proc. Roy. Soc., A* **214**, 137.
- Bethe, H. A. (1947). The E.M. Shift of Energy Levels. *Phys. Rev.*, **72**, 339.
- Epstein, S. T. (1948). Remarks on Lewis paper (below). *Phys. Rev.*, **73**, 177.
- Frohlich, H., Heitler, W., and Kemmer, N. (1938). On the nuclear forces and magnetic moments of the neutron and the proton. *Proc. Roy. Soc.*, **166**, 154.
- Gunther, M. (1949). On the Zero-field Fluctuation energy correction for the ground state of the Helium atom. *Physica*, **15**, 675.
- Koba, Z. (1949). Semi-classical treatment of the Reactive corrections. *Prog. Theor. Phys.*, **4**, 319.
- Lewis, H. W. (1948). On the Reactive terms in Quantum Electrodynamics. *Phys. Rev.*, **73**, 173.
- Luttinger, J. M. (1948). A note on the Magnetic Moment of the electron. *Phys. Rev.*, **74**, 893.
- March, A. (1951). Quantum Mechanics of Particles and Wave-fields.
- Singh, Inderjit (1955). Effect of E.M. Radiation on Lamb Shift. *PNISPS*, **21**, 280-289.
- Weisskopf, V. F. (1949). Recent developments in the Theory of the Electron. *Rev. Mod. Phys.*, **21**, 1.
- Welton, T. A. (1948). Some observable effects of the Quantum-mechanical fluctuations of the E.M. Field. *Phys. Rev.*, **74**, 1157.

ON THE RADIAL PULSATIONS OF AN INFINITE CYLINDER WITH A MAGNETIC FIELD PARALLEL TO ITS AXIS

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1. INTRODUCTION

Chandrasekhar and Fermi (1953) deduced the general equations governing the adiabatic radial pulsations of an infinitely conducting, infinite cylinder, under its own gravity, and with a constant magnetic field parallel to its axis. They obtained an integral formula for the period of the pulsations. We have, in this note, deduced the pulsation equation for an axial field varying with the distance from the axis. An expression for the amplitude of the magnetic variation (as a consequence of the radial pulsations) in terms of the characteristic amplitudes of the pulsations is also obtained. We have obtained explicit expressions for the characteristic functions and values for the particular case in which the magnetic field is assumed to be proportional to the square root of the pressure inside the cylinder (with a finite magnetic field at the surface). The numerical calculations for six different models of such a field have been carried out. The dependence of the period of pulsations on the prevalent magnetic field is investigated. It is found to decrease with the magnetic field.

2. THE PULSATION EQUATION

Following Chandrasekhar and Fermi, we assume the gas to be compressible, of uniform density ρ , and having an infinite electrical conductivity. The effect of the last assumption is that the lines of force shall be pushed aside along with the matter in motion, which shall result in the magnetic pulsations of the same frequency as that of the radial pulsations of the cylinder.

Imagine an element of cylinder, of radius r and unit height, to be displaced through δr . From the conservation of mass, the equation of continuity can be written as

$$\frac{d}{dm}(\pi r^2) = \frac{1}{\rho} \quad \dots \quad (1)$$

where $m = \pi r^2 \rho$, is the mass of the element under consideration. The equation of motion can be written as

$$\frac{\partial^2 r}{\partial t^2} = -2\pi r \frac{\partial P}{\partial m} - \frac{2Gm}{r} \quad \dots \quad (2)$$

where the total pressure is

$$P = p + \frac{H^2}{8\pi} \quad \dots \quad (3)$$

p = gas pressure, $\frac{H^2}{8\pi}$ = magnetic pressure, and H being the magnetic field at the distance r from the centre. Let δP and δp denote the corresponding changes in the total pressure, density and the magnetic field at the distance r from the axis.

Chandrasekhar and Fermi gave the following equations governing the radial oscillations of small amplitude

$$\frac{\partial}{\partial m}(2\pi r \delta r) = -\frac{\delta p}{\rho^2} \quad \dots \quad (4)$$

and
$$\frac{\partial^2}{\partial t^2} \delta r = -2\pi r \frac{\partial}{\partial m} \delta P + \frac{4Gm}{r^2} \delta r. \quad \dots \quad (5)$$

From equation (3)

$$\begin{aligned} \delta P &= \delta p + \frac{H \delta H}{4\pi} \\ &= \frac{\gamma p}{\rho} \delta \rho + \frac{H \delta H}{4\pi} \quad \dots \quad (6) \end{aligned}$$

where γ is the ratio of specific heats. It is, however, assumed to be constant within the cylinder.

In a medium of infinite electric conductivity, the change δH in the magnetic field, as we follow the motion, is given by

$$\delta H = \text{curl}(\delta r \times H) + (\delta r \cdot \text{grad})H \quad \dots \quad (7)$$

Since H is in the z -direction and δr is radial, the only non-vanishing component of δH is

$$\begin{aligned} \delta H_z &= -\frac{1}{r} \frac{\partial}{\partial r}(Hr \delta r) + \delta r \cdot \frac{\partial H}{\partial r} \\ &= -\frac{H}{r} \frac{\partial}{\partial r}(r \delta r) \quad \dots \quad (8) \end{aligned}$$

and
$$\frac{H \delta H}{4\pi} = -\frac{H^2}{4\pi r} \frac{\partial}{\partial r}(r \delta r) \quad \dots \quad (9)$$

Substituting from (9) in (6), we get

$$\delta P = \frac{\gamma p}{\rho} \delta \rho - \frac{H^2}{4\pi r} \frac{d}{dr}(r \delta r) \quad \dots \quad (10)$$

which on substitution in (5) yields

$$\frac{d^2}{dt^2} \delta r = -\frac{1}{\rho} \frac{d}{dr} \left[\frac{\gamma p}{\rho} \delta \rho - \frac{H^2}{4\pi r} \frac{d}{dr}(r \delta r) \right] + \frac{4Gm}{r^2} \delta r.$$

With the help of equations (3) and (4), it reduces to

$$\left(\frac{d^2}{dt^2} - \frac{4Gm}{r^2} \right) \delta r = \frac{1}{\rho} \frac{d}{dr} \left[\frac{1}{r} \left\{ \gamma P + \frac{H^2}{8\pi} (2-\gamma) \right\} \frac{d}{dr}(r \delta r) \right]$$

or since $m = \pi r^2 \rho$

$$\left(\frac{d^2}{dt^2} - 4\pi G \rho \right) \delta r = \frac{1}{\rho} \frac{d}{dr} \left[\frac{1}{r} \left\{ \gamma P + \frac{H^2}{8\pi} (2-\gamma) \right\} \frac{d}{dr}(r \delta r) \right] \quad \dots \quad (11)$$

The gravitational pressure, at a distance r from the axis, is given by

$$P = - \int_r^R \rho \cdot \frac{2Gm}{r} dr = \pi G \rho^2 (R^2 - r^2) \quad \dots \quad (12)$$

If we assume that all the physical variables vary with time as $e^{i\omega t}$, then equation (11), with the help of equation (12), will reduce to

$$(\omega^2 + 4\pi G\rho)\delta r = -\frac{1}{\rho} \frac{d}{dr} \left[\left\{ \gamma \pi G \rho^2 (R^2 - r^2) + \frac{H^2}{8\pi} (2 - \gamma) \right\} \frac{1}{r} \frac{d}{dr} (r \delta r) \right] \quad \dots \quad (13)$$

For convenience, we put

$$A = \frac{1}{\gamma} \left(\frac{\omega^2}{\pi G \rho} + 4 \right) \quad \text{and} \quad f = \left(\frac{2}{\gamma} - 1 \right) \cdot \frac{H^2}{8\pi^2 R^2 \rho^2 G} \quad \dots \quad (14)$$

and introduce new variables

$$x = \frac{r}{R}, \quad \Psi = \frac{\delta r}{R} \quad \dots \quad (15)$$

The differential equation (13) takes the form

$$A\Psi = -\frac{d}{dx} \left\{ (1 - x^2 + f) \frac{1}{x} \frac{d}{dx} (x\Psi) \right\} \quad \dots \quad (16)$$

Let us define a function Φ such that

$$\frac{d\Phi}{dx} = \Psi \quad \dots \quad (17)$$

then the equation (16), on integration, gives

$$A\Phi = -(1 - x^2 + f) \frac{1}{x} \frac{d}{dx} \left(x \frac{d\Phi}{dx} \right) \quad \dots \quad (18)$$

The constant of integration is zero, because

$$\text{at } x = 0, \quad \frac{d\Phi}{dx} = 0 \quad \dots \quad (19)$$

The equation (18) is independent of the manner of variation of H .

3. EXPRESSION OF δH IN TERMS OF x AND Ψ

Using the substitutions (15), the equation (8) can be written as:—

$$\delta H_s = -\frac{H}{x} \frac{d}{dx} (x\Psi) \quad \dots \quad (20)$$

4. CHARACTERISTIC VALUES FOR THE MAGNETIC FIELD PROPORTIONAL TO THE SQUARE ROOT OF PRESSURE

We assume that the magnetic field is proportional to the square root of pressure, such that

$$H^2 = H_s^2 + (H_0^2 - H_s^2)(1 - x^2) \quad \dots \quad (21)$$

The values of the magnetic field at the centre and surface being given by

$$H = H_0 \text{ at } x = 0; \quad H = H_s \text{ at } x = 1.$$

Then

$$f = \frac{\left(\frac{2}{\gamma} - 1\right)}{8\pi^2 G \rho^2 R^2} \left\{ H_s^2 + (H_0^2 - H_s^2)(1 - x^2) \right\}$$

$$= C + B(1 - x^2)$$

where

$$C = \frac{H_s^2 \left(\frac{2}{\gamma} - 1\right)}{8\pi^2 G \rho^2 R^2} \quad \dots \quad \dots \quad \dots \quad \dots \quad (22)$$

and

$$B = \frac{(H_0^2 - H_s^2) \left(\frac{2}{\gamma} - 1\right)}{8\pi^2 G \rho^2 R^2}$$

Let us assume that the solution of the equation (20) is of the series form

$$\Phi = \sum_{k=0}^{\infty} a_k x^{k+\alpha} \quad \dots \quad \dots \quad \dots \quad \dots \quad (23)$$

The indicial equation

$$(C + 1 + B)x^2 a_0 = 0$$

gives $\alpha = 0$, so that

$$\Phi = \sum_{k=0}^{\infty} a_k x^k$$

is the solution of equation (20). We obtain the following recurrence formula for the coefficients:—

$$\frac{a_{k+2}}{a_k} = \frac{k^2}{(k+2)^2} \cdot \frac{1+B}{1+B+C} \left[1 - \frac{A}{(1+B)k^2} \right]$$

$$= \frac{k^2}{(k+2)^2} \cdot L \left[1 - \frac{A}{(1+B)k^2} \right] \quad \dots \quad \dots \quad \dots \quad (24)$$

where

$$L = \frac{1+B}{1+B+C}$$

The coefficients a_1, a_3, a_5, a_7 , etc., are all zero.

It is clear from the recurrence relation (24), that the series terminates for

$$A = (1+B)k^2; \quad k = 0, \text{ or even}$$

$$= 4j^2(1+B), \quad j = \text{integer or zero.} \quad \dots \quad \dots \quad (25)$$

Normalization of the characteristic values and functions

If the n -th characteristic function is denoted by ϕ_n , then the corresponding amplitude ψ_n is given by

$$\psi_n = \frac{d\phi_n}{dx} \quad \dots \quad \dots \quad \dots \quad \dots \quad (26)$$

subject to the orthogonality relation

$$\int_0^1 \psi_j \psi_n x dx = 0 \quad \dots \quad \dots \quad \dots \quad \dots \quad (27)$$

Partially integrating and remembering that $A = 4j^2(1+B)$ we obtain

$$\int_0^1 \psi_j \psi_k x dx = 4j^2 L \int_0^1 \frac{\phi_j \phi_k}{1-Lx^2} x dx \quad \dots \quad (28)$$

since this formula must be valid even if j and k are interchanged, the integrals vanish if $j \neq k$. We choose the arbitrary factor in the characteristic functions so that we have

$$\left. \begin{aligned} \int_0^1 \psi_j \psi_k x dx &= 4j^2 L \int_0^1 \frac{\phi_j \phi_k}{1-Lx^2} x dx = \delta_{jk} \\ \text{where } \delta_{jk} &= 1 \quad \text{for } j = k \\ &= 0 \quad \text{for } j \neq k \end{aligned} \right\} \dots \quad (29)$$

Characteristic functions normalized in this fashion are :

$$\left. \begin{aligned} \phi_1 &= \frac{1}{\sqrt{L(2-L)}} (1-Lx^2) \\ \phi_2 &= \frac{1-4Lx^2+3L^2x^4}{\sqrt{2L(4-14L+20L^2-9L^3)}} = \frac{(1-3Lx^2)(1-Lx^2)}{\sqrt{2L(4-14L+20L^2-9L^3)}} \\ \phi_3 &= \frac{1-9Lx^2+18L^2x^4-10L^3x^6}{\sqrt{3L(6-51L+200L^2-366L^3+312L^4-100L^5)}} \\ &\dots \end{aligned} \right\} \dots \quad (30)$$

while the corresponding amplitudes are :

$$\left. \begin{aligned} \psi_1 &= \frac{1}{\sqrt{L(2-L)}} (-2Lx) \\ \psi_2 &= \frac{1}{\sqrt{2L(4-14L+20L^2-9L^3)}} (-8Lx+12L^2x^3) \\ \psi_3 &= \frac{(-18Lx+72L^2x^3-60L^3x^5)}{\sqrt{3L(6-51L+200L^2-366L^3+312L^4-100L^5)}} \end{aligned} \right\} \dots \quad (31)$$

5. DISCUSSION

(a) *Dependence of period of pulsations on the magnetic field.*

From equations (14) and (15), we have

$$\frac{1}{\gamma} \left(\frac{\omega^2}{\pi G \rho} + 4 \right) = A_n = 4n^2(1+B),$$

$$B = \frac{(H_0^2 - H_i^2) \left(\frac{2}{\gamma} - 1 \right)}{8\pi^2 G \rho^2 R^2}$$

which clearly shows that cylinder is stable for radial oscillations.

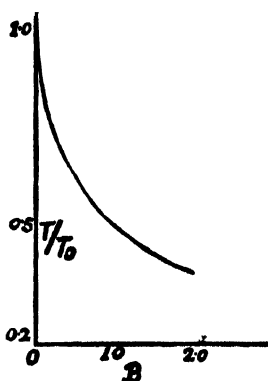
If T denotes the period in the presence of the magnetic field, and T_0 without the magnetic field, then it follows that

$$\frac{T}{T_0} = \sqrt{\frac{\gamma-1}{\gamma(1+B)-1}} \quad \dots \quad (32)$$

This relation shows that the time period decreases with magnetic field. The Table I and figure 1 exhibit the variation of $\frac{T}{T_0}$ with B for $\gamma = 1.5$.

TABLE I
The Dependence of Period on Magnetic Field

B	0.0	0.5	1.0	1.5	2.0
T/T_0	1.0	0.6325	0.5000	0.4372	0.3780



TEXT-FIG. 1.

(b) The amplitudes of radial pulsations

In Table III are listed the calculated values of the amplitudes ψ , corresponding to the following six different models of the magnetic field:—

Model I. $H_0 = 0$, $H_s = 0$; which corresponds to the case of zero magnetic field.

Model II. $H_s = 0$, $H^2 = H_0^2 (1-x^2)$; which corresponds to the case 'magnetic field proportional to the square root of pressure inside the cylinder, but zero at the surface'.

Model III. $H_s = H_0$; which corresponds to the case of uniform field.

Model IV } $H^2 = H_s^2 + (H_0^2 - H_s^2)(1-x^2)$; $H_s = 10^{-1} H_0$ }
 Model V } $H_s = 10^{-2} H_0$ }
 Model VI } $H_s = 10^{-3} H_0$ }

The Models IV, V, VI correspond to the case 'magnetic field proportional to the square root of pressure within the cylinder but finite at the surface'.

The values of ψ change linearly with x (as shown in figure 2).

TABLE II
Values of L

H_s/H_0 b	1.0	0.1	0.01	0.001
0.0	1.000	1.0000	1.000000	1.00000000
1.0	0.500	0.9950	0.999950	0.99999950
1.5	0.400	0.9940	0.999940	0.99999940
3.0	0.250	0.9925	0.999925	0.99999925

TABLE III
The Values of Characteristic Amplitudes
Mode I ($n = 1$)

Model	$H = 0$	$H_s = 0$	$H_s/H_0 = 1$		
L x	1	1	0.5	0.4	0.25
0.00	0.00	0.00	0.00	0.00	0.00
0.20	-0.40	-0.40	-0.2309	-0.20	-0.1511
0.40	-0.80	-0.80	-0.4619	-0.40	-0.3023
0.60	-1.20	-1.20	-0.6928	-0.60	-0.4535
0.80	-1.60	-1.60	-0.9238	-0.80	-0.6047
1.0	-2.00	-2.00	-1.1547	-1.00	-0.7559

TABLE III—(contd.)

Model	$H_s/H_0 = 10^{-1}$			$H_s/H_0 = 10^{-2}$		
L x	0.995	0.994	0.9925	0.999950	0.999940	0.9999925
0.00	0.0	0.0	0.0	0.0	0.0	0.0
0.20	-0.3980	-0.3976	-0.3970	-0.3999	-0.3999	-0.3999
0.40	-0.7960	-0.7952	-0.7940	-0.7999	-0.7999	-0.7999
0.60	-1.1940	-1.1928	-1.1910	-1.1999	-1.1999	-1.1999
0.80	-1.5920	-1.5904	-1.5880	-1.5999	-1.5999	-1.5998
1.00	-1.9900	-1.9880	-1.9850	-1.9999	-1.9998	-1.9998

TABLE III—(contd.)

Model	$H_s/H_0 = 10^{-3}$		
L x	0.99999950	0.99999940	0.99999925
0.00	0.0	0.0	0.0
0.20	-0.3999	-0.3999	-0.3999
0.40	-0.7999	-0.7999	-0.7999
0.60	-1.1999	-1.1999	-1.1999
0.80	-1.5999	-1.5999	-1.5999
1.00	-1.9999	-1.9999	-1.9999

TABLE III—(contd.)

Mode II ($n = 2$)

Model	$H = 0$	$H_s = 0$	$H_s/H_0 = 1$		
$\begin{smallmatrix} L \\ x \end{smallmatrix}$	1	1	0.5	0.4	0.25
0.00	0.0	0.0	0.0	0.0	0.0
0.20	-1.0634	-1.0634	-2.6333	-2.1824	-1.3889
0.40	-1.7197	-1.7970	-4.7498	-4.0428	-2.6509
0.60	-1.5613	-1.5613	-5.9228	-5.2592	-3.6591
0.80	-0.1810	-0.1810	-5.6353	-5.5096	-4.2866
1.00	-2.8284	+2.8284	-3.3906	-4.4721	-4.4065

TABLE III—(contd.)

Model	$H_s/H_0 = 10^{-1}$			$H_s/H_0 = 10^{-2}$		
$\begin{smallmatrix} L \\ x \end{smallmatrix}$	0.995	0.994	0.9925	0.999950	0.999940	0.999925
0.00	0.0	0.0	0.0	0.0	0.0	0.0
0.20	-1.0586	-1.0576	-1.0563	-1.0634	-1.0634	-1.0634
0.40	-1.7139	-1.7128	-1.7113	-1.7196	-1.7196	-1.7196
0.60	-1.5627	-1.5630	-1.5636	-1.5613	-1.5611	-1.5613
0.80	-0.2017	-0.2058	-0.2120	-0.1812	-0.1812	-0.1813
1.00	+2.7723	+2.7612	+2.7448	+2.8278	+2.8277	+2.8276

TABLE III—(contd.)

Model	$H_s/H_0 = 10^{-3}$		
$\begin{smallmatrix} L \\ x \end{smallmatrix}$	0.99999950	0.99999940	0.99999925
0.00	0.0	0.0	0.0
0.20	-1.0634	-1.0634	-1.0634
0.40	-1.7196	-1.7196	-1.7196
0.60	-1.5612	-1.5612	-1.5612
0.80	-0.1810	-0.1810	-0.1810
1.00	+2.8284	+2.8284	+2.8283

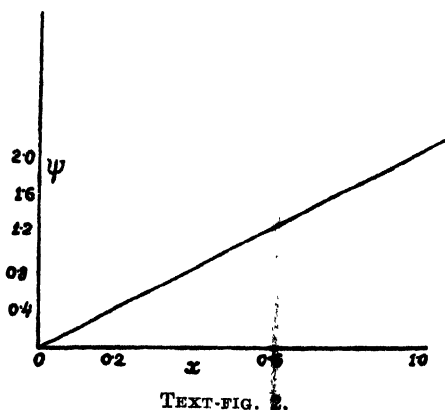
TABLE IV

Values of $\delta H/H$ at the surface ($x = 1$)

Model	$H = 0$	$H_s = 0$	$H_s/H_0 = 1$			$H_s/H_0 = 10^{-1}$		
$L \rightarrow$ $\downarrow n$								
1	1	1	0.50	0.40	0.25	0.9950	0.9940	0.9925
	4.00	4.00	2.3094	0.20	0.1512	3.9800	3.9760	3.9701
2	-22.6276	-22.6276	-16.0750	-4.4721	+3.5252	-22.3477	-22.2923	-22.2115

TABLE—IV (contd.)

Model	$H_s/H_0 = 10^{-2}$			$H_s/H_0 = 10^{-3}$		
$L \rightarrow$ $\downarrow n$						
1	0.999950	0.999940	0.999925	0.9999950	0.9999940	0.9999925
	3.9998	3.9997	3.9997	3.999998	3.999998	3.999997
2	-22.6248	-22.6242	-22.6234	-22.62758	-22.62756	-22.62755



(c) *The amplitude of the magnetic pulsations*

The equation (20) can be rewritten as

$$\frac{\delta H}{H} = -\frac{1}{x} \frac{d}{dx}(x\psi) = \left(-\frac{\psi}{x} + \frac{d\psi}{dx} \right)$$

For $n = 1$

$$\frac{\delta H}{H} = \frac{4L}{\sqrt{L(2-L)}}$$

which is independent of x .

For $n = 2$

$$\frac{\delta H}{H} = -2 \left[\psi + \frac{24L^2x^2}{\sqrt{2L(4-14L+20L^2-9L^3)}} \right]$$

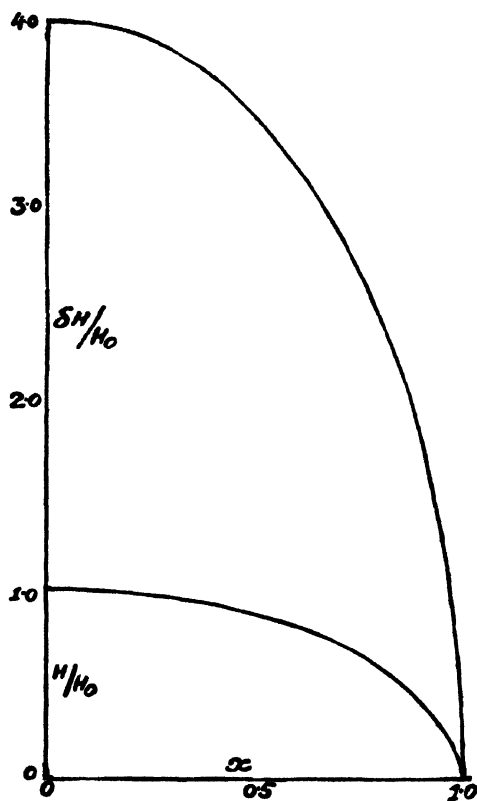
which depends both on x and L .

In Table IV, we give the evaluated values of $\frac{\delta H}{H}$ for $x = 1$ for all the models, whereas in Table V, we calculate H/H_0 and $\delta H/H_0$ for several values of x (only, for $n = 1$). Figures (3) and (4) demonstrate the respective nature of variations.

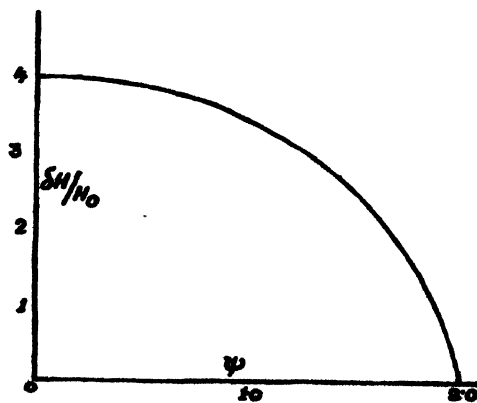
TABLE V

The Amplitude of Magnetic Pulsations

	x	ψ	H/H_0	$\delta H/H_0$
1	0.2	-0.4000	0.980	3.919
2	0.4	-0.8000	0.916	3.666
3	0.6	-1.1999	0.800	3.200
4	0.8	-1.5999	0.600	2.400
5	0.9	-1.7999	0.436	1.744
6	1.0	-1.9999	0.010	0.040



TEXT-FIG. 3.



TEXT-FIG. 4.

ABSTRACT

The expression for the amplitudes of the radial adiabatic pulsations and consequent magnetic variations of an infinitely conducting, infinite cylinder, subject to a variable axial field, are obtained. The numerical calculations of the characteristic amplitudes are carried out for six different models of the particular case in which the magnetic field is assumed to be proportional to the square root of pressure. Further, the period of pulsation is found to decrease with the magnetic field.

ACKNOWLEDGEMENTS

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REFERENCE

Chandrasekhar, S., and Fermi, E. (1953). Problems of gravitational stability in the presence of a magnetic field. *Ap. J.*, 118, 116.

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ON THE RADIAL ADIABATIC PULSATIONS OF AN INFINITE CYLINDER IN THE PRESENCE OF MAGNETIC FIELD PARALLEL TO ITS AXIS—II

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1. INTRODUCTION

S. Chandrasekhar and E. Fermi (1953) deduced the general equations governing the radial adiabatic pulsations of an infinite cylinder of compressible medium of infinite conductivity under its own gravity, with a constant magnetic field parallel to its axis. They obtained an integral formula for the period of pulsations, viz.

$$\omega^2 = \int_0^M r^2 dm = 4\{(\gamma-1) + \mathbf{M}\} \quad \dots \quad (1)$$

where

ω = frequency of pulsations.

$\gamma = C_p/C_v$ (the ratio of the specific heats at constant pressure and at constant volume).

r = radial distance from the axis.

$m = \pi r^2 \rho$ (mass per unit length of the cylinder, interior to r).

M = Mass per unit length of the cylinder.

\mathbf{M} = total magnetic energy.

\mathbf{U} = heat energy.

E. Lytkens (1954) obtained the explicit solutions of the pulsation equation for the case where there is prevalent a magnetic field proportional to the square root of pressure, and vanishing at the surface. K. P. Chopra and S. P. Talwar (1955) have shown in a recent note, hereinafter called Paper I, that the amplitude equation, viz.

$$A\psi = -\frac{d}{dx} \left\{ (1-x^2+f) \frac{1}{x} \frac{d}{dx} (x\psi) \right\} \quad \dots \quad (2)$$

where

$$\psi = \delta r/R; \quad x = r/R$$

and

$$f = \frac{H^2}{8\pi^2 G \rho^2 R^2} \left(\frac{2}{\gamma} - 1 \right)$$

holds in general for any field. The corresponding equation in terms of the characteristic functions ϕ (defined by $\psi = \frac{d\phi}{dx}$) is

$$A\phi = -(1-x^2+f) \frac{1}{x} \frac{d}{dx} \left(x \frac{d\phi}{dx} \right) \quad \dots \quad (3)$$

They further obtained the explicit solutions of the pulsation equation in the special case where the variation of the magnetic field with distance from the axis is given by

$$H^2 = (H_0^2 - H_s^2)(1 - x^2) + H_s^2 \quad \dots \quad (4)$$

where H_s = field at the surface of the cylinder (i.e. at $x = 1$),
 H_0 = field along the axis of the cylinder (i.e. at $x = 0$).

The mechanical and magnetic pulsations occur in phase (on account of infinite conductivity) with a frequency ω given by

$$A = \frac{1}{\gamma} \left(\frac{\omega^2}{\pi G \rho} + 4 \right) \quad \dots \quad (5)$$

The frequency parameter A can take up the values

$$A = 4j^2(1+B); \quad j = 1, 2, 3, \dots \quad \dots \quad (6)$$

where

$$B = \frac{H_0^2 - H_s^2}{8\pi^2 G \rho^2 R^2} \quad \dots \quad (7)$$

The characteristic functions ϕ 's and their amplitudes ψ 's satisfy the normality and orthogonality relations.

$$\int_0^1 \psi_j \psi_k x dx = 4j^2(1+B) \int_0^1 \frac{\phi_j \phi_k}{1-x^2+j} x dx = \delta_{jk} \quad \dots \quad (8)$$

where δ_{jk} is the Kronecker delta, satisfying the properties

$$\left. \begin{aligned} \delta_{jk} &= 1 & \text{for } j &= k \\ \delta_{jk} &= 0 & \text{for } j &\neq k. \end{aligned} \right\} \quad \dots \quad (9)$$

The characteristic functions normalized in this way are:

$$\left. \begin{aligned} \phi_1 &= \frac{1-Lx^2}{\sqrt{L(2-L)}} \\ \phi_2 &= \frac{(1-Lx^2)(1-3Lx^2)}{\sqrt{2L(4-14L+20L^2-9L^3)}} \\ \phi_3 &= \frac{(1-Lx^2)(1-8Lx^2+10L^2x^4)}{\sqrt{3L(6-51L+200L^2-366L^3+312L^4-100L^5)}} \end{aligned} \right\} \quad \dots \quad (10)$$

with their characteristic amplitudes,

$$\left. \begin{aligned} \psi_1 &= \frac{-2Lx}{\sqrt{L(2-L)}} \\ \psi_2 &= \frac{-8Lx+12L^2x^3}{\sqrt{2L(4-14L+20L^2-9L^3)}} \\ \psi_3 &= \frac{-18Lx+72L^2x^3-60L^3x^5}{\sqrt{3L(6-51L+200L^2-366L^3+312L^4-100L^5)}} \end{aligned} \right\} \quad \dots \quad (11)$$

where

$$L = \frac{1+B}{1+B+C} \quad \dots \quad (12)$$

2. FORMULATION OF THE PROBLEM

In this note, called Paper II, I will make use of the well-known Rietz method of evaluating Eigen-values in calculating the higher approximations to the characteristic function A , and the contributory influence of the other modes corresponding to the values of $j = 1, 2, 3$,

In general, it is reasonable to put

$$\Phi = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + \dots + a_n\phi_n \quad \dots \quad (13)$$

where $a_1, a_2, a_3, \dots, a_n$ are constants, to be chosen suitably. The expression (13) gives a very good approximation to the characteristic functions. The corresponding expression for the amplitudes is

$$\Psi = a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + \dots + a_n\psi_n \quad \dots \quad (14)$$

$$\text{Let} \quad x\Psi = \chi. \quad \dots \quad (15)$$

Then we can write (2) in the form:

$$\frac{A}{x}\chi = -\frac{d}{dx}\left\{(1-x^2+f)\frac{1}{x}\frac{d\chi}{dx}\right\} \quad \dots \quad (16)$$

while the characteristic functions of this equation are given by

$$\chi = a_1x\psi_1 + a_2x\psi_2 + a_3x\psi_3 + \dots + a_nx\psi_n. \quad \dots \quad (17)$$

3. THE CHARACTERISTIC FUNCTION A

Let us form the quotient,

$$A = -\frac{\int_0^1 \chi \frac{d}{dx} \left\{ (1-x^2+f) \frac{1}{x} \frac{d\chi}{dx} \right\} dx}{\int_0^1 \frac{1}{x} \chi^2 dx} \quad \dots \quad (18)$$

Substituting for χ from (17) and differentiating with respect to a_j we get

$$\frac{dA}{da_j} = \frac{\left[C \sum_{h=1}^n a_h \left\{ \psi_j \frac{d}{dx} (x\psi_h) + \psi_h \frac{d}{dx} (x\psi_j) \right\} \right]_0^1 - 2 \sum_{h=1}^n a_h \int_0^1 (1-x^2+f) \frac{1}{x} \times \\ \times \frac{d}{dx} (x\psi_j) \cdot \frac{d}{dx} (x\psi_h) dx - 2A \sum_{h=1}^n a_h \int_0^1 \psi_h \psi_j x dx}{\sum_{j=1}^n \sum_{h=1}^n \int_0^1 a_j a_h \psi_j \psi_h x dx} \quad (19)$$

If A is to be minimum, then this expression should vanish.

Since ψ 's are normalized and orthogonal, equation (19) reduces to

$$Aa_j + \frac{1}{2}C \sum_{h=1}^n a_h \left\{ \psi_j \frac{d}{dx} (x\psi_h) + \psi_h \frac{d}{dx} (x\psi_j) \right\} \Big|_0^1 \\ - \sum_{h=1}^n a_h \int_0^1 (1-x^2+f) \frac{1}{x} \frac{d}{dx} (x\psi_j) \frac{d}{dx} (x\psi_h) dx = 0. \quad \dots \quad (20)$$

On replacing ψ by the derivative of ϕ we have

$$Aa_j + \frac{1}{2}C \sum_{k=1}^n a_k \left\{ \psi_j \frac{d}{dx} \left(x \frac{d\phi_k}{dx} \right) + \psi_k \frac{d}{dx} \left(x \frac{d\phi_j}{dx} \right) \right\} \\ - \sum_{k=1}^n a_k \int_0^1 (1-x^2+f) \frac{1}{x} \frac{d}{dx} \left(x \frac{d\phi_j}{dx} \right) \frac{d}{dx} \left(x \frac{d\phi_k}{dx} \right) dx = 0. \quad (21)$$

With the help of the equation (3), the second and the third terms in (21) can be reduced to:

$$\frac{1}{2}C \sum_{k=1}^n a_k \left\{ \psi_j \frac{d}{dx} \left(x \frac{d\phi_k}{dx} \right) + \psi_k \frac{d}{dx} \left(x \frac{d\phi_j}{dx} \right) \right\} \\ = -2C \sum_{k=1}^n a_k \left[k^2 \frac{\psi_j \phi_k x}{1-x^2} + j^2 \psi_k \phi_j \frac{x}{1-x^2} \right] \Big|_{x=1}$$

(provided $C < B$)

and
$$- \sum_{k=1}^n a_k \int_0^1 (1-x^2+f) \frac{1}{x} \frac{d}{dx} \left(x \frac{d\phi_j}{dx} \right) \frac{d}{dx} \left(x \frac{d\phi_k}{dx} \right) dx = -4j^2(1+B) a_j$$

so that (21) can be written as:

$$\{A - 4j^2(1+B)\} a_j - 2C \sum_{k=1}^n a_k \left\{ k^2 \frac{\psi_j \phi_k x}{1-x^2} + j^2 \frac{\psi_k \phi_j x}{1-x^2} \right\} \Big|_{x=1} = 0. \quad (22)$$

In Table I are given the values of the two terms within the parenthesis of the second term in (22)

TABLE I

j	k	$k^2 \psi_j \phi_k \frac{x}{1-x^2} \Big _{x=1}$	$j^2 \psi_k \phi_j \frac{x}{1-x^2} \Big _{x=1}$
1	1	-2	-2
1	2	$8\sqrt{2}$	$2\sqrt{2}$
1	3	$-18\sqrt{3}$	$-2\sqrt{3}$
2	1	$2\sqrt{2}$	$8\sqrt{2}$
2	2	-16	-16
2	3	$18\sqrt{6}$	$8\sqrt{6}$
3	1	$-2\sqrt{3}$	$-18\sqrt{3}$
3	2	$8\sqrt{6}$	$18\sqrt{6}$
3	3	-54	-54

Substituting the corresponding values from Table I (for $j = 1, 2, 3$; $k = 1, 2, 3$) in the equation (21) we get the following linear homogeneous equations for the coefficients a_1, a_2, a_3 :-

$$\left. \begin{array}{l} \text{For } j = 1 \\ \text{For } j = 2 \\ \text{For } j = 3 \end{array} \right\} \begin{array}{l} \{A - 4(1+B-2C)\}a_1 - 20\sqrt{2}Ca_2 + 40\sqrt{3}Ca_3 = 0 \\ -20\sqrt{2}Ca_1 + \{A - 16(1+B-4C)\}a_2 - 52\sqrt{6}Ca_3 = 0 \\ 40\sqrt{3}Ca_1 - 52\sqrt{6}Ca_2 + \{A - 36(1+B-6C)\}a_3 = 0 \end{array} \quad \dots (23)$$

$$\left. \begin{array}{l} \text{If only the first characteristic function was retained} \\ \text{If only the second characteristic function was retained} \\ \text{If only the third characteristic function was retained} \end{array} \right\} \begin{array}{l} A_1 = 4(1+B-2C) \\ A_2 = 16(1+B-4C) \\ A_3 = 36(1+B-6C) \end{array} \quad \dots \quad (24)$$

The exact values of A_1, A_2, A_3 (when the influence of the other two is also taken into account) are obtained by solving the determinant:

$$\begin{vmatrix} A - 4(1+B-2C) & -20\sqrt{2}C & 40\sqrt{3}C \\ -20\sqrt{2}C & A - 16(1+B-4C) & -52\sqrt{6}C \\ 40\sqrt{3}C & -52\sqrt{6}C & A - 36(1+B-6C) \end{vmatrix} = 0 \quad \dots (25)$$

On expansion the determinant yields a cubic in A

$$A^3 - (\alpha + \beta + \gamma)A^2 + \{(\alpha\beta + \beta\gamma + \gamma\alpha) - 21824C^2\}A - \{\alpha\beta\gamma - 32(507\alpha + 150\beta + 25\gamma + 15600C)C^2\} = 0 \quad \dots (26)$$

where

$$\left. \begin{array}{l} \alpha = 4(1+B-2C) \\ \beta = 16(1+B-4C) \\ \gamma = 36(1+B-6C) \end{array} \right\} \quad \dots \quad (27)$$

Let the roots of (26) be α', β', γ' , such that

$$\left. \begin{array}{l} \alpha' = \alpha + \epsilon_1 \\ \beta' = \beta + \epsilon_2 \\ \gamma' = \gamma + \epsilon_3 \end{array} \right\} \quad \dots \quad (28)$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ remain to be determined. But a cubic whose roots are α', β', γ' is

$$A^3 - (\alpha' + \beta' + \gamma')A^2 + (\alpha'\beta' + \beta'\gamma' + \gamma'\alpha')A - \alpha'\beta'\gamma' = 0 \quad \dots (29)$$

Comparison of (26) and (29) yields, with the help of (23) and (24),

$$\left. \begin{array}{l} \epsilon_1 = -217 \frac{C^2}{1+B} \\ \epsilon_2 = -744 \frac{C^2}{1+B} \\ \epsilon_3 = 961 \frac{C^2}{1+B} \end{array} \right\} \quad \dots \quad (30)$$

Thus the exact values of the characteristic functions are:

$$\left. \begin{aligned} A_1 &= 4 \left(1+B-2C-54 \frac{C^2}{1+B} \right) \\ A_2 &= 16 \left(1+B-4C-46.5 \frac{C^2}{1+B} \right) \\ A_3 &= 36 \left(1+B-6C+27 \frac{C^2}{1+B} \right) \end{aligned} \right\} \dots \dots \dots (31)$$

In Table II are given the values of A_1 , A_2 and A_3 for two different models:
(1) $C/B = 10^{-2}$; (2) $C/B = 10^{-4}$.

TABLE II
Characteristic Functions

B	C/B = 10 ⁻²			C/B = 10 ⁻⁴		
	A ₁	A ₂	A ₃	A ₁	A ₂	A ₃
0.20	4.7832	19.0695	42.7712	4.7991	19.1962	43.1956
0.40	5.5655	22.1355	49.5470	5.5972	22.3889	50.3913
0.60	6.3471	25.1992	56.3258	6.3946	25.5794	57.5870
0.80	7.1282	28.2615	63.1065	7.1916	28.7684	64.7827
1.00	7.9091	31.3228	69.8886	7.9883	31.9564	71.9784
1.20	8.7898	34.3833	74.0796	8.7848	35.1436	79.1740
1.40	9.4702	37.4432	83.4553	9.4821	38.3302	86.3697
1.60	10.2506	40.5027	90.2397	10.2783	41.5265	93.5654
1.80	11.0308	43.5617	97.0246	11.0744	44.7122	100.7251
2.00	11.8110	46.6208	103.8096	11.8704	47.8880	107.9568
2.20	12.5908	49.6784	110.2996	12.3680	51.0723	115.1524
2.40	13.4712	52.7379	117.3805	13.4622	54.2585	122.3481
2.60	14.1512	55.7963	124.1665	14.1551	57.4536	129.5438
2.80	14.9312	59.8545	130.9525	14.9549	60.6285	136.7395
3.00	15.7111	61.9126	137.7387	15.7507	63.8134	143.9352

Substituting the values of A from (31) in (23) we obtain, for C , sufficiently less than B , the values of ratios of coefficients:—

For the first characteristic function:

$$\frac{a_2}{a_1} = -2.357 \frac{C}{1+B}; \quad \frac{a_3}{a_1} = 2.165 \frac{C}{1+B}.$$

For the second characteristic function:

$$\frac{a_1}{a_2} = 2.357 \frac{C}{1+B}, \quad \frac{a_3}{a_2} = -6.212 \frac{C}{1+B}.$$

For the third characteristic function:

$$\frac{a_2}{a_3} = 6.368 \frac{C}{1+B}, \quad \frac{a_1}{a_3} = -2.165 \frac{C}{1+B}.$$

In Table III the calculated values of the ratios of coefficients for the three characteristic functions are given.

TABLE III

B	First Characteristic Function		Second Characteristic Function		Third Characteristic Function	
	a_2/a_1	a_3/a_1	a_1/a_2	a_3/a_2	a_1/a_3	a_2/a_3
0.20	-0.0039	0.0036	0.0039	-0.0103	0.0106	-0.0039
0.40	-0.0067	0.0061	0.0067	-0.0177	0.0181	-0.0067
0.60	-0.0088	0.0081	0.0088	-0.0232	0.0238	-0.0088
0.80	-0.0104	0.0096	0.0104	-0.0276	0.0282	-0.0104
1.00	-0.0117	0.0108	0.0117	-0.0310	0.0318	-0.0117
1.20	-0.0128	0.0118	0.0128	-0.0338	0.0347	-0.0128
1.40	-0.0137	0.0126	0.0137	-0.0362	0.0371	-0.0137
1.60	-0.0145	0.0133	0.0145	-0.0382	0.0391	-0.0145
1.80	-0.0151	0.0139	0.0151	-0.0399	0.0409	-0.0151
2.00	-0.0157	0.0144	0.0157	-0.0414	0.0424	-0.0157
2.20	-0.0163	0.0150	0.0163	-0.0430	0.0441	-0.0163
2.40	-0.0166	0.0152	0.0166	-0.0438	0.0449	-0.0166
2.60	-0.0170	0.0156	0.0170	-0.0448	0.0459	-0.0170
2.80	-0.0173	0.0159	0.0173	-0.0457	0.0469	-0.0173
3.00	-0.0176	0.0162	0.0176	-0.0465	0.0477	-0.0176

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ABSTRACT

The radial adiabatic pulsations of an infinite cylinder with a prevalent magnetic field had already been considered in a previous paper. In this note the higher approximations to the characteristic functions and the contributions of other modes are dealt with.

REFERENCES

- Chandrasekhar, S., and Fermi, E. (1953). Problems of gravitational stability in the presence of a magnetic field. *Ap. J.*, 118, 116.
 Chopra, K. P., and Talwar, S. P. (1955). On the radial pulsations of an infinite cylinder with a magnetic field parallel to its axis. *PNISPS*, 21, 302-13.
 Lytkens, E. (1954). On the radial pulsations of an infinite cylinder with a magnetic field parallel to its axis. *Ap. J.*, 119, 413.

INTERNAL BALLISTICS FOR COMPOSITE CHARGE

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1. INTRODUCTION

The problem of composite charge, which consists of a mixture of grains of two or more nominal sizes with the same or different composition, has been discussed by Corner (1950), Hunt, Hinds and Clemmow (1951) and Venkatesan and Patni (1953).

The general problem of composite charge as it stands is quite a difficult one. Corner, Hunt, Hinds and Clemmow obtained an approximate solution by reducing the problem of composite charge to a single equivalent charge with adjusted parameters. Venkatesan and Patni have given a direct treatment to the problem on the following assumptions:—

- (i) $\gamma_1 = \gamma_2 = \gamma$, i.e. the ratio of the two specific heats for the two propellants is the same;
- (ii) the co-volume of the gases equals the specific volume of each propellant; and
- (iii) a linear law of burning has been assumed, i.e. $r = \beta p$.

In this paper the author has dealt with the problem of composite charge under less restricted conditions and the effect of different gammas for the two propellants on the ballistics of the gun have been discussed.

2. BASIC EQUATIONS

There are four basic equations of internal ballistics.

(i) Energy equation:

Total energy given out by the two propellants is

$$\frac{F_1 C_1 Z_1}{\gamma_1 - 1} + \frac{F_2 C_2 Z_2}{\gamma_2 - 1}.$$

Thermal energy of the propellant gases is given by

$$J\sigma'_g T C_1 Z_1 + J\sigma'_g T C_2 Z_2.$$

We have equation of state as

$$p(V-b) = nRT,$$

where b is the weighted mean of the two co-volumes, and

$$n = \frac{n_1 C_1 Z_1 + n_2 C_2 Z_2}{C_1 Z_1 + C_2 Z_2},$$

n_1 and n_2 being the number of gram-molecules per gram of the two propellant gases respectively.

Also we know that

$$\gamma_1 - 1 = \frac{Rn_1}{J\sigma_v}$$

and

$$\gamma_2 - 1 = \frac{Rn_2}{J\sigma_v}.$$

With the help of these equations, the thermal energy is written as

$$J\sigma_v' T C_1 Z_1 + J\sigma_v'' T C_2 Z_2 = \frac{p(V-b) \left(\frac{n_1 C_1 Z_1}{\gamma_1 - 1} + \frac{n_2 C_2 Z_2}{\gamma_2 - 1} \right) (C_1 Z_1 + C_2 Z_2)}{(n_1 C_1 Z_1 + n_2 C_2 Z_2)}$$

Energy of the shot, taking into account the frictional and rotational energy, and kinetic energy of the propellant gases, is given by

$$\frac{1}{2} w_1 v^2, \text{ where } w_1 = 1.05 W + \frac{1}{8} (C_1 + C_2).$$

Hence the energy equation becomes

$$\frac{F_1 C_1 Z_1}{\gamma_1 - 1} + \frac{F_2 C_2 Z_2}{\gamma_2 - 1} = \frac{p(V-b)(C_1 Z_1 + C_2 Z_2) \left(\frac{n_1 C_1 Z_1}{\gamma_1 - 1} + \frac{n_2 C_2 Z_2}{\gamma_2 - 1} \right)}{(n_1 C_1 Z_1 + n_2 C_2 Z_2)} + \frac{1}{2} w_1 v^2.$$

Now

$$(C_1 Z_1 + C_2 Z_2) V = K_0 + Ax - \frac{C_1(1-Z_1)}{\delta_1} - \frac{C_2(1-Z_2)}{\delta_2}$$

$$\begin{aligned} \therefore (V-b)(C_1 Z_1 + C_2 Z_2) &= K_0 + Ax - \frac{C_1(1-Z_1)}{\delta_1} - \frac{C_2(1-Z_2)}{\delta_2} - b(C_1 Z_1 + C_2 Z_2) \\ &= \left(K_0 - \frac{C_1}{\delta_1} - \frac{C_2}{\delta_2} \right) + Ax - C_1 Z_1 \left(b - \frac{1}{\delta_1} \right) - C_2 Z_2 \left(b - \frac{1}{\delta_2} \right). \end{aligned}$$

Neglecting co-volume correction we get

$$(V-b)(C_1 Z_1 + C_2 Z_2) = A(l+x),$$

$$\text{where } Al = K_0 - \frac{C_1}{\delta_1} - \frac{C_2}{\delta_2}.$$

Therefore the energy equation in the final form is

$$\frac{F_1 C_1 Z_1}{\gamma_1 - 1} + \frac{F_2 C_2 Z_2}{\gamma_2 - 1} = \frac{Ap(l+x) \left(\frac{n_1 C_1 Z_1}{\gamma_1 - 1} + \frac{n_2 C_2 Z_2}{\gamma_2 - 1} \right)}{(n_1 C_1 Z_1 + n_2 C_2 Z_2)} + \frac{1}{2} w_1 v^2 \quad \dots \quad (1)$$

or

$$F_1 C_1 Z_1 + k F_2 C_2 Z_2 = \frac{Ap(l+x)(n_1 C_1 Z_1 + k n_2 C_2 Z_2)}{(n_1 C_1 Z_1 + n_2 C_2 Z_2)} + \frac{\gamma_1 - 1}{2} w_1 v^2, \quad \dots \quad (1a)$$

$$\text{where } k = \frac{\gamma_1 - 1}{\gamma_2 - 1}.$$

(ii) Dynamical equation is

$$w_1 \frac{dv}{dt} = Ap. \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$$

(iii) Form functions are given by

$$Z_1 = (1-f_1)(1+\theta_1 f_1) \quad \dots \quad (3a)$$

$$Z_2 = (1-f_2)(1+\theta_2 f_2). \quad \dots \quad (3b)$$

(iv) Rate of burning equations are

$$D_1 \frac{df_1}{dt} = -\beta_1 p \quad \dots \quad (4a)$$

$$D_2 \frac{df_2}{dt} = -\beta_2 p, \quad \dots \quad (4b)$$

where $C_1, F_1, \beta_1, D_1, \theta_1, f_1, Z_1, n_1, \delta_1, \gamma_1$ refer to first charge and $C_2, F_2, \beta_2, D_2, \theta_2, f_2, Z_2, n_2, \delta_2, \gamma_2$ refer to second charge.

3. SOLUTION OF THE EQUATIONS

Eliminating p from (2) and (4a) and integrating we get

$$v = -\frac{A}{\beta' w_1} + \text{constant},$$

where $\beta' = \frac{\beta_1}{D_1}$.

Applying initial conditions that at

$$x = 0, v = 0, f_1 = f_{10} \text{ and } Z_1 = Z_{10},$$

we get

$$v = \frac{A}{\beta' w_1} (f_{10} - f_1). \quad \dots \quad (5a)$$

Similarly from (2) and (4b) and applying initial conditions

$$x = 0, v = 0, f_2 = f_{20} \text{ and } Z_2 = Z_{20},$$

we get

$$v = \frac{A}{\beta'' w_1} (f_{20} - f_2) \quad \dots \quad (5b)$$

where $\beta'' = \frac{\beta_2}{D_2}$.

From (3a) and (5a) we obtain

$$Z_1 = Z_{10} + \frac{\beta' w_1}{A} v (1 - \theta_1 + 2\theta_1 f_{10}) - \frac{\beta'^2 w_1^2}{A^2} \theta_1 v^2. \quad \dots \quad (6a)$$

Similarly from (3b) and (5b) we have

$$Z_2 = Z_{20} + \frac{\beta'' w_1}{A} v (1 - \theta_2 + 2\theta_2 f_{20}) - \frac{\beta''^2 w_1^2}{A^2} \theta_2 v^2. \quad \dots \quad (6b)$$

Putting $\xi = 1 + \frac{x}{l}$, equation (2) becomes

$$\frac{w_1}{Al} v \frac{dv}{d\xi} = p. \quad \dots \quad (7)$$

Substituting the values of Z_1 and Z_2 from (6a) and (6b) respectively in (1a) and then putting the value of p from (7), we obtain, after simplification,

$$\frac{d\xi}{\xi} = \frac{K_1(a_1-v)(b_1+v)v dv}{KK_2(a-v)(b+v)(a_2-v)(b_2+v)}, \quad \dots \quad (8)$$

where

$$(1) \quad \begin{cases} K = \frac{F_1 C_1 \beta'^2 w_1}{A^2} \theta_1 + \frac{F_2 C_2 \beta''^2 w_1}{A^2} \theta_2 + \frac{\gamma_1 - 1}{2} \\ a - b = \frac{1}{K} \left[\frac{F_1 C_1 \beta'}{A} (1 - \theta_1 + 2\theta_1 f_{10}) + \frac{k F_2 C_2 \beta''}{A} (1 - \theta_2 + 2\theta_2 f_{20}) \right] \\ ab = \frac{1}{K} \left[\frac{F_1 C_1 Z_{10} + k F_2 C_2 Z_{20}}{w_1} \right] \end{cases}$$

$$(2) \quad \begin{cases} K_1 = \frac{n_1 C_1 \beta'^2 w_1^2}{A^2} \theta_1 + \frac{k n_2 C_2 \beta''^2 w_1^2}{A^2} \theta_2 \\ a_1 - b_1 = \frac{1}{K_1} \left[\frac{n_1 C_1 \beta' w_1}{A} (1 - \theta_1 + 2\theta_1 f_{10}) + \frac{k n_2 C_2 \beta'' w_1}{A} (1 - \theta_2 + 2\theta_2 f_{20}) \right] \\ a_1 b_1 = \frac{1}{K_1} [n_1 C_1 Z_{10} + k n_2 C_2 Z_{20}] \end{cases}$$

and

$$(3) \quad \begin{cases} K_2 = \frac{n_1 C_1 \beta'^2 w_1^2}{A^2} \theta_1 + \frac{n_2 C_2 \beta''^2 w_1^2}{A^2} \theta_2 \\ a_2 - b_2 = \frac{1}{K_2} \left[\frac{n_1 C_1 \beta' w_1}{A} (1 - \theta_1 + 2\theta_1 f_{10}) + \frac{n_2 C_2 \beta'' w_1}{A} (1 - \theta_2 + 2\theta_2 f_{20}) \right] \\ a_2 b_2 = \frac{1}{K_2} [n_1 C_1 Z_{10} + n_2 C_2 Z_{20}]. \end{cases}$$

Breaking the R.H.S. of equation (8) into partial fractions, we get

$$\lambda_1 \frac{d\xi}{\xi} = \frac{A_1}{a-v} + \frac{B_1}{b+v} + \frac{C_1}{a_2-v} + \frac{D_1}{b_2+v}, \quad \dots \quad (9)$$

where $\lambda_1 = \frac{KK_2}{K_1}$

$$A_1 = \frac{a(a_1-a)(b_1+a)}{(b+a)(a_2-a)(b_2+a)}$$

$$B_1 = \frac{b(a_1+b)(b-b_1)}{(a+b)(a_2+b)(b_2-b)}$$

$$C_1 = \frac{a_2(a_1-a_2)(b_1+a_2)}{(a-a_2)(b+a_2)(a_2+b_2)}$$

$$D_1 = \frac{b_2(a_1+b_2)(b_2-b_1)}{(a+b_2)(b-b_2)(a_2+b_2)}.$$

Integrating equation (9) we get

$$\lambda_1 \log \xi = [-A \log (a-v) + B \log (b+v) - C \log (a_2-v) + D \log (b_2+v)] + \text{constant}. \quad \dots \quad (10)$$

Initially $\xi = 1$, $v = 0$, so we get

$$\xi^{\lambda_1} = \left(\frac{a}{a-v}\right)^{A_1} \left(\frac{b+v}{b}\right)^{B_1} \left(\frac{a_2}{a_2-v}\right)^{C_1} \left(\frac{b_2+v}{b_2}\right)^{D_1}.$$

Therefore
$$\xi = \left[\left(\frac{a}{a-v}\right)^{A_1} \left(\frac{b+v}{b}\right)^{B_1} \left(\frac{a_2}{a_2-v}\right)^{C_1} \left(\frac{b_2+v}{b_2}\right)^{D_1} \right]^{1/\lambda_1} \quad \dots (10a)$$

This equation expresses shot-travel as a function of v .

From equations (8) and (7) we get

$$p = \frac{\lambda_1 w_1}{Al} \frac{(a-v)(b+v)(a_2-v)(b_2+v)}{\xi(a_1-v)(b_1+v)}. \quad \dots (11)$$

This equation gives pressure as a function of v (ξ being a function of v).

These equations are valid so long as both the propellants are burning.

There are three possible ways of burning of the two charges :

- (i) Charge C_1 burns out first and charge C_2 continues to burn till all-burnt position. The condition for this is $\beta' f_{20} > \beta'' f_{10}$.
- (ii) Charge C_2 burns out first and charge C_1 continues to burn till all-burnt position. The condition for this is $\beta' f_{20} < \beta'' f_{10}$.
- (iii) Charges C_1 and C_2 burn out at the same time.

Hence two cases arise :

Case I. Both the propellants burn out simultaneously.

Case II. The two propellants burn out at different times.

Case I. Simultaneous burning of the two propellants.

For this case all the equations up to (11) hold good.

The velocity at 'all-burnt' is given by

$$v_2 = \frac{A}{\beta' w_1} f_{10} = \frac{A}{\beta'' w_1} f_{20} \quad \dots (12)$$

(where suffix 2 denotes the all-burnt position).

The shot-travel is given by equation (10a) as

$$\xi_2 = \left[\left(\frac{a}{a-v_2}\right)^{A_1} \left(\frac{b+v_2}{b}\right)^{B_1} \left(\frac{a_2}{a_2-v_2}\right)^{C_1} \left(\frac{b_2+v_2}{b_2}\right)^{D_1} \right]^{1/\lambda_1} \quad \dots (13)$$

and the pressure is given by

$$p_2 = \frac{\lambda_1 w_1}{Al} \frac{(a-v_2)(b+v_2)(a_2-v_2)(b_2+v_2)}{\xi_2(a_1-v_2)(b_1+v_2)}. \quad \dots (14)$$

At 'all-burnt' position equation (1a) becomes

$$F_1 C_1 + k F_2 C_2 = \frac{Ap(l+x)(n_1 C_1 + k n_2 C_2)}{(n_1 C_1 + n_2 C_2)} + \frac{\gamma_1 - 1}{2} w_1 v^2. \quad \dots (15)$$

With the help of equation (7) this can be written as

$$\xi \frac{v}{d\xi} \left(\frac{n_1 C_1 + k n_2 C_2}{n_1 C_1 + n_2 C_2} \right) = \frac{F_1 C_1}{w_1} + \frac{k F_2 C_2}{w_1} - \frac{\gamma_1 - 1}{2} v^2 \quad \dots (15a)$$

or
$$\xi \frac{v}{d\xi} \left(\frac{n_1 C_1 + k n_2 C_2}{n_1 C_1 + n_2 C_2} \right) = \frac{\gamma_1 - 1}{2} [L - v^2], \quad \dots \quad (16)$$

where
$$L = \frac{2}{\gamma_1 - 1} \left[\frac{F_1 C_1}{w_1} + \frac{k F_2 C_2}{w_1} \right].$$

Therefore $\frac{2v}{L - v^2} \frac{dv}{d\xi} = \lambda_2 \frac{d\xi}{\xi}$, where $\lambda_2 = \frac{(n_1 C_1 + n_2 C_2)(\gamma_1 - 1)}{(n_1 C_1 + k n_2 C_2)}$.

Integrating this we get

$$-\log [L - v^2] = \lambda_2 \log \xi + \text{constant}.$$

Initially

$$v = v_2 \quad \text{and} \quad \xi = \xi_2.$$

Therefore
$$\left(\frac{\xi}{\xi_2} \right)^{\lambda_2} = \left[\frac{L - v_2^2}{L - v^2} \right]. \quad \dots \quad (17)$$

The pressure is given by

$$p = \frac{\lambda_2 w_1}{2Al} \frac{(L - v^2)}{\xi}. \quad \dots \quad (18)$$

Let v_3 and ξ_3 denote the values at the muzzle; then the muzzle velocity is given by

$$\left(\frac{\xi_3}{\xi_2} \right)^{\lambda_2} = \left[\frac{L - v_2^2}{L - v_3^2} \right].$$

Hence
$$v_3^2 = L \left[1 - \left(\frac{\xi_2}{\xi_3} \right)^{\lambda_2} \right] + v_2^2 \left(\frac{\xi_2}{\xi_3} \right)^{\lambda_2}. \quad \dots \quad (19)$$

Case II. Non-simultaneous burning of the two propellants.

Let us suppose for the sake of definiteness that charge C_1 burns out first. We will have to deal this under two headings:

- (i) When only C_2 is burning and C_1 is burnt out.
- (ii) When C_2 is also burnt out.

Part (i): In this case equation (1a) becomes

$$F_1 C_1 + k F_2 C_2 Z_2 = \frac{Ap(l+x)(n_1 C_1 + k n_2 C_2 Z_2)}{(n_1 C_1 + n_2 C_2 Z_2)} + \frac{\gamma_1 - 1}{2} w_1 v^2 \quad \dots \quad (20)$$

which can be written as

$$\xi v \frac{dv}{d\xi} \left(\frac{n_1 C_1 + k n_2 C_2 Z_2}{n_1 C_1 + n_2 C_2 Z_2} \right) = \frac{F_1 C_1}{w_1} + \frac{k F_2 C_2 Z_2}{w_1} - \frac{\gamma_1 - 1}{2} v^2. \quad \dots \quad (20a)$$

This can be put as

$$\lambda_3 \frac{d\xi}{\xi} = \frac{v(a'_1 - v)(b'_1 + v)}{(a' - v)(b' + v)(a'_2 - v)(b'_2 + v)}, \quad \dots \quad (21)$$

where $\lambda_3 = \frac{K'K'_2}{K'_1}$

$$(1) \quad \begin{cases} K' = \frac{\gamma_1 - 1}{2} + \frac{kF_2 C_2 \beta'^2 w_1}{A^2} \theta_2 \\ a' - b' = \frac{1}{K'} \left[\frac{kF_2 C_2 \beta''}{A} (1 - \theta_2 + 2\theta_2 f_{20}) \right] \\ a'b' = \frac{1}{K'} \left[\frac{F_1 C_1}{w_1} + \frac{kF_2 C_2 Z_{20}}{w_1} \right] \end{cases}$$

$$(2) \quad \begin{cases} K'_1 = \frac{kn_2 C_2 \beta'^2 w_1^2}{A^2} \theta_2 \\ a'_1 - b'_1 = \frac{1}{K'_1} \left[\frac{kn_2 C_2 \beta'' w_1}{A} (1 - \theta_2 + 2\theta_2 f_{20}) \right] \\ a'_1 b'_1 = \frac{1}{K'_1} [n_1 C_1 + kn_2 C_2 Z_{20}] \end{cases}$$

and $(3) \quad \begin{cases} K'_2 = \frac{n_2 C_2 \beta'^2 w_1^2}{A^2} \theta_2 \\ a'_2 - b'_2 = \frac{1}{K'_2} \left[\frac{n_2 C_2 \beta'' w_1}{A} (1 - \theta_2 + 2\theta_2 f_{20}) \right] \\ a'_2 b'_2 = \frac{1}{K'_2} [n_1 C_1 + n_2 C_2 Z_{20}]. \end{cases}$

Putting R.H.S. of equation (21) into partial fractions, we get

$$\lambda_3 \frac{d\xi}{\xi} = \frac{A'}{a' - v} + \frac{B'}{b' + v} + \frac{C'}{a'_2 - v} + \frac{D'}{b'_2 + v}, \quad \dots \quad (22)$$

where $A' = \frac{a'(a'_1 - a')(b'_1 + a')}{(b' + a')(a'_2 - a')(b'_2 + a')}$

$$B' = \frac{b'(a'_1 + b')(b' - b'_1)}{(a' + b')(a'_2 + b')(b'_2 - b')}$$

$$C' = \frac{a'_2(a'_1 - a'_2)(b'_1 + a'_2)}{(a' - a'_2)(b' + a'_2)(a'_2 + b'_2)}$$

and $D' = \frac{b'_2(a'_1 + b'_2)(b'_2 - b'_1)}{(a' + b'_2)(b' - b'_2)(a'_2 + b'_2)}$

Integrating equation (22) and applying initial conditions, $v = v_{2,1}$, $\xi = \xi_{2,1}$, we get

$$\left(\frac{\xi}{\xi_{2,1}} \right)^{\lambda_3} = \left[\left(\frac{a' - v_{2,1}}{a' - v} \right)^{A'} \left(\frac{b + v}{b + v_{2,1}} \right)^{B'} \left(\frac{a_2 - v_{2,1}}{a_2 - v} \right)^{C'} \left(\frac{b_2 + v}{b_2 + v_{2,1}} \right)^{D'} \right] \quad \dots \quad (23)$$

which gives shot-travel as a function of v .

From equation (20a) pressure becomes

$$p = \frac{\lambda_3 w_1}{Al} \frac{(a'-v)(b'+v)(a'_2-v)(b'_2+v)}{\xi(a'_1-v)(b'_1+v)} \quad \dots \quad (24)$$

Part (ii): In this case equation (1a) reduces to

$$F_1 C_1 + k F_2 C_2 = \frac{Ap(l+x)(n_1 C_1 + k n_2 C_2)}{(n_1 C_1 + n_2 C_2)} + \frac{\gamma_1 - 1}{2} w_1 v^2. \quad \dots \quad (25)$$

As in Case I it can be written as

$$\frac{2v dv}{L-v^2} = \lambda_2 \frac{d\xi}{\xi}, \quad \dots \quad (25a)$$

where $\lambda_2 = \frac{(n_1 C_1 + n_2 C_2)(\gamma_1 - 1)}{(n_1 C_1 + k n_2 C_2)}.$

Integrating and applying initial conditions, $v = v_2$, $\xi = \xi_2$, we get

$$\left(\frac{\xi}{\xi_2}\right)^{\lambda_2} = \left[\frac{L-v_2^2}{L-v^2}\right]. \quad \dots \quad (26)$$

The pressure is given by

$$p = \frac{\lambda_2 w_1}{2Al} \frac{(L-v^2)}{\xi}$$

and the muzzle velocity by

$$v_3^2 = L \left[1 - \left(\frac{\xi_2}{\xi_3}\right)^{\lambda_2} \right] + v_2^2 \left(\frac{\xi_2}{\xi_3}\right)^{\lambda_2}, \quad \dots \quad (27)$$

where ξ_2 is obtained by putting v_2 in (23), and

$$v_2 = \frac{A}{\beta^* w_1} f_{20}.$$

4. MAXIMUM PRESSURE

In the following cases the maximum pressure can occur:—

- Case (a) both the charges are burning;
- Case (b) C_1 is burnt out and C_2 is burning; and
- Case (c) at the position of 'all-burnt'.

Case (a). From equation (11) we get

$$p = \frac{\lambda_1 w_1}{Al} \frac{(a-v)(b+v)(a_2-v)(b_2+v)}{\xi(a_1-v)(b_1+v)}. \quad \dots \quad (28)$$

For maximum pressure $dp = 0$. Therefore differentiating (28) and simplifying we get the equation giving the value of v at maximum pressure as

$$v = \frac{\lambda_1}{(a_1-v)(b_1+v)} \left[(a-v)(b+v)(a_2-v)(b_2+v) \left\{ -\frac{1}{(a-v)} + \frac{1}{(b+v)} - \frac{1}{(a_2-v)} + \frac{1}{(b_2+v)} - \frac{a_1-b_1-2v}{(a_1-v)(b_1+v)} \right\} \right] \quad \dots \quad (29)$$

Putting $\gamma_1 = \gamma_2 = \gamma$ as a first approximation, we get

$$v_{(1)} = \frac{K(a-b)}{2K+1}, \quad \dots \dots \dots (30)$$

where $k = 1$. This is the first approximation to the value of v .

Now from Newton-Raphson's method of iterative process we get the $(n+1)^{\text{th}}$ approximation as

$$v_{(n+1)} = v_{(n)} - \frac{f(v)}{f'(v)} \quad \dots \dots \dots (31)$$

where $v_{(n)}$ is the solution to the n^{th} approximation.

Let

$$v = v_{(2)} \quad \dots \dots \dots (32)$$

be the second approximation to the value of v .

Therefore the maximum pressure is

$$p_1 = \frac{\lambda_1 w_1}{Al} \frac{[a-v_{(2)}][b+v_{(2)}][a_2-v_{(2)}][b_2+v_{(2)}]}{\xi_{(2)}[a_1-v_{(2)}][b_1+v_{(2)}]}, \quad \dots \dots (33)$$

where $\xi_{(2)}$ is obtained from equation (10a) by putting $v = v_{(2)}$.

Therefore

$$\xi_{(2)} = \left[\left\{ \frac{a}{a-v_{(2)}} \right\}^{A_1} \left\{ \frac{b+v_{(2)}}{b} \right\}^{B_1} \left\{ \frac{a_2}{a_2-v_{(2)}} \right\}^{C_1} \left\{ \frac{b_2+v_{(2)}}{b_2} \right\}^{D_1} \right]^{1/\lambda_1} \dots (34)$$

For the occurrence of maximum pressure in this case, the conditions are

$$f_{11} > 0 \quad \text{and} \quad f_{21} > 0.$$

With the help of (5a) and (5b), these become

$$f_{10} > \frac{\beta' w_1}{A} v_{(2)},$$

and

$$f_{20} > \frac{\beta'' w_1}{A} v_{(2)}.$$

Case (b). From equation (24) we get

$$p = \frac{\lambda_3 w_1}{Al} \frac{(a'-v)(b'+v)(a'_2-v)(b'_2+v)}{\xi(a'_1-v)(b'_1+v)}. \quad \dots \dots (35)$$

For maximum pressure $dp = 0$. Therefore differentiating (35) and simplifying we get

$$v = \frac{\lambda_3}{(a'_1-v)(b'_1+v)} \left[(a'-v)(b'+v)(a'_2-v)(b'_2+v) \left\{ -\frac{1}{(a'-v)} + \frac{1}{(b'+v)} \right. \right. \\ \left. \left. - \frac{1}{(a'_2-v)} + \frac{1}{(b'_2+v)} - \frac{a'_1-b'_1-2v}{(a'_1-v)(b'_1+v)} \right\} \right]. \quad \dots (36)$$

As done before, putting $\gamma_1 = \gamma_2 = \gamma$ as a first approximation, we get

$$v_{(1)} = \frac{K_1(a'-b')}{2K_1+1}, \quad \dots \dots \dots (37)$$

where $k = 1$.

As indicated before here also we apply Newton-Raphson's iterative process and get the desired second approximation.

Therefore the maximum pressure is

$$p_1 = \frac{\lambda_3 w_1}{A l} \frac{[a' - v_{(2)}][b' + v_{(2)}][a'_2 - v_{(2)}][b'_2 + v_{(2)}]}{\xi_{(2)}[a'_1 - v_{(2)}][b'_1 + v_{(2)}]}, \quad \dots \quad (38)$$

where

$$\xi_{(2)} = \xi_{2,1} \left[\left\{ \frac{a' - v_{2,1}}{a' - v_{(2)}} \right\}^{A'} \left\{ \frac{b' + v_{(2)}}{b' + v_{2,1}} \right\}^{B'} \left\{ \frac{a'_2 - v_{2,1}}{a'_2 - v_{(2)}} \right\}^{C'} \left\{ \frac{b'_2 + v_{(2)}}{b'_2 + v_{2,1}} \right\}^{D'} \right]^{1/\lambda_3}$$

For the occurrence of maximum pressure in this case, the conditions are

$$f_{11} = 0 \quad \text{and} \quad f_{21} > 0.$$

Hence

$$f_{10} = \frac{\beta' w_1}{A} v_{(2)}$$

$$f_{20} > \frac{\beta'' w_1}{A} v_{(2)},$$

where $v_{(2)}$ is the second approximation in this case,

and $\beta' f_{20} > \beta'' f_{10}$.

In view of this condition, f_{21} cannot be equal to zero.

Case (c). In this case two possibilities are there: (i) maximum pressure can occur when both the propellants burn out simultaneously and (ii) at the position of 'all-burnt' when the two propellants burn out at different times.

(i) This has already been dealt with in case (a).

(ii) For this case the conditions are

$$f_{11} = 0 \quad \text{and} \quad f_{21} = 0$$

which reduce to

$$f_{10} = \frac{\beta' w_1}{A} v_{(2)}$$

$$f_{20} = \frac{\beta'' w_1}{A} v_{(2)},$$

where $v_{(2)}$ is the value of case (b).

Hence maximum pressure in this is given by equation (38).

ACKNOWLEDGEMENTS

The author is extremely grateful to Dr. D. S. Kothari for his permission to publish this paper. He is deeply indebted to Dr. R. S. Varma for his valuable guidance and keen interest in this work.

SUMMARY

In this communication the author has dealt the problem of internal ballistics for composite charge under less restricted conditions. The propellants having different gammas (ratio of the two specific heats C_p/C_v) have been considered and a linear rate of burning has been assumed.

REFERENCES

- Clemmow, C. A. (1951). *Internal Ballistics*. H.M.S.O. Publication, London.
 Corner, J. (1950). *Theory of Internal Ballistics of Guns*. John Wiley, New York.
 Venkatesan, N. S., and Patni, G. C. (1953). Influence of Composite Charges on Maximum Pressure and Muzzle Velocity. *Def. Sc. Jour.*, 3, 51-60.

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A NOTE ON THE (RELATIVISTIC) STATISTICAL MECHANICS OF AN ASSEMBLY IN MASS-MOTION

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(Communicated by D. S. Kothari, F.N.I.)

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The present paper deals with a relativistic study of the Statistical Mechanics of an ideal gaseous assembly in mass-motion. In order to determine the distribution appropriate to the laboratory system S , we may proceed by introducing the conservation of the net linear momentum \vec{P} of the assembly as an additional constraint besides the usual constraints of the total number N of the particles and the total energy E . It is shown in the first section that the distribution law so derived is formally the same as an observer in the rest system S^0 would obtain without introducing the momentum-conservation (since $\vec{P} = 0$ in S^0): in fact, the one follows from the other by applying the Lorentz transformations for the energy of the particle, the components of its momentum and the temperature of the assembly. The second section deals with the derivation of the equations connecting the macroscopic quantities referring to the two systems of co-ordinates. It is thereby shown that it is the quantities $\left[\vec{P}, \frac{i}{c}(E + pV) \right]$, and not $\left(\vec{P}, \frac{i}{c}E \right)$, that transform like the components of a four-vector.

1. The restrictive conditions controlling the distribution with respect to the laboratory system are

$$\left. \begin{aligned} \sum_j n_j &= N \\ \sum_j n_j \epsilon_j &= E \\ \sum_j n_j \vec{p}_j &= \vec{P} \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

Then, the distribution law states that n_j , the number of particles possessing an energy $\epsilon_j = c(p_j^2 + m^2 c^2)^{1/2}$, where \vec{p}_j has the components p_x , p_y and p_z as measured in S , is given by

$$n_j = \frac{g_j}{\exp \left(-\alpha + \frac{\epsilon_j}{kT} - \gamma \cdot \vec{p}_j \right) + \beta} \quad (2)$$

where g_j is the number of wave functions or states of the particle with eigenvalue ϵ_j of the energy. β depends upon the type of statistics obeyed by the particles. γ is the Lagrange's undetermined multiplier that takes care of the conservation

of net momentum \vec{P} just as α is for the conservation of N . Evidently $\vec{\gamma}$, whose magnitude γ has the dimensions of (momentum) $^{-1}$, is a vector parallel to the only privileged direction, i.e. that of the mass-motion. We take the z -axis of our co-ordinate system in this direction so that if \vec{k} is the unit vector along the z -axis, we have

$$\vec{\gamma} = \gamma \vec{k}, \quad \vec{P} = P \vec{k} \quad \text{and} \quad \vec{\gamma} \cdot \vec{p}_z = \gamma p_z.$$

We can, therefore, put the distribution law in the form

$$n(\vec{p}) d^3p = \frac{V}{h^3} \frac{d^3p}{\exp\left(-\alpha + \frac{\epsilon}{kT} - \gamma p_z\right) + \beta}. \quad \dots \quad (3)$$

In order to determine γ as a function of the macroscopic properties of the assembly, we calculate the value of v , the velocity of mass-motion. This will obviously be equal to \bar{u}_z , the z -component of the particle velocity averaged over the whole assembly in its equilibrium state.

Now

$$u_z = \frac{\partial \epsilon}{\partial p_z} = \frac{c^2 \cdot p_z}{\epsilon}. \quad \dots \quad (4)$$

Hence

$$v = \frac{\int_{-\infty}^{\infty} \frac{d^3p}{\exp\left(-\alpha + \frac{\epsilon}{kT} - \gamma p_z\right) + \beta} \cdot \frac{c^2 p_z}{\epsilon}}{\int_{-\infty}^{\infty} \frac{d^3p}{\exp\left(-\alpha + \frac{\epsilon}{kT} - \gamma p_z\right) + \beta}}. \quad \dots \quad (5)$$

To simplify the integrals involved in this equation, we make the substitutions

$$\left. \begin{aligned} p_x &= p_x^0, \quad p_y = p_y^0, \quad p_z = \frac{p_z^0 + \frac{\gamma kT}{c^2} \cdot \epsilon^0}{\left(1 - \frac{\gamma^2 k^2 T^2}{c^2}\right)^{\frac{1}{2}}} \\ \epsilon &= \frac{\epsilon^0 + \gamma kT \cdot p_z^0}{\left(1 - \frac{\gamma^2 k^2 T^2}{c^2}\right)^{\frac{1}{2}}} \end{aligned} \right\}. \quad \dots \quad (6)$$

so that

$$c^2(p_x^2 + p_y^2 + p_z^2) - \epsilon^2 = c^2(p_x^{02} + p_y^{02} + p_z^{02}) - \epsilon^{02} = -m^2 c^4. \quad \dots \quad (7)$$

$$\begin{aligned} d^3p &= d^3p^0 \frac{\partial(p_x, p_y, p_z)}{\partial(p_x^0, p_y^0, p_z^0)} \\ &= d^3p^0 \frac{1 + \gamma kT \cdot p_z^0 / \epsilon^0}{\left(1 - \frac{\gamma^2 k^2 T^2}{c^2}\right)^{\frac{1}{2}}} \quad \dots \quad (8) \end{aligned}$$

and

$$\frac{\epsilon}{kT} - \gamma p_z = \frac{\epsilon^0}{kT} \left(1 - \frac{\gamma^2 k^2 T^2}{c^2}\right)^{\frac{1}{2}} \quad \dots \quad (9)$$

The equation (5) thus becomes

$$v = \frac{\int_{-\infty}^{\infty} \frac{d^3 p^0 \cdot (c^2 p_z^0 / \epsilon^0 + \gamma kT)}{\exp \left[-\alpha + \frac{\epsilon^0}{kT} \left(1 - \frac{\gamma^2 k^2 T^2}{c^2}\right)^{\frac{1}{2}} \right] + \beta} \cdot \left(1 - \frac{\gamma^2 k^2 T^2}{c^2}\right)^{-\frac{1}{2}}}{\int_{-\infty}^{\infty} \frac{d^3 p^0 (1 + \gamma kT \cdot p_z^0 / \epsilon^0)}{\exp \left[-\alpha + \frac{\epsilon^0}{kT} \left(1 - \frac{\gamma^2 k^2 T^2}{c^2}\right)^{\frac{1}{2}} \right] + \beta} \cdot \left(1 - \frac{\gamma^2 k^2 T^2}{c^2}\right)^{-\frac{1}{2}}} \quad \dots \quad (10)$$

Remembering the equation (7) whereby ϵ^0 is an even function of p_z^0 , the quotient on the right hand side of (10) immediately turns out to be γkT . This is, therefore, the velocity of the rest system S^0 with respect to the laboratory system S . Consequently the microscopic quantities with the superscript 0 occurring in (6) and (7) are appropriate to S^0 . We thus obtain the relation

$$\vec{\gamma} = \frac{\vec{v}}{kT} \quad \dots \quad (11)$$

Now the distribution law as stated in (2) becomes

$$n_j = \frac{g_j}{\exp \left(-\alpha + \frac{\epsilon_j - v p_z}{kT} \right) + \beta} \quad \dots \quad (12)$$

For an observer in the rest system, $v = 0$ and hence the distribution law in S^0 would be

$$n_j^0 = \frac{g_j^0}{\exp \left(-\alpha^0 + \frac{\epsilon_j^0}{kT_0} \right) + \beta} \quad \dots \quad (13)$$

The same is the distribution law arrived at by conserving N and E_0 in the rest system.

Further, in order that the laws of thermodynamics be form-invariant under a Lorentz transformation, we have for the transformation of temperature (Tolman, 1934)

$$T = T_0 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \quad \dots \quad (14)$$

This relation, coupled with the special Lorentz transformations (6), immediately gives

$$\frac{\epsilon - v p_z}{kT} = \frac{\epsilon^0}{kT_0} \quad \dots \quad (15)$$

Thus we see that the distribution function derived in S by introducing momentum-conservation is formally the same as that derived in S^0 without conserving the momentum, except for the appropriate Lorentz transformations.

2. Let us now define certain macroscopic quantities referring to S^0 . With the help of the distribution function (13), we have

$$N = \frac{V_0}{h^3} \int_{-\infty}^{\infty} \frac{d^3 p^0}{\exp\left(-\alpha^0 + \frac{\epsilon^0}{kT_0}\right) + \beta} \quad \dots \quad (16)$$

$$E_0 = \frac{V_0}{h^3} \int_{-\infty}^{\infty} \frac{d^3 p^0}{\exp\left(-\alpha^0 + \frac{\epsilon^0}{kT_0}\right) + \beta} \epsilon^0 \quad \dots \quad (17)$$

Moreover if u^0 and p^0 represent the velocity and the momentum of a particle respectively, then by the usual arguments of the kinetic theory, the pressure p_0 of the assembly will be given by (Kothari and Singh, 1942)

$$p_0 = \frac{1}{3} \int_{mc^2}^{\infty} \frac{u^0 \cdot p^0 \cdot n(\epsilon^0) d\epsilon^0}{V_0}$$

$$\therefore p_0 V_0 = \frac{V_0}{h^3} \int_{-\infty}^{\infty} \frac{c^2 p_z^2}{\epsilon^0} \frac{d^3 p^0}{\exp\left(-\alpha^0 + \frac{\epsilon^0}{kT_0}\right) + \beta} \quad \dots \quad (18)$$

Having set up the integrals for the 'proper' macroscopic quantities N , E_0 and $p_0 V_0$, we now take up the evaluation of the macroscopic quantities referring to S in terms of the 'proper' ones.

From the distribution function (12) and the restrictive conditions (1), one gets

$$N = \frac{V}{h^3} \int_{-\infty}^{\infty} \frac{d^3 p}{\exp\left[-\alpha + \frac{\epsilon - vp_z}{kT}\right] + \beta} \quad \dots \quad (19)$$

$$P = \frac{V}{h^3} \int_{-\infty}^{\infty} \frac{d^3 p}{\exp\left[-\alpha + \frac{\epsilon - vp_z}{kT}\right] + \beta} \cdot p_z \quad \dots \quad (20)$$

$$E = \frac{V}{h^3} \int_{-\infty}^{\infty} \frac{d^3 p}{\exp\left[-\alpha + \frac{\epsilon - vp_z}{kT}\right] + \beta} \cdot \epsilon \quad \dots \quad (21)$$

Rewriting the substitutions (6) and (8) with the help of (11), we have

$$\left. \begin{aligned} p_x &= p_x^0, & p_y &= p_y^0, & p_z &= \frac{p_z^0 + \frac{v}{c^2} \epsilon^0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \\ \epsilon &= \frac{\epsilon^0 + vp_z^0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \end{aligned} \right\} \quad \dots \quad (6')$$

and

$$d^3p = d^3p^0 \frac{1 + vp_z^0/\epsilon^0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \cdot \dots \dots \dots (8')$$

Also as a consequence of the Lorentz contraction

$$V = V_0 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \cdot \dots \dots \dots (22)$$

Using these relations in collaboration with (15), the equations (19), (20) and (21) become

$$N = \frac{V_0}{h^3} \int_{-\infty}^{\infty} \frac{d^3p^0 (1 + vp_z^0/\epsilon^0)}{\exp\left(-\alpha + \frac{\epsilon^0}{kT_0}\right) + \beta} \cdot \dots \dots \dots (23)$$

$$P = \frac{V_0}{h^3} \int_{-\infty}^{\infty} \frac{d^3p^0 (1 + vp_z^0/\epsilon^0)}{\exp\left(-\alpha + \frac{\epsilon^0}{kT_0}\right) + \beta} \cdot \frac{p_z^0 + \frac{v}{c^2}\epsilon^0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \cdot \dots \dots (24)$$

$$E = \frac{V_0}{h^3} \int_{-\infty}^{\infty} \frac{d^3p^0 (1 + vp_z^0/\epsilon^0)}{\exp\left(-\alpha + \frac{\epsilon^0}{kT_0}\right) + \beta} \cdot \frac{\epsilon^0 + vp_z^0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \cdot \dots \dots (25)$$

The second part in the integral (23) vanishes and the remaining integral, when compared with its equal (16), yields the result

$$\alpha = \alpha^0.$$

This means that the Lagrange's multiplier for the conservation of N is an invariant under a Lorentz transformation provided that we include the rest energy of a particle into ϵ_j . Remembering (17) and (18) along with this result, it follows from (24) and (25) that

$$P = \frac{E_0 + p_0 V_0}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \cdot v \cdot \dots \dots \dots (26)$$

$$E = \frac{E_0 + \frac{v^2}{c^2} p_0 V_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \cdot \dots \dots \dots (27)$$

Further, the transformation equation for pressure, i.e.

$$p = p_0, \quad \dots \dots \dots (28)$$

coupled with (22) and (27) gives

$$E + pV = \frac{E_0 + p_0 V_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \cdot \dots \dots \dots (29)$$

From the relations (26) and (27) it follows that the total momentum \vec{P} and the total energy E of the assembly do not transform like the components of a four-vector, rather the combination of (26) and (29) suggests that the quantities $\left[\vec{P}, \frac{i}{c}(E + pV) \right]$ form a four-vector. This is due to the fact that the assembly under consideration is not a closed system in that it consists of a thermodynamic fluid contained in a vessel under the influence of the 'external' pressure from the walls of the vessel.

In fact, the very result that $\left(P, \frac{i}{c}E \right)$ is not a four-vector may, in general, be taken as a proof that the system considered is non-closed (Möller, 1952). The transformations obtained above from the statistical considerations of the microscopic properties of the assembly are well-known transformations in relativistic thermodynamics and can also be established macroscopically by treating a perfect fluid in motion according to the principles of special relativity (Tolman, 1934, and Möller, 1952):

It may be mentioned in passing that the treatment of the text involves the following limiting cases of general interest:

- (i) $kT_0 \gg mc^2$, so that the single-particle energy spectrum may be taken to be

$$\epsilon^0 = p^0 \cdot c.$$

Evidently a moving 'Hohlraum', being an assembly of particles with zero rest-mass, suggests itself as an example of this case.

- (ii) $kT_0 \ll mc^2$, so that the energy-momentum relation becomes

$$\epsilon^0 = mc^2 + \frac{1}{2m} p^0{}^2.$$

An ideal gas for which the particle velocities in S^0 are non-relativistic is an example of this case. Further, the velocity of mass-motion may also be either relativistic or non-relativistic.

The application of the general results to these limiting cases is quite straightforward.

The Statistical Mechanics of a gaseous assembly in rotation is being studied by introducing the angular-momentum conservation and will be discussed in subsequent papers.

3. It is, indeed, a pleasure to acknowledge my deep gratitude to Prof. D. S. Kothari for his kind interest and constant encouragement and to Prof. F. C. Auluck for his valuable guidance.

SUMMARY

Linear momentum-conservation is applied to an ideal relativistic gaseous assembly in mass-motion. It is shown that the distribution function thus obtained is formally the same as one would obtain in the rest-system without conserving the momentum, except for the appropriate Lorentz transformations. The four-vector character of the quantities $\left[\vec{P}, \frac{i}{c}(E + pV) \right]$ is thereby established.

REFERENCES

- Kothari, D. S., and Singh, B. N. (1942). Thermodynamics of a relativistic Fermi-Dirac gas. *Proc. Roy. Soc. A*, 180, 414-423.
 Möller, C. (1952). *The Theory of Relativity* (Oxford: Clarendon Press).
 Tolman, R. C. (1934). *Relativity, Thermodynamics and Cosmology* (Oxford: Clarendon Press).

Note added in proof: Expressed in terms of inverse hyperbolic functions, the equations of transformation, from one inertial system to another, of the macroscopic quantities pertaining to the assembly can be shown to take an interesting form.* As is customary in the special relativity, we have two inertial systems S and S' such that the velocity of S' with respect to S is v_1 in the direction of one of the common axes, say the z -axis. Let the rest system S^0 of the gaseous assembly have a parallel velocity v_2 in S' . Its velocity v in S is then given by

$$\tanh^{-1} \frac{v}{c} = \tanh^{-1} \frac{v_1}{c} + \tanh^{-1} \frac{v_2}{c}.$$

From (26) and (29), however, one readily obtains

$$\tanh^{-1} \frac{v}{c} = \sinh^{-1} \frac{Pc}{E_0 + p_0 V_0} = \cosh^{-1} \frac{E + pV}{E_0 + p_0 V_0}.$$

We can, therefore, write

$$\sinh^{-1} \frac{Pc}{E_0 + p_0 V_0} = \sinh^{-1} \frac{P_1 c}{E_0 + p_0 V_0} + \sinh^{-1} \frac{P_2 c}{E_0 + p_0 V_0}$$

and

$$\cosh^{-1} \frac{E + pV}{E_0 + p_0 V_0} = \cosh^{-1} \frac{E_1 + p_1 V_1}{E_0 + p_0 V_0} + \cosh^{-1} \frac{E_2 + p_2 V_2}{E_0 + p_0 V_0}.$$

In these equations v , P and $(E + pV)$ are the macroscopic quantities of the assembly, as observed in S ; v_2 , P_2 and $(E_2 + p_2 V_2)$ are those observed in S' while v_1 , P_1 and $(E_1 + p_1 V_1)$ are those observed in S for an identical assembly at rest in S' , i.e. they appear by virtue of the relative motion of the two systems.

* The corresponding treatment for a single particle will be appearing in an early issue of the *American Journal of Physics*.

EFFECT OF CATALYTIC AGENTS ON THE PHASE-TRANSFORMATION OF GÖTHITE

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INTRODUCTION

Recently we (1955) have studied the complete phase-transformations of the monohydrated iron-oxide minerals, viz. Limonite and Göthite, during thermal treatments. The common faith of the mineralogists that Limonite was amorphous, was proved to be baseless by Posnjack and Merwin (1922). We (1952) have also shown that Limonite is definitely crystalline. Göthite and Limonite have the identical molecular structures but their physical appearances and properties are quite different from each other. The complete chemical analysis also gives different amounts of iron-oxide and water in different samples of Limonite and Göthite. But their molecular structures being similar, they must have the same molecular formula. But so far the earlier workers depended on the chemical analysis alone, they could not determine the exact percentage of adsorbed water in Limonite and Göthite. Moreover, the common impurities present in those minerals were SiO_2 , TiO_2 , MnO_2 , etc. It is likely that these impurities and the adsorbed water enter the pores and capillaries of those minerals thereby taking no part in the structural distribution. In order to determine the amount of adsorbed water in those minerals, we had carried out differential thermal, thermal and X-ray analysis of Limonite and Göthite. It was seen that with the rise of temperature both Limonite and Göthite lost some percentage of water, which was definitely due to the expulsion of water from the minerals. X-ray analysis showed that though Göthite and Limonite lost a certain percentage of water during heating, the original structure did not break down up to a certain temperature, which was different for Limonite and Göthite. The transition temperature for Limonite was 210°C . and for Göthite it was 250°C . The only explanation for this variation in the transition temperature may be the different amounts of adsorbed water and impurities present in the minerals and also the different particle sizes of Limonite and Göthite. But a look at Table I would show that the amount of adsorbed water and impurities in different samples of Limonite and Göthite are different.

TABLE I
The actual percentage composition of Göthite and Limonite

Samples	% of Fe_2O_3	% of impurities SiO_2	Water	
			Crystalline	Adsorbed
Göthite (Alabama) ..	86.1	..	9.7	4.0
Limonite (Alabama) ..	81.9	2.78	9.32	6.0
Limonite (Jubbulpur) ..	84.44	5.78	9.68	0.1

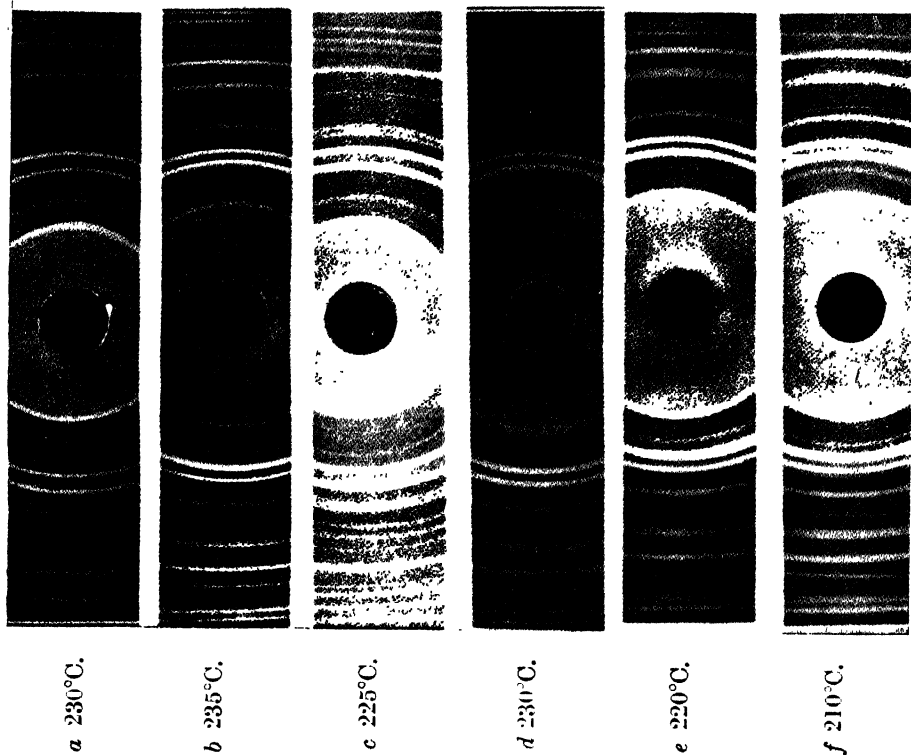


FIG. 1 (A). Powder diffraction patterns of Görlite and amorphous silica heated at different temperatures.
Silica added (a) 0.5%, (b) 0.5%, (c) 1.5%, (d) 1.5%, (e) 2.0%, (f) 2.5%.

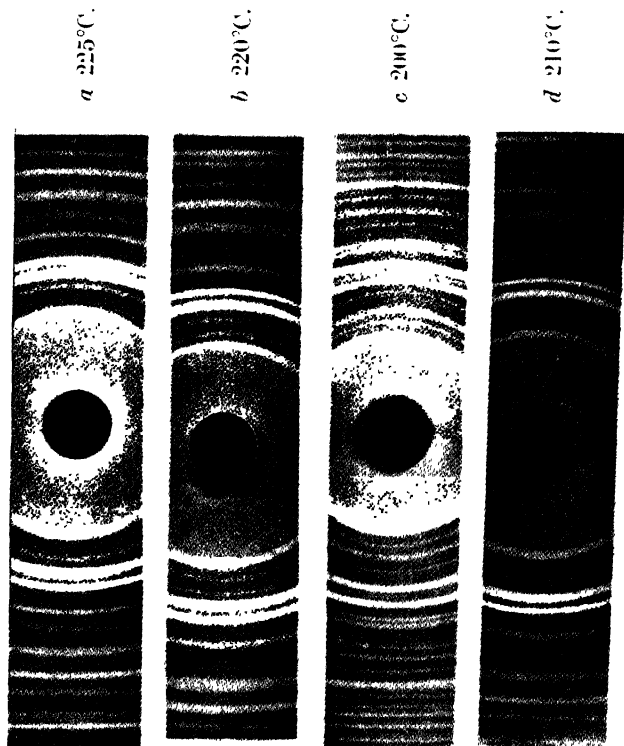


FIG. 1 (B). Powder diffraction patterns of Görlite and crystal line silica heated at different temperatures.
Silica added (a) 1.5%, (b) 2.0%, (c) 2.5%, (d) 2.5%.

Had the water (adsorbed) been responsible for the variation of transition temperature we would get, for different samples of Limonite, different transition temperature. But the transition temperature for all samples of Limonite was found to be 210°C . So we could easily discard the effect of the adsorbed water on the phase-transformation of Limonite. Moreover, the different particle sizes and amount of impurities in those minerals, may have some catalytic influence on the phase-transformation of those minerals. It was for this reason, i.e. to see the rôle played by the silica on the phase-transformation of Göthite, the present work was taken in hand.

EXPERIMENTAL

Göthite was crushed to fine powder in an agate mortar. Known weight of silica was added to a known weight of powdered Göthite. This was then heated in a furnace at a particular temperature for at least twenty-four hours. Different samples, having different percentages of silica and Göthite, were thus prepared. All these samples were then heated at various temperatures. Silica was added both in amorphous and crystalline forms. X-ray powder diffraction photographs of all these samples, having different thermal history, were taken in cylindrical camera, using Fe K -radiations from a Hadding-tube, run at a voltage of 40 kV with a tube current of 10 ma. The exposure time of each photograph was at least eight hours. The following table (Table II) and figures (Plate XV, Figs. 1 (A) and 1 (B)) will give the experimental results.

TABLE II

(A) *Effect of amorphous silica on Göthite*

% of silica added	Temperature to which the mixture was heated	Structure after thermal treatment
0.5	230°C .	Same as Göthite
0.5	235°C .	Same as Fe_2O_3
1.5	225°C .	Same as Göthite
1.5	230°C .	Same as Fe_2O_3
2.0	220°C .	Same as Fe_2O_3
2.5	210°C .	Same as Fe_2O_3

(B) *Effect of quartz on Göthite*

% of quartz added	Temperature to which the mixture was heated	Structure after thermal treatment
1.5	225°C .	Same as Fe_2O_3
2.0	220°C .	Same as Fe_2O_3
2.5	200°C .	Same as Göthite
2.5	210°C .	Same as Fe_2O_3

DISCUSSIONS

The main object of the present work was to see whether silica added to Göthite could form any chemical compound with Göthite when heated up to high temperatures or they simply behaved as a catalytic agent so as to hasten the reaction during which Göthite was being transformed into Fe_2O_3 . That silica could not form any chemical compounds with Göthite is quite evident from the fact that we could not obtain any

diffraction patterns which were quite different either from that of Göthite or of Fe_2O_3 . Had it formed any definite chemical compound we could have found a different diffraction pattern.

Moreover, it has been stated earlier that the transition temperature for Göthite is 250°C . But we, in the present investigation, have found that this transition can be made to take place at lower temperatures by gradually increasing the amount of silica added. It is known that the presence of any catalytic agent can cause a change in the transition phenomenon. It is likely that in this case the presence of silica has a catalytic influence on the phase-transformation of Göthite. But in all the pictures we have not found any line or band due to the presence of silica in the mixture with Göthite. The cause for this was that the percentage of silica added was too small to give any line or band in the diffraction pattern.

Actually it was found that with the increment of the amount of silica the transition temperature changes. With 0.5% of silica the transition temperature was found to be shifted from 250°C . to 235°C . (Table III). With 1.5% of silica it was 230°C ., when the transition temperature came down to 210°C . at a concentration of 2.5% of silica in the mixture. By increasing the amount of silica, it was found not possible to lower down the transition temperature below 210°C . Similar was the case when amorphous silica was replaced by crystalline quartz. Thus it was seen that the saturation limit to cause any change in the transition temperature of Göthite was reached when the presence of silica in the mixture was 2.5%.

TABLE III

Bragg angles and spacings of the diffraction pattern of the mixture of Göthite and silica (0.5%) heated to 235°C .

Bragg angle	Intensity	Spacings	Spacings of Fe_2O_3 by Hanwalt, Rinn and Fravel
$21^\circ 5'$	s	2.698	2.69
$22^\circ 43'$	s	2.512	2.51
$26^\circ 5'$	m	2.207	2.20
$30^\circ 41'$	w	1.847	1.84
$34^\circ 55'$	m	1.695	1.69
$40^\circ 55'$	m	1.452	1.452

The spacings and Bragg angles of all the other patterns of the mixture of silica and Göthite (heat treated) where Göthite is said to be changed are similar to the above.

From this study, we can also account for the different type of transformation in Göthite and Limonite, both having the similar molecular structure. Chemical analysis of Limonite (Table I) shows that in the Jubbulpur sample, the amount of silica present was 5.78% and 2.78% in the Alabama sample. It has been already seen that the presence of 2.5% of silica in Göthite was sufficient to change the transition temperature to 210°C . Thus it is seen that the amount of silica present in Limonite is quite greater than the limiting amount of silica necessary to transform Göthite to Fe_2O_3 at 210°C . It is also likely that the amount of silica present in Limonite is responsible for its different physical properties and appearances from those of Göthite, though Limonite has the same structure as that of Göthite.

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ABSTRACT

During the phase-transformation study of monohydrated iron-oxide minerals, Limonite and Göthite, on thermal treatments, it was found that though the two minerals have identical structures, the transition temperatures for Limonite and Göthite are quite different. The cause for this difference was attributed to the different amounts of impurities present in the minerals. In the present investigation the impurities, which were present in Limonite, were added to powdered Göthite by different amounts and their effects on the phase-transformation of Göthite during thermal treatments were studied by X-ray diffraction method. It was found that, with the increasing amount of the impurities added, the transition temperature was also lowered. But up to a certain amount of impurities added, the transition temperature came down as low as 210°C. from 250°C. in the case of Göthite. After that further addition of impurities did not cause any change in the transition temperature. From this study the cause for Limonite having a lower transition temperature was also well established.

REFERENCES

- Dasgupta, D. R. (1952). X-ray diffraction study of Limonite. *Acta. Cryst.*, 5, 85.
——— (1955). X-ray and physico-chemical studies of the monohydrated iron-oxide minerals—Limonite and Göthite of different geological origin. *Proc. Nat. Inst. Sci. Ind.* (communicated).
Hanwalt, J. D., Rinn, H. W., and Fravel, L. K. (1933). Chemical analysis by X-ray diffraction. *Jour. Indust. Eng. Chem.*, 10, 506.
Posnjak, E., and Merwin, H. E. (1922). The System, $\text{Fe}_2\text{O}_3\text{—SO}_3\text{—H}_2\text{O}$. *Jour. Amer. Chem. Soc.*, 44, 1771.

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ON BALLISTICS OF COMPOSITE CHARGES FOR POWER LAW OF BURNING TAKING ACCOUNT OF DIFFERENT GAMMAS

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1. INTRODUCTION

The theory of composite charge for linear rate of burning has been discussed by various authors. Corner (1950), Hunt, Hinds and Clemmow (1951) dealt with the problem of composite charge for linear rate of burning by reducing the problem to a single equivalent charge with adjusted propellant parameters. Venkatesan and Patni (1953) gave a direct treatment to the problem on the assumptions that (i) the ratio of the two specific heats for the two propellants is equal, i.e. $\gamma_1 = \gamma_2 = \gamma$, (ii) the covolume of gases equals the specific volume of each propellant and (iii) a linear rate of burning has been assumed. This was extended by the author (S. P. Aggarwal) of this paper for different values of gammas, i.e. relaxing the first assumption. Further, the author and A. K. Mehta (1955) gave a direct treatment to the problem of composite charge of two tubular propellants for power law of burning under the following assumptions:

- (a) $\gamma_1 = \gamma_2 = \gamma$ for the two propellants,
- (b) index of rate of burning of the two propellants is the same.

In the present paper the author has dealt with the problem under less restricted conditions, i.e., the condition that the ratio of the two specific heats for the two propellants is equal has been relaxed, and has assumed different values of gammas for the two propellants, γ_1 and γ_2 respectively. The index of rate of burning of the two propellants has been assumed to be the same.

2. FUNDAMENTAL EQUATIONS

The fundamental equations of internal ballistics for composite charge taking different values of γ have been deduced by the author and are as follows:—

(i) Energy equation is

$$\frac{F_1 C_1 Z_1}{\gamma_1 - 1} + \frac{F_2 C_2 Z_2}{\gamma_2 - 1} = \frac{AP(l+x) \left(\frac{n_1 C_1 Z_1}{\gamma_1 - 1} + \frac{n_2 C_2 Z_2}{\gamma_2 - 1} \right)}{(n_1 C_1 Z_1 + n_2 C_2 Z_2)} + \frac{1}{2} w_1 v^2 \quad \dots (1)$$

which can be written as

$$F_1 C_1 Z_1 + k F_2 C_2 Z_2 = \frac{Ap(l+x)(n_1 C_1 Z_1 + k n_2 C_2 Z_2)}{(n_1 C_1 Z_1 + n_2 C_2 Z_2)} + \frac{\gamma_1 - 1}{2} w_1 v^2 \quad \dots (1a)$$

where n_1 and n_2 are the number of gram moles per gram of the two propellant gases, also $k = \frac{\gamma_1 - 1}{\gamma_2 - 1}$, and $w_1 = 1.05W + \frac{1}{2}(C_1 + C_2)$.

(ii) Dynamical equation is

$$w_1 \frac{dv}{dt} = Ap \quad \dots \quad (2)$$

(iii) Form functions are given by

$$Z_1 = (1-f_1)(1+\theta_1 f_1) \quad \dots \quad (3a)$$

$$Z_2 = (1-f_2)(1+\theta_2 f_2) \quad \dots \quad (3b)$$

(iv) Rate of burning equations are

$$D_1 \frac{df_1}{dt} = -\beta_1 p^\alpha \quad \dots \quad (4a)$$

$$D_2 \frac{df_2}{dt} = -\beta_2 p^\alpha \quad \dots \quad (4b)$$

where $C_1, F_1, \beta_1, D_1, \theta_1, f_1, Z_1, n_1, \gamma_1$ and δ_1 refer to the first charge and $C_2, F_2, \beta_2, D_2, \theta_2, f_2, Z_2, n_2, \gamma_2$ and δ_2 refer to the second charge.

We make the following dimensionless transformations

$$\xi = 1 + \frac{x}{l}$$

$$\eta_1 = \frac{vAD_1}{F_1 C_1 \beta_1} \left(\frac{F_1 C_1}{Al} \right)^{1-\alpha}$$

$$\eta_2 = \frac{vAD_2}{F_2 C_2 \beta_2} \left(\frac{F_2 C_2}{Al} \right)^{1-\alpha}$$

$$\zeta_1 = p \frac{Al}{F_1 C_1}$$

$$\zeta_2 = p \frac{Al}{F_2 C_2}$$

$$M_1 = \frac{A^2 D_1^2}{F_1 C_1 \beta_1^2 w_1} \left(\frac{F_1 C_1}{Al} \right)^{2-2\alpha}$$

$$\text{and } M_2 = \frac{A^2 D_2^2}{F_2 C_2 \beta_2^2 w_1} \left(\frac{F_2 C_2}{Al} \right)^{2-2\alpha}$$

and the equations (1) to (4) reduce to

$$Z_1 + \frac{k F_2 C_2}{F_1 C_1} Z_2 = \zeta_1 \xi \frac{(n_1 C_1 Z_1 + k n_2 C_2 Z_2)}{(n_1 C_1 Z_1 + n_2 C_2 Z_2)} + \frac{\gamma_1 - 1}{2M_1} \eta_1^2 \quad \dots \quad (5)$$

$$\eta_1 \frac{d\eta_1}{d\xi} = M_1 \zeta_1 \quad \dots \quad (6a)$$

$$\eta_2 \frac{d\eta_2}{d\xi} = M_2 \zeta_2 \quad \dots \quad (6b)$$

$$Z_1 = (1-f_1)(1+\theta_1 f_1) \quad \dots \quad (7a)$$

$$Z_2 = (1-f_2)(1+\theta_2 f_2) \quad \dots \quad (7b)$$

$$\eta_1 \frac{df_1}{d\xi} = -\zeta_1^\alpha \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (8a)$$

$$\text{and } \eta_2 \frac{df_2}{d\xi} = -\zeta_2^\alpha \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (8b)$$

3. SOLUTION OF EQUATIONS

Dividing equations (6) by (8) we get

$$\frac{d\eta_1}{df_1} = -M_1 \zeta_1^{1-\alpha} \quad \dots \quad \dots \quad \dots \quad \dots \quad (9a)$$

$$\frac{d\eta_2}{df_2} = -M_2 \zeta_2^{1-\alpha} \quad \dots \quad \dots \quad \dots \quad \dots \quad (9b)$$

Equations (8) can be written as

$$\eta_1 = -\zeta_1^\alpha \frac{d\xi}{df_1} \quad \dots \quad \dots \quad \dots \quad \dots \quad (10a)$$

$$\eta_2 = -\zeta_2^\alpha \frac{d\xi}{df_2} \quad \dots \quad \dots \quad \dots \quad \dots \quad (10b)$$

Substituting the values of η_1 and η_2 from (10) in equations (9) we get

$$\frac{d}{df_1} \left(\zeta_1^\alpha \frac{d\xi}{df_1} \right) = M_1 \zeta_1^{1-\alpha} \quad \dots \quad \dots \quad \dots \quad (11a)$$

$$\frac{d}{df_2} \left(\zeta_2^\alpha \frac{d\xi}{df_2} \right) = M_2 \zeta_2^{1-\alpha} \quad \dots \quad \dots \quad \dots \quad (11b)$$

4. CONSTANT BURNING SURFACE WITH THE COVOLUME COEFFICIENT NEGLECTED

The energy equation can be written as

$$Z_1 + RZ_2 = \zeta_1 \xi \left(\frac{n_1 C_1 Z_1 + k n_2 C_2 Z_2}{n_1 C_1 Z_1 + n_2 C_2 Z_2} \right) + \frac{\gamma_1 - 1}{2M_1} \eta_1^2 \quad \dots \quad \dots \quad (12)$$

where

$$R = \frac{kF_2 C_2}{F_1 C_1}.$$

For constant burning surface we have

$$\theta_1 = 0$$

and

$$\theta_2 = 0.$$

Therefore form factors reduce to

$$Z_1 = (1 - f_1) \quad \dots \quad \dots \quad \dots \quad \dots \quad (13a)$$

$$Z_2 = (1 - f_2) \quad \dots \quad \dots \quad \dots \quad \dots \quad (13b)$$

Differentiating these equations (13) we get

$$dZ_1 = -df_1 \quad \dots \quad \dots \quad \dots \quad \dots \quad (14a)$$

$$dZ_2 = -df_2 \quad \dots \quad \dots \quad \dots \quad \dots \quad (14b)$$

So that equations (11) become

$$\frac{d}{dZ_1} \left(\zeta_1^\alpha \frac{d\xi}{dZ_1} \right) = M_1 \zeta_1^{1-\alpha} \quad \dots \quad (15a)$$

$$\frac{d}{dZ_2} \left(\zeta_2^\alpha \frac{d\xi}{dZ_2} \right) = M_2 \zeta_2^{1-\alpha} \quad \dots \quad (15b)$$

To find the relation between Z_1 and Z_2 , we get on dividing (4a) by (4b)

$$\frac{df_1}{df_2} = \frac{D_2 \beta_1}{\beta_2 D_1} = \frac{1}{T} \text{ (say)} \quad \dots \quad (16)$$

Substituting the values of df_1 and df_2 from equations (14) we get

$$\frac{dZ_1}{dZ_2} = \frac{1}{T} \quad \dots \quad (17)$$

or
$$dZ_2 = T dZ_1 \quad \dots \quad (18)$$

Integrating this and applying initial conditions we get

$$Z_2 = T Z_1 \quad \dots \quad (19)$$

With the help of equation (19) energy equation (12) becomes

$$Z_1(1+R) = \zeta_1 \xi_1 \left(\frac{n_1 C_1 + T k n_2 C_2}{n_1 C_1 + T n_2 C_2} \right) + \frac{\gamma_1 - 1}{2M_1} \eta_1^2 \quad \dots \quad (20)$$

Differentiating both sides of this equation we get

$$dZ_1 (1+R) = Q_1 (d\zeta_1 \xi + \zeta_1 d\xi) + \frac{\gamma_1 - 1}{M_1} \eta_1 d\eta_1,$$

where

$$Q_1 = \frac{n_1 C_1 + T k n_2 C_2}{n_1 C_1 + T n_2 C_2}$$

With the help of equation (6a) the above equation reduces to

$$Q dZ_1 = Q_1 \left[d\zeta_1 \xi + \frac{\gamma_1 - 1 + Q_1}{Q_1} \zeta_1 d\xi \right], \text{ where } Q = (1+R)$$

or

$$Q dZ_1 = Q_1 [d\zeta_1 \xi + \bar{\gamma} \zeta_1 d\xi]$$

where

$$\bar{\gamma} = \frac{\gamma_1 - 1 + Q_1}{Q_1}$$

This further becomes as

$$Q_2 dZ_1 = \xi^{1-\bar{\gamma}} d(\zeta_1 \xi^{\bar{\gamma}}), \text{ where } Q_2 = \frac{Q}{Q_1} \quad \dots \quad (21)$$

This is the energy equation in the final form and which is of the same form as deduced in the previous paper (Aggarwal and Mehta).

Here

$$Q_2 = \frac{\left(1 + k \frac{F_2 C_2}{F_1 C_1} \right) (n_1 C_1 + T n_2 C_2)}{(n_1 C_1 + T k n_2 C_2)}$$

$$\text{and } \bar{\gamma} = \frac{\left[\gamma_1 - 1 + \frac{n_1 C_1 + T k n_2 C_2}{n_1 C_1 + T n_2 C_2} \right] (n_1 C_1 + T n_2 C_2)}{(n_1 C_1 + T k n_2 C_2)}$$

Now we have to proceed in the same way as in the previous paper with modified values of Q and γ , as Q_2 and $\bar{\gamma}$. Making the substitutions,

$$\xi = \left(\frac{X}{Q_2} \right)^m \text{ and } \zeta_1 \xi^{\bar{\gamma}} = \left(\frac{Y}{M_1} \right)^n \quad \dots \quad (22)$$

where we have to choose proper values of m and n , we get

$$\zeta_1 = \left(\frac{Y}{M_1} \right)^n \left(\frac{X}{Q_2} \right)^{-\bar{\gamma}m} \quad \dots \quad (23)$$

$$\text{and } dZ_1 = n \left(\frac{X}{Q_2} \right)^{(1-\bar{\gamma})m} \cdot Y^{n-1} \cdot M_1^{-n} dY \quad \dots \quad (23a)$$

Substituting these values in (15a) and taking $n = \frac{1}{3-2\alpha}$, $m = \frac{2n}{\bar{\gamma}-n}$ we obtain

$$XY \frac{d^2 X}{dY^2} - \frac{(\bar{\gamma}-1)n}{(\bar{\gamma}-n)} \left(\frac{dX}{dY} \right)^2 + \frac{1}{2}(1+n)X \frac{dX}{dY} = \frac{1}{2}n(\bar{\gamma}-n) \quad \dots \quad (24)$$

Further making the substitution, $Y = \frac{1+n}{n(\bar{\gamma}-n)} Z$, the above equation reduces to

$$\frac{2XZ}{1+n} \frac{d^2 X}{dZ^2} - \frac{2n(\bar{\gamma}-1)}{(1+n)(\bar{\gamma}-n)} \left(\frac{dX}{dZ} \right)^2 + X \frac{dX}{dZ} = 1 \quad \dots \quad (25)$$

This is exactly the same equation as deduced earlier. Hence the problem of composite charge for different gammas can now be tackled very simply without introducing any difficulty.

We can solve equation (25) by numerical methods such as Runge Kutta. Now we consider two cases.

Case I. Finite shot-start pressure.

$$\text{For this case, } X_0 = Q_2 = \frac{\left(1 + k \frac{F_2 C_2}{F_1 C_1} \right) (n_1 C_1 + T n_2 C_2)}{(n_1 C_1 + T k n_2 C_2)}$$

$$Z_0 = \frac{n(\bar{\gamma}-n)M_1}{(1-n)} \zeta_0^{1/n}$$

and the velocity is given as

$$\eta_1 = 2 \left[\frac{X \left(1 + k \frac{F_2 C_2}{F_1 C_1} \right) (n_1 C_1 + T n_2 C_2)}{(n_1 C_1 + T k n_2 C_2)} \right]^{-\frac{1}{2}n(\bar{\gamma}-1)} \left[\frac{Z}{\bar{\gamma}-n} \right]^{\frac{1}{2}(1+n)} \left[\frac{nM}{1+n} \right]^{\frac{1}{2}(1-n)} \frac{dX}{dZ} \quad (26)$$

So that $\frac{dX}{dZ}$ is zero initially.

Case II. Zero shot-start pressure.

For this case $X_0 = Q_2$ and $Z_0 = 0$.

$$\text{The initial value of } \frac{dX}{dZ} = \frac{1}{Q_2} = \frac{(n_1 C_1 + T k n_2 C_2)}{\left(1 + k \frac{F_2 C_2}{F_1 C_1}\right) (n_1 C_1 + T n_2 C_2)}$$

Series solution for equation (25) for zero shot-start pressure is

$$X = Q_2 + \frac{Z}{Q_2} - \frac{(1-n)(\bar{\gamma}+n)Z^2}{2(3+n)(\bar{\gamma}-n)Q_2^3} \cdot \left[1 - \frac{\{(1-n)(7+3n) + (\bar{\gamma}-1)(7-5n)\}Z}{3(5+n)(\bar{\gamma}-n)Q_2^2} \right] + \quad (27)$$

5. CONDITIONS FOR SIMULTANEOUS AND NON-SIMULTANEOUS BURNING OF THE TWO CHARGES

From (19) condition for simultaneous burning is $T = 1$ and for non-simultaneous burning $T < 1$ or > 1 according as charge C_1 burns out first or charge C_2 burns out first.

Simultaneous burning of the two charges.

All the above equations from (1) to (27) are true for $T = 1$. In this case maximum pressure is given by

$$p_{\max.} = \left[\frac{F_1^2 C_1^2 \beta_1^2 w_1}{A^3 D_1^2 l} \right]^n \left[\frac{(1+n)Z_{\max}}{n(\bar{\gamma}-n)} \right]^n \left[\frac{\left(1 + k \frac{F_2 C_2}{F_1 C_1}\right) (n_1 C_1 + T n_2 C_2)}{(n_1 C_1 + T k n_2 C_2)} \right]^{\bar{\gamma} m} X_{\max.}^{-\bar{\gamma} m} \quad (28)$$

where X_{\max} , Z_{\max} , is the solution of

$$\frac{dX}{dZ} = \frac{nX}{\bar{\gamma} m Z} = \frac{\bar{\gamma} - n}{2\bar{\gamma}} \frac{X}{Z}$$

which is obtained by differentiating equation (23a).

The shot-travel at all-burnt is

$$x_b = l \left[\left\{ \frac{Xb}{Q_2} \right\}^m - 1 \right] \quad \dots \quad \dots \quad \dots \quad (29)$$

The velocity as obtained in the previous paper is given by

$$v = \frac{F_1 C_1 \beta_1}{A D_1} \left[\frac{A^3 l D_1^2}{F_1^2 C_1^2 \beta_1^2 w_1} \right]^{\frac{1}{2}(1-n)} \left[\frac{\left(1 + k \frac{F_2 C_2}{F_1 C_1}\right) (n_1 C_1 + T n_2 C_2)}{(n_1 C_1 + T k n_2 C_2)} \right]^{\frac{1}{2}m(\bar{\gamma}-1)} \cdot V(Z). \quad (30)$$

where

$$V(Z) = \frac{2}{X^{\frac{1}{2}m(\bar{\gamma}-1)}} \left[\frac{Z}{\bar{\gamma}-n} \right]^{\frac{1}{2}(1+n)} \left[\frac{n}{1+n} \right]^{\frac{1}{2}(1-n)} \frac{dX}{dZ}$$

Muzzle velocity is given by

$$\frac{1}{2}(\bar{\gamma}-1) \frac{\eta_{1,3}^2}{M_1} = 1 - \left[\frac{(1+n)Z_3}{n(\bar{\gamma}-n)M_1} \right]^n \xi_3^{1-\bar{\gamma}} \quad \dots \quad \dots \quad (31)$$

Non-simultaneous burning of the two charges.

For the sake of definiteness suppose charge C_1 burns out first, the condition for which is $T < 1$.

For this case, i.e. when charge C_1 is burnt out, the equations (1) to (4) reduce to

$$\frac{F_1 C_1}{\gamma_1 - 1} + \frac{F_2 C_2 Z_2}{\gamma_2 - 1} = \frac{Ap(l+x) \left(\frac{n_1 C_1}{\gamma_1 - 1} + \frac{n_2 C_2 Z_2}{\gamma_2 - 1} \right)}{(n_1 C_1 + n_2 C_2 Z_2)} + \frac{1}{2} w_1 v^2 \quad \dots (32)$$

$$w_1 \frac{dv}{dt} = Ap, \quad \dots \dots \dots (33)$$

$$Z_2 = (1 - f_2) \quad \dots \dots \dots (34)$$

$$D_2 \frac{df_2}{dt} = -\beta_2 p^\alpha \quad \dots \dots \dots (35)$$

These become with usual transformations as

$$\frac{F_1 C_1}{k F_2 C_2} + Z_2 = \frac{\zeta_1 \xi \left(\frac{n_1 C_1}{k} + n_2 C_2 Z_2 \right)}{(n_1 C_1 + n_2 C_2 Z_2)} + \frac{\gamma_2 - 1}{2M_2} \eta_2^2 \quad \dots \dots (36)$$

$$\eta_2 \frac{d\eta_2}{d\xi} = M_2 \zeta_2 \quad \dots \dots \dots (37)$$

$$Z_2 = (1 - f_2) \quad \dots \dots \dots (38)$$

$$\eta_2 \frac{df_2}{d\xi} = -\zeta_2^\alpha \quad \dots \dots \dots (39)$$

Energy equation can be written as

$$\frac{F_1 C_1}{k F_2 C_2} + Z_2 = k' \zeta_1 \xi + \frac{\gamma_2 - 1}{2M_2} \eta_2^2$$

where k' is the mean value of $\left[\frac{\frac{n_1 C_1}{k} + n_2 C_2 Z_2}{n_1 C_1 + n_2 C_2 Z_2} \right]$ between the points where C_1 is burnt out and the all-burnt position. Differentiating both sides of this equation and using (37) we get

$$dZ_2 = k' \left[d\zeta_2 \xi + \frac{\gamma_1 - 1 + k'}{k'} \zeta_1 d\xi \right]$$

$$\text{or} \quad \frac{1}{k'} dZ_2 = \xi^{1-\gamma'} d(\zeta_2 \xi^{\gamma'}) \quad \dots \dots \dots (40)$$

$$\text{where} \quad \gamma' = \frac{\gamma_1 - 1 + k'}{k'}$$

Making the substitutions

$$\xi = (k' X)^{m'} \quad \text{and} \quad \zeta_2 \xi^{\gamma'} = \left(\frac{Y}{M_2} \right)^{n'}$$

we get the usual equation

$$\frac{2XZ}{1+n'} \frac{d^2X}{dZ^2} - \frac{2n'(\gamma'-1)}{(1+n')(\gamma'-n')} \left(\frac{dX}{dZ} \right)^2 + X \frac{dX}{dZ} = 1 \quad \dots \quad (41)$$

where

$$m' = \frac{2n'}{\gamma'-n'} \text{ and } n' = \frac{1}{3-2\alpha}$$

also

$$Y = \frac{1+n'}{n'(\gamma'-n')} Z.$$

Now we have to solve this equation (41), numerically, as done before, with the initial conditions obtained at the point where C_1 is burnt out.

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SUMMARY

This paper is an extension of the previous paper, 'On Ballistics of Composite Charges for Power Law of Burning', written by the author and A. K. Mehta. In this the author deals with the problem of internal ballistics for composite charges for power law of burning by considering different values of gammas, γ_1 and γ_2 , for the two propellants respectively.

REFERENCES

- Aggarwal, S. P., and Mehta, A. K. (1955). On ballistics of composite charges for power law of burning. *PNISIPS*, 21, 55.
 Aggarwal, S. P. (1955). Internal ballistics for composite charge. *PNISIPS*, 21, 321.
 Corner, J. (1950). Theory of internal ballistics of guns. John Wiley, New York.
 Hunt, Hinds (1951). H.M.S.O.; Internal Ballistics.
 Venkatesan, N. S., and Patni, G. C. (1953). Influence of composite charge on maximum pressure and muzzle velocity. *Def. Sci. Jour.*, 3, 51.

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INTERNAL BALLISTICS OF H/L GUN USING PROPELLANT OF ANY SHAPE

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1. INTRODUCTION

In World War II the German engineers made a notable contribution in the field of low pressure guns by closing the front of the cartridge case with plate having one or many nozzles usually in the form of holes. Suitably choosing the nozzle area the chamber pressure was kept considerably high while the projectile suffered a low pressure in the bore. Thus the ignition properties and regularities were improved without increasing the barrel length. The reduction of the volume of the cartridge case, thus achieved, was a decided advantage.

A simple theory of the internal ballistics of the high and low pressure gun for tubular ($\theta = 0$) propellant has been given by Corner. In this communication the author has presented a generalization of Corner's theory for a propellant of any shape, i.e., for any value of θ .

2. NOTATIONS AND ASSUMPTIONS

Fig. 1 illustrates an idealized model of the gun.

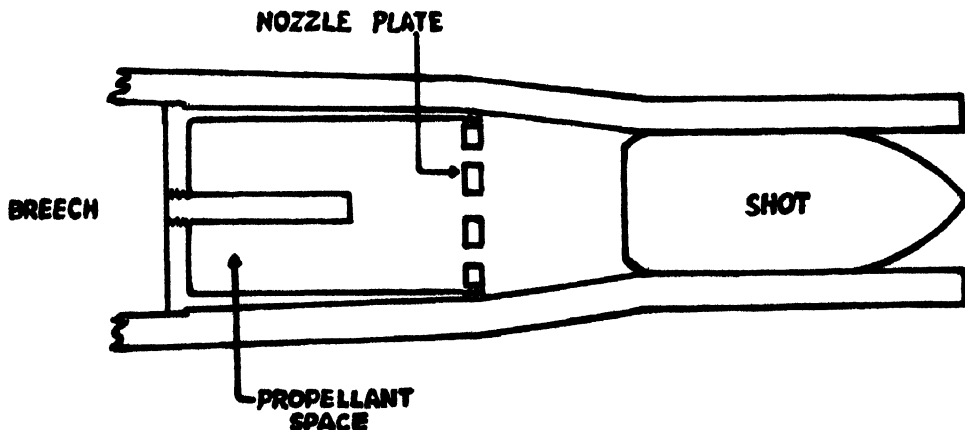


FIG. 1.

W = the projectile weight.

C = the charge weight.

K = the volume of the first chamber containing the charge.

K_0 = the volume of the second chamber.

S = the area of the venturi or nozzles connecting the two chambers.

A = the bore area.

P = the pressure in the main chamber taken to be uniform.

p = the pressure in the second chamber and the bore.

v = the shot velocity.

x = the shot travel.

CZ = amount of propellant burnt up to time t .

CN = amount of gas in the first chamber at the instant t .

In this treatment the usual isothermal assumption has been made and the mean force constant is denoted by λ . The approximation is valid in both the chambers and bore and we may neglect small regions near the venturi where the approximation is not valid.

The initial resistance to the motion of the shot can be accounted by adjusting the rate of burning constant β_1 . The pressures at inlet and exit of the nozzle are P and p as near as matters.

If $p/P < \left(\frac{2}{\gamma+1}\right)^{\gamma-1}$ then the rate of flow is determined by P alone. For $\gamma = 1.25$ the condition reduces to

$$p/P < 0.555,$$

then the rate of flow is given by $\frac{\psi SP}{\sqrt{\lambda}}$ where ψ is the numerical factor which is a function of γ but lies within 1% of 0.66 for all service propellants. A reasonable correction for friction and heat losses in the nozzle make ψ about 0.63.

If p/P is greater than the limit mentioned above, the rate of flow for $\gamma = 1.25$ is given by

$$\left(\frac{\chi SP}{\sqrt{\lambda}}\right) \left(\frac{p}{P}\right)^{0.8} \left[1 - \left(\frac{p}{P}\right)^{0.2}\right]^{\frac{1}{2}}$$

where $\chi = 3.162$ when heat and friction losses are not accounted. If this is done $\chi = 3.00$.

Further two valid simplifying assumptions have been made,

(i) $p/P < 0.555$

throughout, and

(ii) no unburnt cordite passes the nozzles.

3. BASIC EQUATIONS UP TO ALL-BURNT

The equation of state for the gas in the first chamber is

$$P \left[K - \frac{C(1-Z)}{\delta} - CNb \right] = CN\lambda \quad \dots \quad (1)$$

and for the gas in the second chamber is

$$p [K_0 + Ax - C(Z-N)b] = C\lambda(Z-N) \quad \dots \quad (2)$$

Neglecting the conventional Lagrange's correction, which is indeed very small, the dynamical equation becomes

$$w_1 \frac{dv}{dt} = Ap \quad \dots \quad (3)$$

where $w_1 = W_1 + \frac{1}{2}C$, W_1 being the effective mass of the shot, allowing for friction.

The rate of burning is given by

$$D \frac{df}{dt} = -\beta P \quad \dots \quad (4)$$

The nozzle flow equation is

$$C \frac{dN}{dt} = C \frac{dZ}{dt} - \frac{\psi SP}{C\sqrt{\lambda}} \quad \dots \quad (5)$$

The form function is given by

$$Z = (1-f)(1+\theta f) \quad \dots \quad (6)$$

4. SOLUTION OF THE EQUATIONS

Equation (6) gives

$$f = \frac{(\theta-1) \pm \sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} \quad \dots \quad (7)$$

Now f being a positive fraction (fraction of web remaining) we have only the positive sign.

Therefore

$$\frac{df}{dt} = - \frac{1}{\sqrt{(\theta+1)^2 - 4\theta Z}} \frac{dZ}{dt} \quad \dots \quad (8)$$

From (4), (5) and (8) we get

$$\frac{dN}{dt} = \frac{dZ}{dt} - \frac{\psi SD}{\beta C \sqrt{\lambda}} \frac{1}{\sqrt{(\theta+1)^2 - 4\theta Z}} \frac{dZ}{dt} \quad \dots \quad (9)$$

Integrating this equation we get

$$N = Z + \frac{\psi SD}{\beta C \sqrt{\lambda}} \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} + \text{constant.}$$

By initial condition, $Z = 0, N = 0$, we get

$$N = Z + \frac{\psi SD}{\beta C \sqrt{\lambda}} \left\{ \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{(\theta+1)}{2\theta} \right\} \quad \dots \quad (10)$$

$$: Z + \Psi \left\{ \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{(\theta+1)}{2\theta} \right\}$$

where $\Psi = \frac{\psi SD}{\beta C \sqrt{\lambda}}$.

The dimensionless parameter Ψ plays an important part in the ballistics of H/L guns as it does in that of RCL guns.

Substituting the value of $(Z-N)$ from (10) into (1) we get

$$C\lambda \left[Z + \Psi \left\{ \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{\theta+1}{2\theta} \right\} \right]$$

$$= P \left[K - \frac{C(1-Z)}{\delta} - cb \left\{ Z + \Psi \left(\frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{\theta+1}{2\theta} \right) \right\} \right] \quad \dots \quad (11)$$

$$\text{or } P = \frac{C\lambda \left[Z + \Psi \left\{ \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{\theta+1}{2\theta} \right\} \right]}{\left[K - \frac{C(1-Z)}{\delta} - Cb \left\{ Z + \Psi \left(\frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{\theta+1}{2\theta} \right) \right\} \right]} \quad \dots (12)$$

From (4), (8) and (12) we get

$$\frac{1}{\sqrt{(\theta+1)^2 - 4\theta Z}} \frac{dZ}{dt} = \frac{\beta}{D} \frac{C\lambda \left[Z + \Psi \left\{ \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{\theta+1}{2\theta} \right\} \right]}{\left[K - \frac{C(1-Z)}{\delta} - Cb \left\{ Z + \Psi \left(\frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{\theta+1}{2\theta} \right) \right\} \right]} \quad \dots (13)$$

$$\text{Put } Z' = \frac{\theta+1}{2\theta} - \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} \quad \dots (14)$$

$$\text{Therefore } \frac{dZ'}{dt} = \frac{1}{\sqrt{(\theta+1)^2 - 4\theta Z}} \frac{dZ}{dt}.$$

From (14) we get

$$\left[Z' - \frac{\theta+1}{2\theta} \right]^2 = \frac{(\theta+1)^2}{4\theta^2} - \frac{4\theta Z}{4\theta^2}.$$

On simplification this gives

$$Z = -\theta Z'^2 + (\theta+1)Z'$$

Substituting these values in (13) we get

$$\frac{dZ'}{dt} = \frac{C\beta\lambda}{D} \frac{[-\theta Z'^2 + (\theta+1)Z' - \Psi Z']}{\left[\left(K - \frac{C}{\delta} \right) + C \left(\frac{1}{\delta} - b \right) \left\{ -\theta Z'^2 + (\theta+1)Z' \right\} + Cb\Psi Z' \right]}$$

Separating the two variables we get

$$\frac{\left[\left(K - \frac{C}{\delta} \right) + C \left(\frac{1}{\delta} - b \right) \left\{ -\theta Z'^2 + (\theta+1)Z' \right\} + Cb\Psi Z' \right] dZ'}{[-\theta Z'^2 + (\theta+1-\Psi)Z']} = \frac{C\beta\lambda}{D} dt$$

or

$$\frac{\left[\left(K - \frac{C}{\delta} \right) + C \left(\frac{1}{\delta} - b \right) \left\{ -\theta Z'^2 + (\theta+1-\Psi)Z' \right\} + \Psi C \left(\frac{1}{\delta} - b \right) Z' + Cb\Psi Z' \right] dZ'}{[-\theta Z'^2 + (\theta+1-\Psi)Z']} = \frac{C\beta\lambda}{D} dt$$

This becomes, after simplification and putting into integrable form,

$$C \left(\frac{1}{\delta} - b \right) dZ' - \frac{\Psi C}{2\theta\delta} \frac{\{-2\theta Z' + (\theta+1-\Psi)\} dZ'}{[-\theta Z'^2 + (\theta+1-\Psi)Z']} + \frac{1}{(\theta+1-\Psi)} \left\{ \frac{\Psi C(\theta+1-\Psi)}{2\theta\delta} + \left(K - \frac{C}{\delta} \right) \right\} \left[\frac{1}{Z'} + \frac{\theta}{-\theta Z' + (\theta+1-\Psi)} \right] dZ' = \frac{C\beta\lambda}{D} dt$$

Integrating this equation we get

$$C \left(\frac{1}{\delta} - b \right) Z' - \frac{\Psi C}{2\theta\delta} \log \left[-\theta Z'^2 + (\theta + 1 - \Psi) Z' \right] \\ + \frac{\left\{ \left(K - \frac{C}{\delta} \right) + \frac{\Psi C(\theta + 1 - \Psi)}{2\theta\delta} \right\}}{(\theta + 1 - \Psi)} \left[\log Z' - \log \left\{ -\theta Z' + (\theta + 1 - \Psi) \right\} \right] \\ = \frac{C\beta\lambda}{D} t + \text{constant.} \quad \dots \quad \dots \quad \dots \quad (15)$$

We have to determine this constant by initial conditions.

Taking the origin of time when $Z = 1$, i.e. $Z' = 1$, the solution is

$$C \left(\frac{1}{\delta} - b \right) (Z' - 1) - \frac{\Psi C}{2\theta\delta} \log \frac{\{-\theta Z'^2 + (\theta + 1 - \Psi) Z'\}}{(1 - \Psi)} \\ + \frac{\left\{ \left(K - \frac{C}{\delta} \right) + \frac{\Psi C(\theta + 1 - \Psi)}{2\theta\delta} \right\}}{(\theta + 1 - \Psi)} \left[\log \left\{ \frac{Z'(1 - \Psi)}{-\theta Z' + (\theta + 1 - \Psi)} \right\} \right] = \frac{C\beta\lambda}{D} t \quad \dots \quad (15a)$$

The pressure in the first chamber is given by (12) and its value at all-burnt, i.e. $Z = 1$ or $Z' = 1$, is given by

$$P_B = \frac{C\lambda \left[1 + \Psi \left\{ \frac{1 - \theta}{2\theta} - \frac{\theta + 1}{2\theta} \right\} \right]}{\left[K - Cb \left\{ \frac{1 - \theta}{2\theta} - \frac{\theta + 1}{2\theta} \right\} \right]} \\ - \frac{C\lambda(1 - \Psi)}{[K - Cb(1 - \Psi)]} \quad \dots \quad \dots \quad \dots \quad (16)$$

As for a tubular charge here the maximum pressure does not exactly occur at all-burnt but for practical cases it is very nearly at all-burnt and equation (16) can be taken to give approximately the maximum pressure.

Second chamber and bore.

The equation of the second chamber and bore are simply

$$w_1 \frac{dv}{dt} = Ap \quad \dots \quad \dots \quad \dots \quad (17)$$

and

$$p = \frac{C\lambda(Z - N)}{[K_0 + Ax - C(Z - N)b]} \quad \dots \quad \dots \quad \dots \quad (18)$$

Substituting the value of $(Z - N)$ in (18) we get

$$p = \frac{C\lambda\Psi \left[\frac{\theta + 1}{2\theta} - \frac{\sqrt{(\theta + 1)^2 - 4\theta Z}}{2\theta} \right]}{\left[K_0 + Ax - Cb\Psi \left\{ \frac{\theta + 1}{2\theta} - \frac{\sqrt{(\theta + 1)^2 - 4\theta Z}}{2\theta} \right\} \right]} \quad \dots \quad \dots \quad (18a)$$

If K_0 is not zero, then for small Z

$$p = \frac{C\lambda\psi \left[\frac{\theta+1}{2\theta} - \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} \right]}{K_0}$$

and

$$P = \frac{C\lambda \left[Z + \psi \left\{ \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{(\theta+1)}{2\theta} \right\} \right]}{\left(K - \frac{C}{\delta} \right)}$$

so that initially

$$\frac{p}{P} = \frac{\psi \left[\frac{\theta+1}{2\theta} - \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} \right] \left(K - \frac{C}{\delta} \right)}{\left[Z + \psi \left\{ \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{\theta+1}{2\theta} \right\} \right] K_0} \quad \dots \quad (19)$$

or

$$\frac{p}{P} = \frac{\psi Z' \left(K - \frac{C}{\delta} \right)}{K_0 [-\theta Z'^2 + (\theta+1-\psi)Z']} \quad \dots \quad (19a)$$

From (17) and (18a) we get

$$\frac{dv}{dZ} = \frac{AC\lambda\psi \left[\frac{\theta+1}{2\theta} - \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} \right]}{w_1 \left[K_0 + Ax - Cb\psi \left\{ \frac{\theta+1}{2\theta} - \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} \right\} \right]} \frac{dt}{dZ} \quad \dots \quad (20)$$

$$\text{or } \frac{dv}{dZ} = \frac{AC\lambda\psi D \left[\frac{\theta+1}{2\theta} - \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} \right] \left[K - \frac{C(1-Z)}{\delta} - \left\{ Z + \psi - \left(\frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{\theta+1}{2\theta} \right) \right\} Cb \right]}{w_1 C\lambda\beta \left[K_0 + Ax - Cb\psi \left\{ \frac{\theta+1}{2\theta} - \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} \right\} \right] \times \sqrt{(\theta+1)^2 - 4\theta Z} \left[Z + \psi \left(\frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} - \frac{\theta+1}{2\theta} \right) \right]} \quad \dots \quad (20a)$$

Transforming this equation to Z' we get

$$\begin{aligned} \frac{dv}{dZ'} &= \frac{AC\lambda\psi D Z' \left[\left(K - \frac{C}{\delta} \right) + C \left(\frac{1}{\delta} - b \right) \left\{ -\theta Z'^2 + (\theta+1)Z' \right\} + Cb\psi Z' \right]}{w_1 C\lambda\beta [K_0 + Ax - Cb\psi Z'] [-\theta Z'^2 + (\theta+1-\psi)Z']} \\ &= \frac{A\psi D \left[\left(K - \frac{C}{\delta} \right) + C \left(\frac{1}{\delta} - b \right) \left\{ -\theta Z'^2 + (\theta+1)Z' \right\} + Cb\psi Z' \right]}{w_1 \beta [K_0 + Ax - Cb\psi Z'] [-\theta Z'^2 + (\theta+1-\psi)Z']} \quad \dots \quad (20b) \end{aligned}$$

Further making the following substitution,

$$X = \frac{\beta}{AD} (K_0 + Ax) \left(\frac{Cw_1\lambda}{\psi} \right)^{\frac{1}{2}}$$

and

$$v = \frac{Cb\beta}{AD} \left(w_1 C \lambda \Psi \right)^{\frac{1}{2}},$$

equation (20b) becomes

$$\begin{aligned} \frac{d}{dZ'} \left[\left\{ \frac{-\theta Z'^2 + (\theta + 1 - \Psi)Z'}{C \left(\frac{1}{\delta} - b \right) \left\{ \theta Z'^2 - (\theta + 1)Z' \right\} - Cb\Psi Z' - \left(K - \frac{C}{\delta} \right)} \right\} \frac{dX}{dZ'} \right] \\ = \frac{1}{(X - vZ')} \left[\frac{C \left(\frac{1}{\delta} - b \right) \left\{ \theta Z'^2 - (\theta + 1)Z' \right\} - Cb\Psi Z' - \left(K - \frac{C}{\delta} \right)}{-\theta Z' + (\theta + 1 - \Psi)} \right] \dots (21) \end{aligned}$$

It is easy to show that $v \approx \frac{2bp}{\lambda}$ and small, where p is the space-mean pressure on the shot up to all-burnt. If $p = 8$ tons/sq. in., which is not likely to be exceeded in H/L guns, $v < 0.3$.

Generally, the integration of equation (21) is best affected numerically. The boundary conditions are $Z' \frac{dX}{dZ'} = 0$ and $X = X_0$ when $Z' = 0$, ($Z = 0$). The pressure, velocity and shot-travel at any value of Z between 0 and 1 are, respectively,

$$\begin{aligned} p &= \frac{\beta C \lambda \Psi Z' \left(\frac{w_1 C \lambda}{\Psi} \right)^{\frac{1}{2}}}{AD(X - vZ')} \\ &= \frac{\beta C \lambda (w_1 C \lambda \Psi)^{\frac{1}{2}}}{AD} \frac{Z'}{(X - vZ')} \dots \dots \dots (22) \end{aligned}$$

$$v = \left(\frac{\Psi}{w_1 C \lambda} \right)^{\frac{1}{2}} C \lambda \frac{dX}{dZ'} \left[\frac{-\theta Z'^2 + (\theta + 1 - \Psi)Z'}{\left(K - \frac{C}{\delta} \right) + C \left(\frac{1}{\delta} - b \right) \left\{ -\theta Z'^2 + (\theta + 1)Z' \right\} + Cb\Psi Z'} \right] \quad (23)$$

$$x = \frac{D}{\beta} [X - X_0] \left(\frac{\Psi}{w_1 C \lambda} \right)^{\frac{1}{2}} \dots \dots \dots (24)$$

where

$$X_0 = \frac{\beta K_0}{AD} \left(\frac{w_1 C \lambda}{\Psi} \right)^{\frac{1}{2}}$$

As the maximum pressure occurs at all-burnt, we are interested in the values of pressure, velocity and shot-travel at all-burnt only.

Series solution.

By Maclaurin's theorem we can express the solution of equation (21) as a power series in Z' as follows:

$$X = X_0 + X_0'Z' + \frac{1}{2}X_0''Z'^2 + \frac{1}{6}X_0'''Z'^3 + \dots$$

where the coefficients are the initial values of successive derivatives of X with respect to Z' , these are obtained by successive differentiation of equation (21). The solution becomes

$$X = X_0 + Z' \left[\frac{1}{X_0} \frac{\alpha_1^2}{\alpha_2^2} \right] \\ + \frac{1}{2} Z'^2 \left[\frac{1}{X_0} \left\{ \frac{3}{2} \frac{\alpha_1}{\alpha_2} \left(\frac{\alpha_3}{\alpha_2} + \frac{\alpha_1 \theta}{\alpha_2^2} \right) \right\} + \frac{1}{X_0^2} \left\{ \frac{1}{2} \frac{\alpha_1^2}{\alpha_2^2} \nu \right\} + \frac{1}{X_0^3} \left\{ -\frac{1}{2} \frac{\alpha_1^4}{\alpha_2^4} \right\} \right] \\ + \frac{1}{6} Z'^3 \left[\frac{1}{X_0} \left\{ \frac{1}{3} \left(\frac{14 \theta \alpha_3 \alpha_1}{\alpha_2^3} + \frac{11 \theta^2 \alpha_1^2}{\alpha_2^4} - \frac{8 \theta \alpha_1 \alpha_3}{\alpha_2^2} \right) + \frac{1}{3} \left(\frac{\alpha_3^2}{\alpha_2^2} \right) \right\} \right. \\ \left. + \frac{1}{X_0^2} \left\{ \frac{5}{3} \nu \left(\frac{\theta \alpha_1^2}{\alpha_2^3} + \frac{\alpha_1 \alpha_3}{\alpha_2^2} \right) \right\} \right. \\ \left. + \frac{1}{X_0^3} \left\{ \frac{1}{3} \left(\frac{2 \nu^2 \alpha_1^2}{\alpha_2^2} - \frac{13}{2} \frac{\alpha_1^4 \theta}{\alpha_2^5} - \frac{13}{2} \frac{\alpha_1^3 \alpha_3}{\alpha_2^4} \right) \right\} \right. \\ \left. + \frac{1}{X_0^4} \left\{ -\frac{3}{2} \frac{\alpha_1^4 \nu}{\alpha_2^4} \right\} \right. \\ \left. + \frac{1}{X_0^5} \left\{ \frac{5}{6} \frac{\alpha_1^6}{\alpha_2^6} \right\} \right] \\ + \dots$$

where $\alpha_1 = \left(K - \frac{C}{\delta}\right)$, $\alpha_2 = (\theta + 1 - \Psi)$, $\alpha_3 = \left\{ C \left(\frac{1}{\delta} - b \right) (\theta + 1) + Cb\Psi \right\}$

and $\alpha_4 = C \left(\frac{1}{\delta} - b \right).$

The corresponding pressure is given as

$$p = \frac{\beta C \lambda (w_1 C \lambda \Psi)^{\frac{1}{2}}}{AD} \left(\frac{Z'}{X - \nu Z'} \right)$$

Substituting the value of X from (25), and then bringing this term up we get

$$p = \frac{\beta C \lambda (w_1 C \lambda \Psi)^{\dagger} Z'}{A D X_0} \left[1 + Z' \left[\frac{1}{X_0^2} \frac{\alpha_1^2}{\alpha_2^2} - \frac{\nu}{X_0} \right] \right. \\ + \frac{1}{2} Z'^2 \left[\frac{1}{X_0^2} \left\{ \frac{3}{2} \frac{\alpha_1}{\alpha_2} \left(\frac{\alpha_3}{\alpha_2} + \frac{\alpha_1 \theta}{\alpha_2^2} \right) \right\} + \frac{1}{X_0^3} \left\{ \frac{1}{2} \frac{\alpha_1^2}{\alpha_2^2} \nu \right\} + \frac{1}{X_0^4} \left\{ -\frac{1}{2} \frac{\alpha_1^4}{\alpha_2^4} \right\} \right] \\ + \frac{1}{6} Z'^3 \left[\frac{1}{X_0^2} \left\{ \frac{1}{3} \left(\frac{\alpha_3^2}{\alpha_2^2} + \frac{11 \theta^2 \alpha_1^2}{\alpha_2^4} + \frac{14 \theta \alpha_3 \alpha_1}{\alpha_2^3} - \frac{8 \theta \alpha_4 \alpha_1}{\alpha_2^2} \right) \right\} \right. \\ \left. \left. + \frac{1}{X_0^3} \left\{ \frac{5}{3} \nu \left(\frac{\theta \alpha_1^2}{\alpha_2^3} + \frac{\alpha_1 \alpha_3}{\alpha_2^2} \right) \right\} \right] \right]$$

$$\begin{aligned}
& + \frac{1}{X_0^4} \left\{ \frac{1}{3} \left(\frac{2\nu^2 \alpha_1^2}{\alpha_2^2} - \frac{13}{2} \frac{\alpha_1^4 \theta}{\alpha_2^5} - \frac{13}{2} \frac{\alpha_3 \alpha_1^3}{\alpha_2^4} \right) \right\} \\
& + \frac{1}{X_0^5} \left\{ -\frac{3}{2} \frac{\alpha_1^4 \nu}{\alpha_2^4} \right\} \\
& + \frac{1}{X_0^6} \left\{ \frac{5}{6} \frac{\alpha_1^6}{\alpha_2^6} \right\} \\
& + \dots \dots \dots \left. \right]^{-1} \dots \dots \dots (26)
\end{aligned}$$

Z' being small we can expand this binomially. After expansion this reduces to

$$\begin{aligned}
p = \frac{\beta C \lambda (w_1 C \lambda \psi)^{\frac{1}{2}}}{AD} \frac{Z'}{X_0} & \left[1 - Z' \left[\left\{ \frac{1}{X_0^2} \frac{\alpha_1^2}{\alpha_2^2} - \frac{\nu}{X_0} \right\} \right] \right. \\
& - \frac{1}{2} Z'^2 \left[\frac{1}{X_0^2} \left\{ \frac{3}{2} \frac{\alpha_1}{\alpha_2} \left(\frac{\alpha_3}{\alpha_2} + \frac{\alpha_1 \theta}{\alpha_2^2} \right) - 2\nu^2 \right\} + \frac{1}{X_0^3} \left\{ \frac{9}{2} \frac{\alpha_1^2}{\alpha_2^2} \nu \right\} + \frac{1}{X_0^4} \left\{ -\frac{5}{2} \frac{\alpha_1^4}{\alpha_2^4} \right\} \right] \\
& - \frac{1}{6} Z'^3 \left[\frac{1}{X_0^2} \left\{ \frac{1}{3} \left(\frac{\alpha_3^2}{\alpha_2^2} + \frac{11\theta^2 \alpha_1^2}{\alpha_2^4} + \frac{14\theta \alpha_3 \alpha_1}{\alpha_2^3} - \frac{8\theta \alpha_4 \alpha_1}{\alpha_2^2} \right) \right\} \right. \\
& \quad + \frac{1}{X_0^3} \left\{ \frac{32}{3} \nu \left(\frac{\theta \alpha_1^2}{\alpha_2^3} + \frac{\alpha_1 \alpha_3}{\alpha_2^2} \right) - 6\nu^3 \right\} \\
& \quad + \frac{1}{X_0^4} \left\{ \frac{1}{3} \left(\frac{65\nu^2 \alpha_1^2}{\alpha_2^2} - \frac{67}{2} \frac{\alpha_1^4 \theta}{\alpha_2^5} - \frac{67}{2} \frac{\alpha_3 \alpha_1^3}{\alpha_2^4} \right) \right\} \\
& \quad + \frac{1}{X_0^5} \left\{ -\frac{51}{2} \frac{\alpha_1^4 \nu}{\alpha_2^4} \right\} \\
& \quad + \frac{1}{X_0^6} \left\{ \frac{59}{6} \frac{\alpha_1^6}{\alpha_2^6} \right\} \\
& \left. + \dots \dots \dots \right] \dots \dots \dots (26a)
\end{aligned}$$

and the velocity is

$$v = \left(\frac{\Psi}{w_1 C \lambda} \right)^{\frac{1}{2}} C \lambda \frac{dX}{dZ'} \left[\frac{-\theta Z'^2 + (\theta + 1 - \Psi) Z'}{\left(K - \frac{C}{\delta} \right) + C \left(\frac{1}{\delta} - b \right) \left\{ -\theta Z'^2 + (\theta + 1) Z' \right\} + C b \Psi Z'} \right]$$

$$= \frac{\beta C \lambda (w_1 C \lambda \Psi)^{\frac{1}{2}}}{2AD} \frac{\alpha_2}{\alpha_1} Z'^{\frac{1}{2}} \left[1 + \frac{2\nu\alpha_2}{5\alpha_1} Z'^{\frac{1}{2}} + \left\{ \frac{18\nu^2\alpha_2^2}{125\alpha_1^2} - \frac{2}{5} \frac{\alpha_5}{\alpha_1\alpha_2} \right\} Z' - \frac{158\nu\alpha_5}{425\alpha_1^2} Z'^{\frac{3}{2}} \right. \\ \left. + \left\{ \frac{17}{130} \frac{\alpha_5^2}{\alpha_1^2\alpha_2^2} - \frac{3\theta\alpha_5}{13\alpha_2^2\alpha_1} + \frac{3\theta\alpha_4}{13\alpha_1} \right\} Z'^2 + \dots \right] \quad \dots \quad (29)$$

For values of $\frac{\alpha_5}{\alpha_1\alpha_2}$ less than 5/6, and ν being small, p is maximum at $Z' = 1$.

Generally for all practical densities of loading the peak of p always occurs at $Z' = 1$.

Therefore

$$p_{\max.} = \frac{\beta C \lambda (w_1 C \lambda \Psi)^{\frac{1}{2}}}{2AD} \frac{\alpha_2}{\alpha_1} \left[1 + \frac{2\nu\alpha_2}{5\alpha_1} + \left\{ \frac{18\nu^2\alpha_2^2}{125\alpha_1^2} - \frac{2}{5} \frac{\alpha_5}{\alpha_1\alpha_2} \right\} - \frac{158\nu\alpha_5}{425\alpha_1^2} \right. \\ \left. + \left\{ \frac{17}{130} \frac{\alpha_5^2}{\alpha_1^2\alpha_2^2} - \frac{3\theta\alpha_5}{13\alpha_2^2\alpha_1} + \frac{3\theta\alpha_4}{13\alpha_1} \right\} + \dots \right] \quad \dots \quad (29a)$$

From (29a), and (16) we get

$$\frac{p_{\max.}}{P_{\max.}} = \frac{\beta (w_1 C \lambda \Psi)^{\frac{1}{2}} \{K - cb(1 - \Psi)\}}{2AD(1 - \Psi)} \frac{\alpha_2}{\alpha_1} \left[1 + \frac{2\nu}{5} \frac{\alpha_2}{\alpha_1} + \left\{ \frac{18\nu^2}{125} \frac{\alpha_2^2}{\alpha_1^2} - \frac{2}{5} \frac{\alpha_5}{\alpha_1\alpha_2} \right\} \right. \\ \left. - \frac{158\nu}{425} \frac{\alpha_5}{\alpha_1^2} + \left\{ \frac{17}{130} \frac{\alpha_5^2}{\alpha_1^2\alpha_2^2} - \frac{3\theta\alpha_5}{13\alpha_2^2\alpha_1} + \frac{3\theta\alpha_4}{13\alpha_1} \right\} + \dots \right] \quad \dots \quad (30)$$

For the velocity corresponding to equation (28) we have

$$v = \left(\frac{\Psi C \lambda}{w_1} \right)^{\frac{1}{2}} \left\{ \frac{-\theta Z'^2 + (\theta + 1 - \Psi)Z'}{\left(\left(K - \frac{c}{\delta} \right) + C \left(\frac{1}{\delta} - b \right) \{ -\theta Z'^2 + (\theta + 1)Z' \} + cb\Psi Z' \right)} \right\} \\ \times \left[\frac{\alpha_1}{\alpha_2} \frac{1}{Z'^{\frac{1}{2}}} + \frac{\nu}{5} + \frac{3}{2} \left\{ \frac{4\nu^2}{125} \frac{\alpha_2}{\alpha_1} + \frac{4}{5} \frac{\alpha_5}{\alpha_2^2} \right\} Z'^{\frac{1}{2}} + \frac{88}{425} \frac{\nu\alpha_5}{\alpha_1\alpha_2} Z' \right. \\ \left. + \frac{5}{2} \left\{ \frac{19}{325} \frac{\alpha_5^2}{\alpha_1\alpha_2^2} + \frac{6\theta\alpha_5}{13\alpha_2^2} - \frac{6\theta\alpha_4}{13\alpha_2} \right\} Z'^{\frac{3}{2}} + \dots \right].$$

This gives on expansion and simplification

$$v = \left(\frac{\Psi C \lambda}{w_1} \right)^{\frac{1}{2}} \left[Z'^{\frac{1}{2}} + \frac{\nu}{5} \frac{\alpha_2}{\alpha_1} Z' + \left\{ \frac{6}{125} \frac{\nu^2\alpha_2^2}{\alpha_1^2} + \frac{1}{5} \frac{\alpha_5}{\alpha_1\alpha_2} \right\} Z'^{\frac{3}{2}} + \frac{3}{425} \frac{\nu\alpha_5}{\alpha_1^2} Z'^{\frac{5}{2}} \right. \\ \left. + \left\{ -\frac{7}{130} \frac{\alpha_5^2}{\alpha_1^2\alpha_2^2} + \frac{2}{13} \frac{\theta\alpha_5}{\alpha_1^2\alpha_2^2} - \frac{2}{13} \frac{\theta\alpha_4}{\alpha_1} \right\} Z'^{\frac{7}{2}} + \dots \right] \quad \dots \quad (31)$$

Therefore the velocity at all-burnt is

$$v_B = \left(\frac{\Psi C \lambda}{w_1} \right)^{\frac{1}{2}} \left[1 + \frac{\nu}{5} \frac{\alpha_2}{\alpha_1} + \frac{6\nu^2}{125} \frac{\alpha_2^2}{\alpha_1^2} + \frac{1}{5} \frac{\alpha_5}{\alpha_1 \alpha_2} + \frac{3\nu}{425} \frac{\alpha_5}{\alpha_1^2} - \frac{7}{130} \frac{\alpha_5^2}{\alpha_1 \alpha_2} + \frac{2}{13} \frac{\theta \alpha_5}{\alpha_1^2 \alpha_2} - \frac{2}{13} \frac{\theta \alpha_4}{\alpha_1} + \dots \right] \quad \dots \quad (32)$$

By putting $\theta = 0$, all the equations deduced so far reduce to the equations for tubular propellant as deduced by Corner.

5. SOLUTION AFTER ALL-BURNT

The gas flow from the first chamber to the second chamber depends upon the ratio p/P , and for $\nu = 1.25$ this ratio p/P is less than 0.555. So long as $p/P < 0.555$ the gas flow from the first chamber is the same as if p were zero. Hugoniot's theory with Rateau's corrections for covolume (Corner, 1950, chapter 9) gives the rate of decay of pressure P .

The solution after burnt in this general case is exactly the same as is for tubular propellant done by Corner. Simply we have to take our all-burnt values for tubular propellant.

6. SUMMARY OF WORKING FORMULAE.

Generally we are interested only in a few important features of the ballistic solution. One of these is the peak pressure which is approximately given by

$$P_{\max} \approx \frac{C \lambda (1 - \Psi)}{K - (1 - \Psi) C b}$$

where

$$\Psi = \frac{\psi S D}{\beta C \sqrt{\lambda}}$$

If $p/P > 0.555$ for some time Ψ is multiplied by an appropriate back pressure factor from Table 2, p. 264 (H.M.S. Office, 1951).

It has been shown in this paper that

$$X_0 = \frac{\beta K_0}{A D} \left(\frac{w_1 C \lambda}{\Psi} \right)^{\frac{1}{2}} \quad \dots \quad (34)$$

where $w_1 = 1.05 W + \frac{1}{3} C$, and can be calculated easily.

We also work out

$$A \sqrt{w_1 C \lambda \Psi}^{\frac{1}{2}}.$$

Now we calculate X_B and $\left(\frac{dX}{dz} \right)_B$ and then the peak pressure

$$p_B = \frac{\beta C \lambda (\Psi w_1 C \lambda)^{\frac{1}{2}}}{A D (X_B - \nu)} \quad \dots \quad (35)$$

the travel at all-burnt is

$$x_B = \frac{D}{\beta} [X_B - X_0] \left[\frac{\Psi}{w_1 C \lambda} \right]^{\frac{1}{2}} \quad \dots \quad (36)$$

and the velocity at all-burnt is

$$v_B = \left(\frac{C \lambda \Psi}{w_1} \right)^{\frac{1}{2}} \left[\frac{(1 - \Psi)}{K_0 - C b (1 - \Psi)} \right] \left[\frac{dX}{dZ'} \right]_B \quad \dots \quad (37)$$

The muzzle velocity can be calculated easily as indicated in section 5.

Now it may be remarked here that the equation (21) can be integrated numerically and tables can be computed corresponding to the tables given by Corner for tubular propellants.

SUMMARY

In this paper the author has extended Corner's theory of internal ballistics of high and low pressure guns using tubular propellants to guns using propellants of any shape. The linear law of burning has been assumed as was done by Corner.

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REFERENCES

- Corner, J. (1945). Hoch-Und-Niederdruck Kanone, A.R.D.Th.M. 22/45.
 — (1948). A theory of the internal ballistics of the 'Hoch-Und-Niederdruck Kanone'. *Jour. Frank. Inst.*, Vol. 246, No. 3 (Sept., 1948).
 — (1950). Theory of Internal Ballistics of Guns, New York.
 I. Stationery Office, London (1951). Internal Ballistics.

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LATENT HEAT OF EVAPORATION OF LIQUID He⁴ AND LIQUID He³.

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1. INTRODUCTION

The latent heat of evaporation of liquid He³ has been calculated by Abraham, Osborne and Weinstock (1950), using the Clausius-Clapeyron equation. The results, however, do not have the certainty of direct experimental observation, as the latent heat of evaporation of liquid He³ has not been determined experimentally. In order to form an idea about the accuracy of the theoretical results, we have calculated the latent heat of evaporation of liquid He⁴ by following the same procedure. The results thus obtained are compared with experimental measurements of Keesom (1942). We have then recalculated the latent heat of evaporation of liquid He³ by using recent values of virial coefficients as computed by Kilpatrick, Keller, Hammel and Metropolis (1954), and liquid densities of He³ as determined recently by Kerr (1954). The calculations have also been extended to low temperatures (1.0 to 0.0° K.) by making use of Chen and London (1953) vapour pressure equation. Using the latent heat values of He⁴ and He³ at absolute zero, the effective mass of He³ atom has been estimated.

2. LATENT HEAT OF LIQUID He⁴

According to Bleaney and Simon (1939), the pressure temperature relation obeyed by He⁴ in the region 0.0° K. to 1.6° K. is

$$\log_{10} p_{mm} = -\frac{3.117}{T_A} + 2.5 \log_{10} T_A + 2.196 - 2.27 \times 10^{-3} \cdot T_A^3 + \Delta, \quad \dots (1)$$

where Δ is a small correction term whose dependence on T_A is given graphically by Keesom (1942) and p_{mm} is in mm. of Hg at 0° C. Here T_A is the 'Agreed Temperature' calculated from the He⁴ pressure and the table given by van Dijk and Shoenberg (1949).

For the region from 1.6 to 4.3° K., Keesom and Lignac have given the following interpolation formula

$$\log_{10} p_{mm} = -\frac{4.7921}{T_A} + 0.00783 T_A + 0.017601 T_A^2 + 3.6730. \quad \dots (2)$$

For the region 4.3° K. to 5.2° K. (the critical point) experimentally determined data of p and T due to Kamerlingh Onnes and Weber can be represented, as pointed out by Squire (1953), by the interpolation formula

$$T = 2.967 + 1.905 \times 10^{-3} p_{mm} - 3.52 \times 10^{-7} p_{mm}^2. \quad \dots (3)$$

We have used for calculating (dp/dT_A) the vapour-pressure formulas given in equations (1) to (3) and the vapour-pressure data of liquid He⁴.

We have calculated V_g , the molar volume of He^4 in the gaseous state, by making use of the equation

$$pV_g = RT \left[1 + \frac{B(T)}{V_g} \right], \quad \dots \dots \dots (4)$$

the values of second virial coefficient $B(T)$ being taken from the computations of Kilpatrick *et al.* (1954).

V_l , the molar volume of liquid He^4 , is obtained from Keesom's (1942) value for density under saturated vapour pressure. Then the latent heat of evaporation can be calculated by using the Clausius-Clapeyron equation:

$$= \frac{T(V_g - V_l)}{T(V_g - V_l)} \cdot \dots \dots \dots (5)$$

The calculations of L could not be carried out for temperatures greater than 4.5°K . for lack of data of liquid densities for He^4 . The results are tabulated in Table I and are also shown in Figure 1.

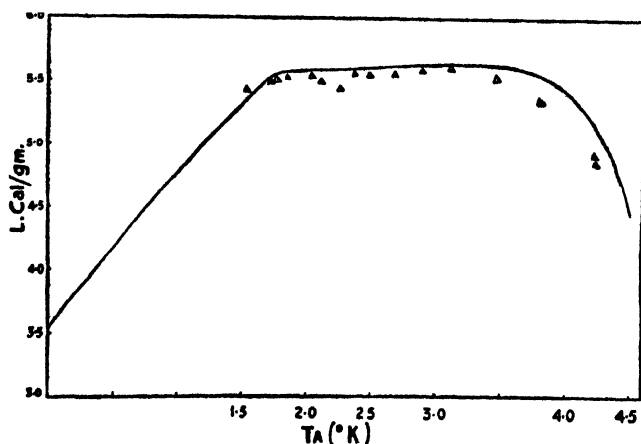


FIG. 1. Latent Heat of Evaporation of Liquid He^4 versus $T_A^\circ \text{K}$. The curve indicates the calculated values, while the points represent experimental results of Keesom (1942).

TABLE I
Latent Heat of Evaporation of Liquid He^4 .

$T_A^\circ \text{K}$.	$T_k^\circ \text{K}$.	L cal./gm. (calculated).	L cal./gm. (experimental).
0.000	0.000	3.56
0.500	0.500	4.18
1.000	1.000	4.78
1.500	1.500	5.32	5.40
2.000	1.989	5.59	5.55
2.500	2.490	5.62	5.54
3.000	2.995	5.64	5.60
3.500	3.500	5.63	5.52
4.000	4.001	5.42	5.17
4.500	4.501	4.46

We notice from Fig. 1 that the calculated values are in good agreement with those determined experimentally. Table I also includes the theoretically extrapolated value of L at 0.0°K . which is 3.56 cal./gm .

3. LATENT HEAT OF LIQUID He^3

The saturated vapour pressure of liquid He^3 was observed by Abraham, Osborne and Weinstock (1950) in the region 1.0 to 3.3°K . The experimental data given by them are represented by the equation

$$\log_{10} p_{\text{mm}} = -0.97796 + 2.5 \log_{10} T_K + 0.000302 T_K^3 + 1.91594, \quad (6)$$

which is valid in the region 1.0 to 3.3°K . Here T_K is the temperature calculated by applying to T_A the corrections of Kistemaker (1946) smoothed as shown by Abraham *et al.* (1950).

The value of (dP/dT) is obtained from equation (6), while V_g , as before, is obtained from (4) making use of virial coefficient computations of Kilpatrick *et al.* (1954). The molar volume V_l of the liquid is obtained from liquid densities measured by Kerr (1954). The heat of evaporation of He^3 is then obtained from the thermodynamical relation (5).

No calculations have been made above 2.5°K . because the equation of state, as given by (4), is inadequate for the saturated vapour above this temperature.

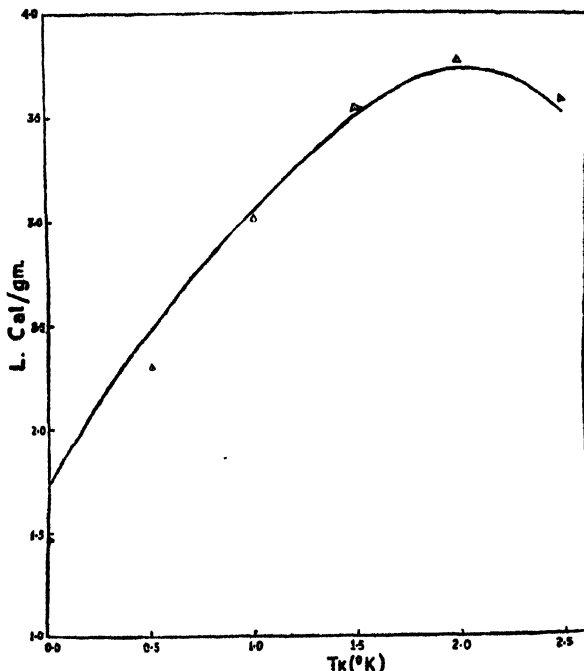


FIG. 2. Latent Heat of Evaporation of Liquid He^3 versus $T_K^\circ \text{K}$. The curve indicates calculated values, while the points represent the computations by Abraham *et al.* (1950).

In order to extend the calculations of latent heat below 1.0° K. we make use of the Chen and London (1953) vapour-pressure equation

$$\log_{10}\left(\frac{p_{\text{mm}}}{T_K^{5/2}}\right) = 2.3126 - \frac{1.1561}{T_K} - 0.25254T_K - 0.00667T_K^2 + 0.05266T_K^3 - 0.01210T_K^4. \quad \dots (7)$$

This equation represents the vapour pressure measurements of Abraham *et al.* satisfactorily up to 2.5° K. The entropy curve of liquid He³ derived from it, in agreement with the third law of thermodynamics, goes smoothly to zero at 0° K. and does not lead to any residual entropy like equation (6) given earlier by Abraham *et al.*

The heat of vaporisation of He³ thus calculated is tabulated in Table 2 and is shown in Fig. 2. It has a maximum value of 3.72 cal./gm. at 2.0° K. Further, we conclude that at 0.0° K. the vaporisation heat is 1.75 cal./gm.

TABLE 2
Latent Heat of Evaporation of Liquid He³.

$T_h^\circ \text{K.}$	$T_A^\circ \text{K.}$	$L \text{ cal./gm.}$	$T_h^\circ \text{K.}$	$T_A^\circ \text{K.}$	$L \text{ cal./gm.}$
0.000	0.000	1.75	1.300	1.300	3.34
0.100	0.100	1.91	1.400	1.400	3.38
0.200	0.200	2.07	1.500	1.500	3.51
0.300	0.300	2.21	1.600	1.603	3.57
0.400	0.400	2.35	1.700	1.705	3.63
0.500	0.500	2.49	1.800	1.807	3.67
0.600	0.600	2.61	1.900	1.909	3.70
0.700	0.700	2.73	2.000	2.011	3.72
0.800	0.800	2.84	2.100	2.112	3.71
0.900	0.900	2.94	2.200	2.213	3.69
1.000	1.000	3.02	2.300	2.312	3.66
1.100	1.100	3.13	2.400	2.411	3.60
1.200	1.200	3.24	2.500	2.510	3.51

As has already been mentioned, the vapour-pressure equation proposed by Abraham *et al.* (1950) leads to a non-vanishing entropy for liquid He³ at the absolute zero. Thus it seems that the extrapolation of their curve to absolute zero is not quite justified and, in fact, various authors have tried to look for a transition in He³ below 1.0° K. in order to account for this non-vanishing entropy at zero degree Kelvin. On the other hand, no such difficulty exists with the Chen and London (1953) vapour-pressure equation. Now Fig. 2 clearly shows that our results (derived on the basis of the Chen and London equation) differ considerably from those of Abraham *et al.* (1950) below 1.0° K. Consequently, it appears that observation on the heat of vaporisation of He³ may provide conclusive evidence regarding the existence of a transition.

4. THE EFFECTIVE MASS OF HE³ ATOM

At the absolute zero the latent heat of He⁴ differs considerably from that of He³. This difference may, in part, be attributed to the 'Statistical Zero Point Energy' of He³ which in case of He⁴ is zero. Equating it to

$$\frac{3N}{5} \left\{ \frac{\hbar^2}{2m_3} \cdot \left(\frac{3N}{8\pi V_3} \right)^{\frac{1}{3}} \right\}, \quad \dots (8)$$

(where N is Loschmidt's number and V_3 the molar volume of liquid He³ at 0° K.), we obtain the ratio of the effective to the actual mass of He³ atom as

$$m_3^*/m_3 = 0.66.$$

This is, however, a crude method of estimating the effective mass of He³ atom, because we have ignored the effect of the interactions. The evaluation of the effective mass, taking interactions into account, will be published in a subsequent paper.

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SUMMARY

Latent heat of evaporation of He⁴ and He³ is calculated by using the Clausius-Clapeyron equation. Since in case of He⁴, satisfactory agreement between the theoretical and the experimental latent heat values is observed, therefore the calculated He³ values can be used with confidence in further thermodynamical computations. The latent heat values for He³, as calculated by using Chen and London (1953) vapour-pressure equation, differ considerably from those calculated by Abraham *et al.* (1950) in the temperature range 1.0° K. to 0.0° K. Thus experimental results of latent heat of He³ below 1.0° K. will enable us to decide whether or not a transition in He³ exists below 1.0° K.

REFERENCES

- Abraham, B. M., Osborne, D. W., and Weinstock, B. (1950). The Vapour Pressure, Critical Point, Heat Vaporisation and Entropy of Liquid He³. *Phys. Rev.*, **80**, 366-371.
- Bleaney, B., and Simon, F. (1939). The Vapour-Pressure Curve of Liquid Helium below the λ -point. *Trans. Faraday Soc.*, **35**, 1205-1214.
- Chen, T. C., and London, F. (1953). The Entropy of Liquid He³. *Phys. Rev.*, **89**, 1038-1040.
- Keesom, W. H. (1942). Helium, Elsevier, Amsterdam.
- Kerr, E. C. (1954). Orthobaric Densities of He³—1.3° K. to 3.2° K. *Phys. Rev.*, **96**, 551-554.
- Kilpatrick, J. E., Keller, W. E., Hammel, E. F., and Metropolis, N. (1954). Second Virial Coefficients of He³ and He⁴. *Phys. Rev.*, **94**, 1103-1110.
- Kistemaker, J. (1946). The Vapour Pressure of Liquid Helium from the λ -point to 1.3° K. *Physica*, **12**, 272-280.
- Squire, C. F. (1953). Low Temperature Physics. MacGraw-Hill.
- Van Dijk, H., and Shoenberg, D. (1949). Tables of Vapour Pressure of Liquid Helium. *Nature*, **164**, 151.

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ON A PROBLEM OF G. KUREPA

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The purpose of this note is to give a characterisation of ramified tables in which there exists no maximal chain intersecting every maximal antichain. Also a minimal ramified set possessing this property is constructed, and a question communicated to me by G. Kurepa is answered in the affirmative.

A chain (antichain) of a partially ordered set P is any subset S of P , every pair (no pair) of elements of which is comparable. If further S is such that $(S \cup p)$ is a chain (antichain) for no element $p \in P - S$, then S is a maximal chain (antichain) of P .

It may be noted that the antichain A is a maximal antichain of P if and only if every element of P is comparable with some element of A .

A ramified partially ordered set is a partially ordered set in which the set of all elements $<$ any given element is a chain. A ramified table is a ramified set in which every descending chain is finite.

A subset Q of the p.o. set P is cofinal with P if every element of P is $<$ some corresponding $q \in Q$. G. Kurepa (1952) has proposed the following questions:—

(1) Does there exist a partially ordered set in which there is no maximal antichain which intersects every maximal chain?

(2) Does there exist a partially ordered set in which there is no maximal chain which intersects every maximal antichain?

Both these questions have been answered in the affirmative by J. C. Sheperdson (1954), who has constructed a ramified set possessing both these properties. These properties (of a p.o. set, referred to in questions (1) and (2) respectively) will be referred to as properties (1) and (2) respectively in the sequel. Necessary and sufficient conditions for a ramified set to possess property (2) are established in 1.1 and 2.2. In 2.1 a minimal ramified set possessing property (2) is constructed by means of partitions of ordertypes (i.e. of completely ordered sets). It is pointed out that every ramified table possessing property (2) has a cofinal subset which can be represented by means of segments of an ordertype, the segments being ordered by set inclusion. Re-ordering of these segments according to their place in the completely ordered set leads to the answer (2.4) of the following question communicated to me by G. Kurepa:

(3) Does there exist a partially ordered set S so that for each maximal antichain $A \subset S$, each maximal chain of $S - A$ is one such in S , too?

1.1. A necessary and sufficient condition for the ramified set P to possess property (2) is that for every element $p \in P$ there exists at least one pair p_1, p_2 of mutually incomparable elements of P , each $> p$.

(A) To prove that the condition is necessary, consider any ramified set P , not satisfying this condition. There exists an element $p \in P$ for which the set C_p of all elements $> p$ is a chain. Also the set C^p of all elements $< p$ is a chain, by the definition of a ramified set. Therefore, since every element of C^p is $<$ every element of C_p , $C^p + C_p = C$, the set of all elements, comparable with p , is a chain. Also no element $x \in C_p$ is comparable with any element $y \in C$, for $y > x$ would imply that $y > x > p$, while $y < x$ would imply that $y \in C^p$, the chain of all elements $< x$, and

$C \supset p$ if $x \neq p$. In either case $y \in C$. Now, if A were a maximal antichain of P , which does not intersect C , every element of P , and hence every element of C_p , is comparable with some element of A . But this has been shown above to be impossible. Therefore, since C_p is nonnull, it follows that C is a chain which intersects every maximal antichain, i.e. P does not possess property (2), which shows that the condition is necessary.

To prove that the condition is sufficient, it is evidently enough to show that

(B) If P is a ramified set satisfying the condition of 1.1, then for every maximal chain C of P , every maximal antichain A of $P-C$ is a maximal antichain of P also.

For, corresponding to every maximal chain C of P , there will be a maximal antichain A of $P-C$, and $A \cap C$ is evidently null. Hence (assuming that $P \not\subset C$, which condition can be easily seen to be true when P satisfies the condition of 1.1), to prove that P possesses property (2), it is sufficient to show that every such maximal antichain A of $P-C$ is a maximal antichain of P .

Consider any ramified set P satisfying the condition of 1.1. Let C be any maximal chain of P and A any maximal antichain of $(P-C)$. Then every element of $(P-C)$ is comparable with some corresponding element of A , and hence to prove that A is a maximal antichain of P , it is sufficient to show that every element p of C is comparable with some corresponding element of A . By hypothesis there exist two mutually incomparable elements p_1, p_2 of P , each $> p$, and at most one of these can belong to C . Let $p_1 \in (P-C)$. Then, since A is a maximal antichain of $(P-C)$, A contains an element a comparable with p_1 . a is comparable with p also, since $a > p_1$ implies that $a > p_1 > p$, while $a < p_1$ implies that $a \in C^{p_1}$, the chain of all elements $< p_1$, which also contains p . Hence corresponding to any element $p \in C$, there exists $a \in A$ such that a is comparable with p , which proves (B). Hence the condition of 1.1 is sufficient for P to possess property (2).

1.2 Corollary:—From (B) and 1.1 it follows immediately that

(C) If P is any ramified set possessing property (2), and C any maximal chain of P , then every maximal antichain of $(P-C)$ is a maximal antichain of P also.

§2. It has been pointed out by G. Kurepa (1935, p. 112) that every set of successive partitions of a completely ordered set corresponds to a ramified table whose elements are the segments of the partitions, the segments being ordered by the relation of set inclusion reversed. Using this notion a minimal ramified set possessing property (2) can be constructed as follows:—

2.1. The ramified table corresponding to the partition Q of the real number segment $0 < x < 1$, in which $0 < x < 1$ is itself a segment, and which contains with every segment, the two subsets formed by bisecting it, each segment of Q being of length $1/2^n$, $n > 0$, is a minimal ramified set possessing property (2) (i.e. Q is imbeddable in every ramified set possessing property (2)).

It may be noted that every minimal ramified table possessing property (2) is also a minimal ramified set possessing property (2), on account of the following 2.2 Lemma. A ramified set P possesses property (2) if and only if there exists a ramified table cofinal with P , possessing property (2).

Since it is known (1950, p. 125) that for every ramified set there exists a cofinal ramified table, it is sufficient to show that P possesses property (2) if and only if every cofinal subset of P possesses this property. Let R be any subset of P , cofinal with P .

Assuming that P possesses property (2), consider any element $r \in R$. Then, since $r \in P$, by 1.1, there exist two mutually incomparable elements p_1, p_2 of P , both $> r$. Since R is cofinal with P , there exist elements r_1, r_2 of R such that $r_1 > p_1, r_2 > p_2$. Therefore, since P is a ramified set and p_1, p_2 are incomparable, r_1, r_2 are incomparable. (If $r_1 > r_2, p_1, p_2$ would both belong to the set of all elements $\leq r_1$, which is a chain.) Also $r_1 > r, r_2 > r$. Hence corresponding to any $r \in R$, there exist two mutually incomparable elements r_1, r_2 of R each $> r$, and hence by 1.1 R

possesses property (2). It follows that every cofinal subset of the ramified set P possessing property (2) possesses property (2).

Conversely, if R possesses property (2), and p is any element of P , there exists an element $r \in R$ such that $r > p$ (R being cofinal with P), and also two mutually incomparable elements r_1, r_2 of R each $> r$ (since R possesses property (2)). Since r_1, r_2 also belong to P , and each $> p$, it follows that P possesses property (2) if it has a cofinal subset possessing this property. This proves 2.2.

Proof of 2.1:—The partition Q defined in the hypothesis of 2.1 can be easily seen to be minimal in the class of partitions P of any ordertype E , where P possesses the following property:—

(D) Every segment I of the ordertype E , which belongs to P , contains more than one subsegment belonging to P .

For, let P be any partition of an ordertype E , satisfying the condition (D). Consider the following mapping ϕ of the partition Q (defined in 2.1) into P : Let the lowest element of Q , namely the segment $0 < x < 1$, be mapped on to any (arbitrarily chosen) element of P . Suppose that the element $\phi(I)$ of P has been defined for all elements I of Q (i.e. segments of $0 < x < 1$ belonging to Q) of length $> 1/2^n$, in such a way that order is preserved, i.e. $\phi(I_1) \subset \phi(I_2)$ in E if and only if $I_1 \subset I_2$. Then by (D), every segment $\phi(I)$ of E contains at least two disjoint subsegments p_1, p_2 both belonging to P , while every $I \in Q$ of length $1/2^n$ contains only two subsegments q_1, q_2 of length $1/2^{n+1}$ each, belonging to Q . Let $\phi(q_1), \phi(q_2)$ be defined as p_1, p_2 respectively. It is evident that order is preserved. Also this defines $\phi(I)$ for all $I \in Q$ of length $> 1/2^{n+1}$. Hence from the definition of $\phi(I)$ for $I = 0 < x < 1, \phi(I)$ can be defined by induction for all $I \in Q$, so that order is preserved, which proves that Q is imbeddable in P .

To prove 2.1 it remains to show that every ramified table possessing property (2) has a subset which can be represented by a partition of an ordertype E satisfying condition (D). This latter result follows from the representation of ramified tables by complexes given by Kurepa (1935, pp. 84–89). Every ramified table P can be represented by a set of complexes $X = (x_0, \dots, x_\beta, \dots)$ $\beta < \xi = \xi(X)$, where each $\xi(X) < \gamma = \gamma(P)$, the rank of P , the complexes being ordered so that $X < Y$ if and only if $\xi(X) < \xi(Y)$ and $x_\beta = y_\beta$ for $\beta < \xi(X)$. In this representation it can be assumed without loss of generality (as can be easily seen from the proof given in (1935, pp. 84–89) that for every $X \in P$ and every nonlimiting ordinal $\beta+1 < \xi(X)$, the complex $(X)^{\beta+1} = (x_0, \dots, x_\beta)$ consisting of the first $\beta+1$ co-ordinates of X , also belongs to P . Let \bar{P} denote the set of all complexes $Y = (y_0, \dots, y_\beta, \dots)$ $\beta < \eta = \eta(Y)$ such that for every nonlimiting ordinal $\beta+1 < \eta(Y)$, the set $(Y)^{\beta+1}$ of first $\beta+1$ co-ordinates of Y belongs to P , and there exists no complex $X \in P$ such that $\xi(X) > \eta(Y)$ and $x_\beta = y_\beta$ for $\beta < \eta(Y)$. For each $Y \in \bar{P}$ and each $\beta < \eta(Y)$ let $L(Y)^\beta$ denote the set of values of the co-ordinate y_β corresponding to the fixed set of first β co-ordinates $(Y)^\beta$. By Zermelo's axiom this set $L(Y)^\beta$ can be completely ordered. Let $M(Y)^\beta$ be any complete ordering of $L(Y)^\beta$. Let \bar{P} be ordered lexicographically, i.e. $Y_1 = (y_0^1, \dots, y_\beta^1, \dots) < Y_2 = (y_0^2, \dots, y_\beta^2, \dots)$ if and only if there exists $\lambda < \min \{\eta(Y_1), \eta(Y_2)\}$ such that $y_\beta^1 = y_\beta^2$ for $\beta < \lambda$, i.e. $(Y_1)^\lambda = (Y_2)^\lambda$, and $y_\lambda^1 < y_\lambda^2$ in the ordertype $M(Y_1)^\lambda = M(Y_2)^\lambda$. From the definition of \bar{P} it is clear that λ exists for every pair of distinct complexes Y_1, Y_2 of \bar{P} [since otherwise $\eta(Y_1) < \eta(Y_2)$ would imply that $Y_1 = (Y_2)^{\eta(Y_1)}$ and hence there exists $X \in P$ such that $\xi(X) > \eta(Y_1)$ and $Y_1 = (X)^{\eta(Y_1)}$]. Hence this is a

complete ordering of \bar{P} . It is clear that \bar{P} is a 'product with variable factors' defined by Hausdorff (1908, p. 470).

The ramified table P has a cofinal subset P_1 which is similar to a partition of the ordertype \bar{P} . For each $Y \in \bar{P}$ and each $\beta < \eta(Y)$, $(Y)^\beta$ represents a segment of \bar{P} , namely the set of all $Z \in \bar{P}$ such that $\eta(Z) > \beta$ and $(Z)^\beta = (Y)^\beta$. For any $Z \in \bar{P}$ and any $\beta > \eta(Z)$ let $(Z)^\beta$ be defined as $(Z)^{\eta(Z)} = Z$. Let $\mu = \text{Max}_{Y \in \bar{P}} \eta(Y)$.

Then for each $\beta < \mu$, every element $Z \in \bar{P}$ belongs to the corresponding segment $(Z)^\beta$ of \bar{P} . Also for $(Y)^\beta \neq (Z)^\beta$ the segments $(Y)^\beta, (Z)^\beta$ are disjoint. Hence for each $\beta < \mu$, the set of all distinct segments $\{(Y)^\beta\}, Y \in \bar{P}$, forms a partition of \bar{P} . Hence the set of all distinct segments $\{(Y)^\beta\}, Y \in \bar{P}$, where β assumes all values $< \mu$, is a set of successive partitions T' of \bar{P} . Every element $X \in P$ is, by definition, = some $(Y)^\beta, \beta < \mu$, and hence corresponds to an element of the partition T' . Let T denote the subset of T' consisting of all such elements which correspond to elements of P . Then T is also a partition of \bar{P} . Also if t_1, t_2 are any two distinct elements of T , i.e. $t_1 = (Y)^\beta \neq t_2 = (Z)^\delta, t_1 < t_2$ in T , i.e. the segment $(Y)^\beta \supset (Z)^\delta$ if and only if $\beta < \delta$ and $(Z)^\beta = (Y)^\beta$, i.e. if and only if the element $(Y)^\beta < (Z)^\delta$ in P . Hence the correspondence of the elements of P with those of T is an order-preserving mapping. But this mapping is not necessarily one-one, since the distinct elements $(Y)^\beta, (Y)^{\beta+1}$ of P correspond to the same segment of \bar{P} if the range of variation $L(Y)^\beta$ of y_β corresponding to $(Y)^\beta$ consists of a single element, i.e. there exists only one element of P next higher than $(Y)^\beta$. For every element $t \in T$ let $p(t)$ denote the lowest element of P which corresponds to the segment t of \bar{P} , [$p(t)$ exists since P satisfies the descending chain condition] and let $P_1 = \sum_{t \in T} p(t)$. Then from the

property (2) of P it will follow, as shown below, that P_1 is cofinal with P , and that T satisfies condition (D).

Consider any element $X \in P$. Since P has property (2), by 1.1, there exist two mutually incomparable elements X_1, X_2 of P both $> X$. The elements t_1, t_2 of T corresponding to X_1, X_2 are disjoint segments of \bar{P} both contained in t , the element of T corresponding to X , since the correspondence between P and T is order-preserving. Therefore the elements $p(t_1), p(t_2)$ of P_1 are incomparable and both $> x$, i.e. for every $x \in P$ there exists an element of $P_1 > x$, which proves that P_1 is cofinal with P . Hence by 2.2 P_1 possesses property (2). Hence by 1.1, for every element $p \in P_1$, there exist two mutually incomparable elements p_1, p_2 of P_1 both $> p$. Therefore, since T is similar to P_1 (being in one-one order-preserving correspondence with the latter set), it follows that for every element $t \in T$ there are two disjoint segments of \bar{P} contained in t , belonging to T . This proves condition (D) for the partition T of \bar{P} . Hence any ramified set P possessing property (2) has a subset P_1 which can be represented by a partition T of \bar{P} satisfying (D). This proves 2.1. It also follows that

2.3. A ramified table (and hence a ramified set) possessing property (2) has a cofinal subset which can be represented by a partition of an ordertype.

2.4. The above arguments also prove that a ramified table P possessing property (2) can be represented by the segments of a partition T of \bar{P} (the correspondence being order-preserving but not necessarily one-one). This result leads to the answer of question (3) of Kurepa. Consider the mapping of the ramified table P on to the partition T of \bar{P} defined in 2.1 above. Let R be a reordering of the aggregate of P , in which the elements X_1, X_2 of P are connected by the relation $X_1 < X_2$ if and only if the corresponding elements t_1, t_2 of T are such that every element of t_1 is <

every element of t_2 in the ordertype \bar{P} . This new relation is evidently transitive and assymmetric, so that by postulating also the relation $X < X$, R is a partial ordering of the aggregate of \bar{P} . Also it is clear that any two elements of the aggregate of P are comparable in P if and only if they are incomparable in R . Hence every maximal chain (antichain) of P corresponds to a maximal antichain (chain) of R and conversely. But by 1.2, P has the following property, which is a consequence of property (2):—

If C is any maximal chain of P , every maximal antichain of $P - C$ is a maximal antichain of P also.

Hence R has the property: —

If A is any maximal antichain of R , every maximal chain of $R - A$ is a maximal chain of R also.

This is the property referred to in question (3) of Kurepa. Hence every ramified table possessing property (2) can be reordered so that every pair of comparable (incomparable) elements of the old ordering becomes incomparable (comparable) in the new ordering, and the new ordering has the property mentioned in question (3).

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REFERENCES

- Hausdorff (1908). Theorie der geordneten Mengen. *Math. Ann.*, 65, 435–505.
 Kurepa, G. (1935). Ensembles ordonnées et ramifiées Thesis (Paris).
 ——— (1950). Ensembles partiellement ordonnés et ensembles partiellement bien ordonnés. Publications de l'institut mathématique, 3, 119–125.
 ——— (1952). On a characteristic property of finite sets. *Pac. Journ. Math.*, 2, 323–326.
 Sheperdson, J. C. (1954). On two problems of G. Kurepa. *Pac. Journ. Math.*, 4, 301.

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AN ESSENTIALLY STATISTICAL APPROACH TO THERMODYNAMIC PROBLEM—II

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INTRODUCTION

In a recent paper * (1953) the author has stressed the necessity of an essentially statistical approach to the thermodynamic problems and has developed a general statistical method for investigating thermodynamic behaviour of a system. There, for simplicity, the system has been assumed to be composed of matter of (chemically) single type, and the contribution of radiation energy, if any, in the behaviour of the system has been taken to be negligibly insignificant. In the present paper, the proposed statistical method will be shown to be equally suitable for investigations of thermodynamic behaviour of system of different types, viz. those of black-body radiations, those of free radiation and matter of single chemical types, and those of matter of chemically different types. In agreement with other usual theories and with the assumptions of the paper I, it is assumed that the amount of energy and matter can only be discrete. In every case, the usual formulæ, as given by Fowler (1936), Tolman (1946) and others, have been obtained.

PART I. AN ASSEMBLY OF FREE RADIATION

1. *Formulation of Problem: The Law of Probability-distribution*

Here, the system means a definite amount of energy E , in the form of free radiation in a definite volume in a definite environment. As in the paper I, the system is to be taken as open, i.e. there is no hindrance of energy-exchange between the system and the environment. Now, since there is no question of matter-exchange in this case with the environment, so in consistency with the trend of discussions made in the paper I, only one a priori probability z for each unit of energy to be in the volume under consideration will be introduced. Then, the probability of occurrences of a system in a state specified by E is given by

$$P(z, E) = \frac{W(E)z^E}{\sum_{E=0}^{\infty} W(E)z^E} = \frac{W(E)z^E}{f(z)}, \quad \dots \dots \dots (1)$$

where
$$f(z) = \sum_{E=0}^{\infty} W(E)z^E \quad \dots \dots \dots (2)$$

The above may be looked upon as a form of Baye's theorem. As before, for determining z in agreement with the usual ideas of statistical equilibrium, the 'real' value E_0 (taken to be known in the sense of classical thermodynamics) corresponds

* This paper will be referred to as the paper I in the present discussion.

$$\forall \theta, \quad \ddot{\theta} = -\frac{g}{L} \sin \theta \quad (7)$$

Thus, $\Phi(z)$, i.e. $P(z, E)$, has only one maximum. Therefore z_0 is the unique value of z corresponding to the stationary state.

4. Influence of Variations of Environments

Now as in the paper I, if the variations of parameters, involved in the definitions of the environments, be taken into account, then the equation of continuity of energy after usual averaging can be written as

$$\overline{dE} = \sum_i \frac{\partial \overline{E}}{\partial x_i} dx_i + d'Q, \quad \dots \quad (8)$$

where

$$\left(-\frac{\partial \overline{E}}{\partial x_i}\right) = \frac{\sum_{E=0}^{\infty} \left(-\frac{\partial E}{\partial x_i}\right) W(E) z_0^E}{\sum_{E=0}^{\infty} W(E) z_0^E} = -\left(\log \frac{1}{z_0}\right) \cdot \frac{\partial}{\partial x_i} \{\log f(z_0)\} \quad (9)$$

Now

$$\begin{aligned} \left(\log \frac{1}{z_0}\right) d'Q &= \left(\log \frac{1}{z_0}\right) \left\{ \overline{dE} - \sum_i \frac{\partial \overline{E}}{\partial x_i} dx_i \right\} \\ &= d \left\{ \left(\log \frac{1}{z_0}\right) \overline{E} \right\} + \sum_i \frac{\partial}{\partial x_i} \{\log f(z_0)\} dx_i + \frac{\partial}{\partial z_0} \{\log f(z_0)\} \\ &= d \left\{ \left(\log \frac{1}{z_0}\right) \overline{E} + \log f(z_0) \right\} \quad \dots \quad (10) \end{aligned}$$

Then, on putting

$$z_0 = e^{-\frac{1}{kT}}, \quad \dots \quad (11)$$

after choosing the scale properly as in the paper I, one has

$$dS = d \left\{ \frac{E}{T} + k \log F(T) \right\}. \quad \dots \quad (12)$$

5. Laws of Microscopic Distributions

Now, here if the radiation be assumed to be composed of the photon and if number of ways in which a particular n_ν (number of photons) with energy ϵ_ν be realised when the total energy of the system E is denoted by $w'(E - n_\nu \epsilon_\nu)$, then the expected value of \bar{n}_ν is given by

$$\begin{aligned} \bar{n}_\nu &= \frac{\sum_{E=0}^{\infty} \sum_{n_\nu=0}^{\infty} n_\nu w'(E - n_\nu \epsilon_\nu) z_0^E}{\sum_{E=0}^{\infty} W(E) z_0^E} \\ &= \frac{\left(\sum_{n_\nu=0}^{\infty} n_\nu z_0^{n_\nu \epsilon_\nu} \right) \left[1 + \sum_E w'(E) z_0^E \right]}{\left(\sum_{n_\nu=0}^{\infty} z_0^{n_\nu \epsilon_\nu} \right) \left[1 + \sum_E w'(E) z_0^E \right]} \end{aligned}$$

$$= z_0^{\epsilon_\nu} \frac{\partial}{\partial (z_0^{\epsilon_\nu})} \left\{ \log \left(1 + \sum_{\nu} z_0^{\epsilon_\nu} \right) \right\} \dots \dots \dots (13)$$

Then, the spectral-distribution of energy becomes

$$\bar{n}_\nu = \frac{A_\nu z_0^{\epsilon_\nu}}{1 - z_0^{\epsilon_\nu}} = \frac{A_\nu}{e^{\epsilon_\nu/kT}} \dots \dots \dots (14)$$

This is the well-known Planck's law.

Remarks

Now, the assumption of existence of $W(E)$, the total number of realizations in which E can be realized in different ways in different frequencies, implies the exchange of energies amongst different frequencies, i.e. the existence of some matter in contact with the system consisting of radiation under consideration has been tacitly assumed. So, the present case is the case of black-body radiation.

PART II. AN ASSEMBLY CONSISTING OF RADIATION AND MATTER

1. Description of Assembly

Here, as in the paper I, a system in a definite thermodynamic state will mean a definite amount of energy and a definite amount of matter observed (in the sense of classical thermodynamics) in a definite volume in a definite environment. Here, as in the paper I, the system will be taken to be open with respect to energy and matter. Of course, for simplicity at present, the system will be assumed to be composed of (chemically) single type of material.

Now, as before, a constant a priori probability for a unit quantity of energy to exist in the volume and another constant a priori probability for a unit quantity of matter to exist in the volume will be introduced. Then, the probability of occurrence of the system in a state specified by E and M is given by

$$P(t, z, M, E) = \frac{W(E, M) t^M z^E}{\sum_E \sum_M W(E, M) t^M z^E} = \frac{W(E, M) t^M z^E}{f(t, z)}, \dots \dots (15)$$

where

$$f(t, z) = \sum_E \sum_M W(E, M) t^M z^E. \dots \dots (16)$$

This expression is same as that in the previous paper (Dutta, 1953). The only difference is that here $W(E, M)$ can always be written as

$$\sum_{E_1 + E_2 = E} W_1(E_1) W_2(E_2, M) = W(E, M), \dots \dots (17)$$

where E_1 is the amount of energy in the form of free radiation and E_2 that as the energy of matter. Obviously $W_1(E_1)$ is the function used in the part I of this paper and $W_2(E_2)$ is that used in the paper I. Then, from the theorem, stated and proved in the paper I, one gets

$$f(t, z) = f_1(z) f_2(t, z). \dots \dots (18)$$

As before, the distribution-parameters t, z corresponding to the real values of E_0, M_0 (taken to be known from the observations in classical sense) are to be

specified from the principle that the real values of E_0 and M_0 correspond to the maximum of the probability of occurrence considered as function of t and z .

Specification and Uniqueness of t_0 and z_0 , etc.

In this case, equations specifying values of t_0 and z_0 and calculations showing the uniqueness of specified values and discussions of the effects of variations in the environments are same in the previous paper and so are not repeated here.

Laws of Microscopic Distributions

The previous discussions are mainly of microscopic nature and have been put forward without any specific assumptions about the microscopic nature of the system. Here, matter will be assumed to be composed of molecules and radiation of photons. As stated in the paper I, E_1 and E_2 are both to be taken as discrete. Let us write

$$w(E, N) = W(E, M) = w_1(E_1)w_2(E_2, N). \quad \dots \quad (19)$$

Let ϵ_ν be the energy of one of the micro-energy state of photons and n_ν be the number of photons with energy ϵ_ν at any instant. Then, on writing

$$t'_0 = t_0^m, \quad \dots \quad (20)$$

one gets

$$\begin{aligned} n_\nu &= \frac{\sum_{E=0}^{\infty} \sum_{N=0}^{\infty} \sum_{n_\nu=0}^{\infty} n_\nu w'_1(E_1 - n_\nu \epsilon_\nu) w_2(E_2, N) t_0'^N z_0^E}{\sum_{E=0}^{\infty} \sum_{N=0}^{\infty} \sum_{E_1+E_2=0}^{\infty} w'_1(E_1) w'_2(E_2, N) t_0'^N z_0^E} \\ &= \frac{\left(\sum_{n_\nu=0}^{\infty} n_\nu z_0^{n_\nu \epsilon_\nu} \right) \left[1 + \sum w'_1(E_1) w'_2(E_2, N) t_0'^N z_0^E \right]}{\left(1 + \sum_{n_\nu=1}^{\infty} z_0^{n_\nu \epsilon_\nu} \right) \left[1 + \sum \sum w'_1(E_1) w'_2(E_2, N) t_0'^N z_0^E \right]} \\ &= z_0^{\epsilon_\nu} \frac{\partial}{\partial z_0^{\epsilon_\nu}} \left\{ \log \left(1 + \sum_{n_\nu=0}^{\infty} z_0^{n_\nu \epsilon_\nu} \right) \right\}, \\ t'_0 &= t_0^m \quad \dots \quad (21) \end{aligned}$$

where $w'_1(E)$ is the number of partition of E_1 , into sets of $\epsilon_0, \epsilon_1, \dots, \epsilon_\nu, \dots$ of which no part is equal to ϵ_ν . Then the spectral-distribution of radiation energy becomes

$$\bar{n}_\nu = \frac{A_\nu z_0^{\epsilon_\nu}}{1 - z_0^{\epsilon_\nu}} = \frac{A_\nu}{e^{\epsilon_\nu/kT} - 1}. \quad \dots \quad (22)$$

Similarly, the microscopic distribution for molecules with respect of energy can be obtained. This is again same as that in the paper I.

Formula for fluctuations can be obtained similarly and similar to these in the previous paper.

PART III. ASSEMBLY COMPOSED OF MATTER OF TWO DIFFERENT TYPES

1. *Description of System: Law of Probability-distribution*

The system means certain amount of E and certain amounts M_1 and M_2 of matters of two (chemically) different types, contained in a definite volume in a definite environment. For simplicity, it will be assumed that there is no chemical reaction in the assembly and that the condition of the system and the environment is such that the contribution in the energy due to the free radiation in the system may be taken as insignificantly small and so will be neglected.

Here, as before, a constant a priori z for each unit of energy to occur within the volume under consideration and constant a priori probabilities t_1 and t_2 for unit masses of matter of types (1) and (2) to be in the volume respectively will be introduced. Then, the probability of occurrences of the system to be in a state specified by E , M_1 and M_2 is given by

$$\begin{aligned} P(t_1, t_2, z, E, M_1, M_2) &= \frac{W(E, M_1, M_2) z^E t_1^{M_1} t_2^{M_2}}{\Sigma \Sigma \Sigma W(E, M_1, M_2) z^E t_1^{M_1} t_2^{M_2}} \\ &= \frac{W(E, M_1, M_2) z^E t_1^{M_1} t_2^{M_2}}{f(t_1, t_2, z)}, \quad \dots \quad (23) \end{aligned}$$

where

$$f(t_1, t_2, z) = \sum_E \sum_{M_1} \sum_{M_2} W(E, M_1, M_2) z^E t_1^{M_1} t_2^{M_2} \quad \dots \quad (24)$$

The above may be looked upon as same form of Baye's theorem. As before, for determining z , t_1 , t_2 for the distribution law, in agreement with usual ideas of statistical equilibrium, the real values of E_0 , M_{10} , M_{20} (taken to be known from the observations in the classical sense) are assumed to correspond to the maximum of the probability of occurrence considered as functions of z , t_1 , t_2 for fixed values of $E = E_0$, $M_1 = M_{10}$, $M_2 = M_{20}$. Thus, the estimation of parameters z , t_1 and t_2 is similar to that in statistics by principle of maximum likelihood.

2. *Behaviour of Functions of $W(E, M_1, M_2)$, $f(t_1, t_2, z)$, etc.*

All these functions are similar to those in the paper I, and in the parts I and II of this paper, and some similar properties. As in the paper I and the part II of this paper, it can be easily shown that

$$f(t_1, t_2, z) = f_1(t_1, z) f_2(t_2, z), \quad \dots \quad (25)$$

where

$$W(E, M_1, M_2) = \sum_{E_1 + E_2 = E} W_1(E_1, M_1) W_2(E_2, M_2), \quad \dots \quad (26)$$

$$f_1(t_1, z) = \sum_{E_1} \sum_{M_1} W_1(M_1, E_1) z^{E_1} t_1^{M_1}, \quad \dots \quad (27)$$

$$f_2(t_2, z) = \sum_{E_2} \sum_{M_2} W_2(M_2, E_2) z^{E_2} t_2^{M_2}, \quad \dots \quad (28)$$

and

$$E = E_1 + E_2, \quad \dots \quad (29)$$

E being the total energy of the system and E_1 , E_2 being energies shared by matters of types (1) and (2).

As before,

$$P(t_1, t_2, z) > 0, \quad \dots \dots \dots (30)$$

and for infinite rarefactions, $t_1 = 0, t_2 = 0, z = 0$,

$$P(t_1, t_2, z) = 0, \quad \dots \dots \dots (31)$$

and for infinite condensation, $t_1 = 1, t_2 = 1, z = 1$,

$$P(t_1, t_2, z) = 0. \quad \dots \dots \dots (32)$$

Then, $P(t_1, t_2, z)$ must have at least one maximum uniqueness of maximum can be tested as in the paper I and the part I of this paper.

3. Influence of Variations of Environments on the System

As before, in order to take into account of slow changes in environment, the equation of continuity (in average) for energy can be written as

$$\overline{dE} = \sum_i \frac{\partial \overline{E}}{\partial x_i} dx_i + d'Q, \quad \dots \dots \dots (33)$$

where x_i 's are the parameters involved in the definitions of the environments and

$$\left(-\frac{\partial \overline{E}}{\partial x_i}\right) = -\frac{1}{\left(\log \frac{1}{z_0}\right)} \frac{\partial}{\partial x_i} \{\log f(t_{10}, t_{20}, z_0)\}. \quad \dots \dots (34)$$

Then,

$$\begin{aligned} \left(\log \frac{1}{z_0}\right) d'Q &= \left(\log \frac{1}{z_0}\right) \left\{ \overline{dE} - \sum \frac{\partial \overline{E}}{\partial x_i} dx_i \right\} \\ &= d \left\{ \left(\log \frac{1}{z_0}\right) \overline{E} \right\} + \frac{\partial}{\partial z_0} \{\log f(t_{10}, t_{20}, z_0)\} + \sum_i \frac{\partial}{\partial x_i} \{\log f(t_{10}, t_{20}, z_0)\} \\ &= d \left\{ \left(\log \frac{1}{z_0}\right) E_0 + \log f(t_{10}, t_{20}, z_0) - M_{10} \log t_{10} - M_{20} \log t_{20} \right\}. \dots (35) \end{aligned}$$

Now, according to the argument advanced in the paper I,

$$z_0 = e^{-\frac{1}{kT}}. \quad \dots \dots \dots (36)$$

Then, the entropy is given by

$$S = \frac{E}{kT} + \log F(t_{10}, t_{20}, T) - M_{10} \log t_{10} - M_{20} \log t_{20}. \quad \dots \dots (37)$$

4. Laws of Microscopic Distributions

Now, as before, for deducing microscopic distributions, the system will be assumed to be composed of N_1 and N_2 numbers of particles, viz. molecules or atoms, where

$$M_1 = N_1 m_1, M_2 = N_2 m_2. \quad \dots \dots \dots (38)$$

If the number of ways, in which a particular $n_{1\epsilon}$, (number of molecules of type (1)) is realized with energy $\epsilon_{1\epsilon}$, when the state of system is specified by M_{10}, M_{20}

and E_0 , is denoted by $w'_1(E - n_{1\nu}\epsilon_{1\nu}, N_1 - n_{1\nu}, N_2)$, then the expected value of $n_{1\nu}$ is given by

$$\begin{aligned}
 n_{1\nu} &= \frac{\sum_{E=0}^{\infty} \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \sum_{n_{1\nu}=0}^{N_1} n_{1\nu} w'_1(E - n_{1\nu}\epsilon_{1\nu}, N_1 - n_{1\nu}, N_2) t_1'^{N_1} t_2'^{N_2} z_0^E}{\sum_{E=0}^{\infty} \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} W(E, N_1, N_2) t_1'^{N_1} t_2'^{N_2} z_0^E} \\
 &= \frac{\left(\sum_{n_{1\nu}=0}^{\infty} n_{1\nu} t_1'^{n_{1\nu}} z_0^{n_{1\nu}\epsilon_{1\nu}} \right) \left[1 + \sum_E \sum_{N_1} \sum_{N_2} w'(E, N_1, N_2) t_1'^{N_1} t_2'^{N_2} z_0^E \right]}{\left(1 + \sum_{n_{1\nu}=1}^{\infty} t_1'^{n_{1\nu}} z_0^{n_{1\nu}\epsilon_{1\nu}} \right) \left[1 + \sum_E \sum_{N_1} \sum_{N_2} w'(E, N_1, N_2) t_1'^{N_1} t_2'^{N_2} z_0^E \right]} \\
 &= \frac{t_1' \frac{\partial}{\partial t_1} \left(1 + \sum_{n_{1\nu}=1}^{\infty} t_1'^{n_{1\nu}} z_0^{n_{1\nu}\epsilon_{1\nu}} \right)}{\left(1 + \sum_{n_{1\nu}=0}^{\infty} t_1'^{n_{1\nu}} z_0^{n_{1\nu}\epsilon_{1\nu}} \right)} \\
 &= t_1' \frac{\partial}{\partial t_1} \left\{ \log \left(1 + \sum_{n_{1\nu}=0}^{\infty} t_1'^{n_{1\nu}} z_0^{n_{1\nu}\epsilon_{1\nu}} \right) \right\} \\
 &= t_1' \frac{\partial}{\partial t_1} \{ \log g_\nu(t_1', z_0) \}, \quad \dots \dots \dots (39)
 \end{aligned}$$

where

$$\left. \begin{aligned} t_1' &= t_{10}^{m_1} \\ t_2' &= t_{20}^{m_2} \end{aligned} \right\} \dots \dots \dots (40)$$

Similar are the expressions for $n_{2\nu}$.

Thus, usual discussions for microscopic distributions will follow as before. The discussion on fluctuations can also be made and standard existing formulae (Tolman, 1946) can be deduced as in the paper I.

General Concluding Remarks

From the discussions in this paper, it is evident that the extension of the method proposed in the paper I to the general cases, viz. of the multi-component systems containing radiation in absence of chemical reactions, is easy and straightforward. Thus, this method, without losing any of the advantageous characteristics, mentioned in the paper I, is powerful enough to yield the usual formulae in the form as given by Fowler (1936) for general cases. The discussions of crystals and of chemical reactions by this method will be subject-matter of future communications.

ABSTRACT

In this paper, the thermodynamic behaviour of systems of different types, viz. those of black-body radiation, of matter and radiation, and of mixtures, has been investigated by the general statistical method, developed in a paper (Dutta, 1953) previously. Expressions for

thermodynamic functions, microscopic distributions, fluctuations, etc., have been obtained in the form given by Fowler. As in the previous paper (Dutta, 1953), it is seen that for investigations of microscopic properties, the introduction of any assumption about microscopic nature is not at all necessary. This assumption is to be introduced only for discussions of microscopic distributions or the like.

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REFERENCES

- Dutta, M. (1953). An Essentially Statistical Approach to the Thermodynamic Problem. *Proc. Nat. Inst. Sc. Ind.*, 19, 109.
Fowler, R. H. (1936). *Statistical Mechanics*, Camb. Univ. Press, 2nd Edn. Cambridge.
Tolman, R. C. (1946). *The Principles of Statistical Mechanics*, Oxford University Press, 929-43.

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SOME PROPERTIES OF GENERALISED LAPLACE TRANSFORM

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1. The Laplace transform is defined by the equation

$$f(p) = p \int_0^{\infty} e^{-px} h(x) dx \quad \dots \quad \dots \quad \dots \quad (1)$$

when the integral is convergent and $R(p) > 0$.

R. S. Varma (1951) generalised (1) by the relation

$$\phi(p) = p \int_0^{\infty} e^{-\frac{1}{2}px} (px)^{m-\frac{1}{2}} W_{k,m}(px) h(x) dx \quad \dots \quad \dots \quad (2)$$

which reduces to (1) when $k = -m + \frac{1}{2}$ by virtue of the identity

$$W_{-m+\frac{1}{2},m}(z) \equiv z^{-m+\frac{1}{2}} e^{-\frac{1}{2}z} \quad \dots \quad \dots \quad \dots \quad (3)$$

We shall denote (2) by

$$\phi(p) \stackrel{v}{=}_{k,m} h(x) \quad \dots \quad \dots \quad \dots \quad (4)$$

and as usual, (1) will be represented by

$$f(p) \doteq h(x).$$

In this paper a few theorems involving (1) and (2) are proved and some interesting results deduced.

2. In this section we collect together the generalised transforms of some functions. These have been obtained by taking suitable functions for $h(x)$ in (2) and evaluating the integrals with the help of known results (Erdelyi, 1939; Goldstein, 1932; Meijer, 1936; Pasricha, 1943; Rathie, 1953, 1954). Some of these will be required in our investigations later on.

$$x^{\nu-1} \stackrel{v}{=}_{k,m} \frac{\Gamma(\nu) \Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})} p^{1-\nu}, \quad R(\nu) > 0, R(\nu+2m) > 0. \quad \dots \quad (5)$$

$$e^{-ax} x^{\nu-1} \stackrel{v}{=}_{k,m} \frac{\Gamma(\nu) \Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})} p^{1-\nu} {}_2F_1\left(\nu, \nu+2m; \nu+m-k+\frac{1}{2}; -\frac{a}{p}\right), \quad (6)$$

$$R(\nu) > 0, R(\nu+2m) > 0, R(p) > R(a) > 0.$$

$$x^{-\lambda-m-\frac{1}{2}} (1+x)^{k+\lambda-1} \stackrel{v}{=}_{k,m} \frac{\Gamma(\frac{1}{2}-\lambda+m) \Gamma(\frac{1}{2}-\lambda-m)}{\Gamma(1-\lambda-k)} p^{m+\frac{1}{2}} e^{\frac{1}{2}p} W_{\lambda,m}(p), \quad (7)$$

$$R(\frac{1}{2}-\lambda \pm m) > 0.$$

$$x^{-\lambda-\mu-2m-\frac{1}{2}} {}_2F_1\left(\frac{1}{2}-\lambda+\mu, 1-\lambda-\mu-k-m; \frac{1}{2}-\lambda-\mu-2m; -x\right) \\ = \sum_{k,m} \frac{\Gamma(\frac{1}{2}-\lambda-\mu) \Gamma(\frac{1}{2}-\lambda-\mu-2m)}{\Gamma(1-\lambda-\mu-k-m)} p^{2m+\mu+\frac{1}{2}} e^{\frac{1}{2}p} W_{\lambda,\mu}(p), \quad \dots \quad (8)$$

$$R(\frac{1}{2}-\lambda-\mu) > 0, R(\frac{1}{2}-\lambda-\mu-2m) > 0.$$

$$x^{c-2m-1} {}_3F_2\left(\frac{1}{2}-\lambda+\mu, \frac{1}{2}-\lambda-\mu, \frac{1}{2}+c-k-m; c, c-2m; -x\right) \\ = \sum_{k,m} \frac{\Gamma(c) \Gamma(c-2m)}{\Gamma(\frac{1}{2}+c-k-m)} p^{2m-c-\lambda+1} e^{\frac{1}{2}p} W_{\frac{1}{2},\mu}(p), \quad \dots \quad (9)$$

$$R(c) > 0, R(c-2m) > 0.$$

$$x^{\nu-1} {}_rF_s(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; \pm x) \\ = \sum_{k,m} \frac{\Gamma(\nu) \Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})} p^{1-\nu} {}_{r+2}F_{s+1} \left\{ \alpha_1, \dots, \alpha_r, \nu, \nu+2m; \beta_1, \dots, \beta_s, \nu+m-k+\frac{1}{2}; \pm \frac{1}{p} \right\}, \quad \dots \quad (10)$$

$$R(\nu) > 0, R(\nu+2m) > 0, R(p) > 0 (r < s); R(p) > 1 (r = s).$$

$$x^{\nu-1} {}_rF_s(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; \pm x^2) \\ = \sum_{k,m} \frac{\Gamma(\nu) \Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})} p^{1-\nu} {}_{r+4}F_{s+2} \left\{ \alpha_1, \dots, \alpha_r, \frac{1}{2}\nu, \frac{1}{2}(\nu+1), \frac{1}{2}(\nu+2m), \frac{1}{2}(\nu+2m+1); \beta_1, \dots, \beta_s, \frac{1}{2}(\nu+m-k+\frac{1}{2}), \frac{1}{2}(\nu+m-k+\frac{3}{2}); \pm \frac{4}{p^2} \right\}, \quad \dots \quad (11)$$

$$R(\nu) > 0, R(\nu+2m) > 0; R(p) > 0 (s > r+1); R(p) > 2 (s = r+1).$$

$$e^{-ax} x^{\nu-1} {}_rF_s(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; \pm x^\lambda) \\ = \sum_{k,m} p^{1-\nu} \sum_{n=0}^{\infty} \left\{ \frac{(\alpha_1)_n \dots (\alpha_r)_n \Gamma(\nu+\lambda n) \Gamma(\nu+2m+\lambda n)}{(\beta_1)_n \dots (\beta_s)_n \Gamma(\nu+m-k+\frac{1}{2}+\lambda n)} (\pm p)^{-\lambda n} \right. \\ \left. \times {}_2F_1\left(\nu+\lambda n, \nu+2m+\lambda n; \nu+m-k+\frac{1}{2}+\lambda n; -\frac{a}{p}\right) \right\}, \quad \dots \quad (12)$$

$$R(\nu) > 0, R(\nu+2m) > 0; R(a+p) > 0 \text{ and } p < 1 \text{ when } \lambda < s-r+1;$$

$$R(a+p) > s-r+1 \text{ and } |p| < 1 \text{ when } \lambda = s-r+1; \lambda \text{ being real and positive.}$$

$$e^{-\frac{a}{2x}} x^{k-m-\frac{1}{2}} = \sum_{k,m} 2a^{k-\frac{1}{2}} p^{m+1} K_{2m}(2\sqrt{ap}) \quad \dots \quad (13)$$

$$e^{-\frac{1}{2x}} x^{\frac{1}{2}(\mu+k-m-\frac{1}{2})} W_{-\frac{1}{2}(3m+\mu+k-\frac{1}{2}), \frac{1}{2}(m+\mu-k+\frac{1}{2})} \left(\frac{1}{x}\right) = \sum_{k,m} 2p^{1-\frac{1}{2}\mu} K_{\mu}(2\sqrt{p}) \quad (14)$$

$$x^{2k-m-\frac{1}{2}} e^{-\frac{1}{2x}} W_{\frac{1}{2}-k, m} \left(\frac{1}{x}\right) = \sum_{k,m} \sqrt{\frac{2}{\pi}} p^{m-k+\frac{1}{2}} K_{2k-1}(\sqrt{p}) K_{2m}(\sqrt{p}) \quad \dots \quad (15)$$

$$x^{-m-1} e^{-\frac{1}{2x}} D_{-2k} \sqrt{\frac{2}{x}} = \sum_{k,m} \sqrt{\pi} 2^{-k} p^{m+\frac{1}{2}} W_{k,m}(2\sqrt{p}) W_{-k,m}(2\sqrt{p}) \quad \dots \quad (16)$$

$$x^{-m} e^{-\frac{1}{2x}} W_{k,m} \left(\frac{1}{x}\right) = \sum_{k,m} \sqrt{\pi} 2^{k-2k} p^{m+\frac{1}{2}} e^{-\sqrt{p}} W_{2k-\frac{1}{2}, 2m}(2\sqrt{p}) \quad \dots \quad (17)$$

$$x^{-m-1} e^{\frac{1}{2x}} D_{2k-1} \left(\sqrt{\frac{2}{x}} \right) \underset{k, m}{=} \frac{\Gamma(\frac{1}{2}+m-k) \Gamma(\frac{1}{2}-m-k)}{\Gamma(1-2k)} \\ \times 2^{-k-\frac{1}{2}} p^{m+\frac{1}{2}} W_{k, m}(2i\sqrt{p}) W_{k, m}(-2i\sqrt{p}), \quad \dots \quad (18)$$

$$R(\frac{1}{2}-k \pm m) > 0.$$

$$e^{\frac{1}{2}x} x^{\lambda-\lambda-m-\frac{1}{2}} W_{\lambda, \mu}(x) \\ \underset{k, m}{=} \frac{\Gamma(k-m-\lambda+\mu) \Gamma(k-m-\lambda-\mu) \Gamma(k+m-\lambda+\mu) \Gamma(k+m-\lambda-\mu)}{\Gamma(\frac{1}{2}-\lambda-\mu) \Gamma(\frac{1}{2}-\lambda+\mu) \Gamma(2k-2\lambda)} \\ \times p {}_2F_1(k-\lambda-m+\mu, k-\lambda-m-\mu; 2k-2\lambda; 1-p), \quad \dots \quad (19)$$

$$R(k-\lambda \pm m \pm \mu) > 0.$$

$$x^{\gamma-1} (1+x)^{-\beta} \underset{k, m}{=} \frac{p^{1-\gamma}}{\Gamma(\beta)} E(\beta, \gamma, \gamma+2m; \gamma+m-k+\frac{1}{2}; p), \quad \dots \quad (20)$$

$$R(\gamma) > 0, R(\gamma+2m) > 0.$$

$$x^{\gamma-1} {}_2F_1(\beta, \gamma+m-k+\frac{1}{2}; \delta; -x) \\ \underset{k, m}{=} \frac{\Gamma(\delta)}{\Gamma(\beta) \Gamma(\gamma+m-k+\frac{1}{2})} p^{1-\gamma} E(\beta, \gamma, \gamma+2m; \delta; p), \quad \dots \quad (21)$$

$$R(\gamma) > 0, R(\gamma+2m) > 0.$$

$$x^{\gamma-1} {}_3F_2(\alpha, \beta, \gamma+m-k+\frac{1}{2}; \delta, \gamma+2m; -x) \\ \underset{k, m}{=} \frac{\Gamma(\delta) \Gamma(\gamma+2m)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma+m-k+\frac{1}{2})} p^{1-\gamma} E(\alpha, \beta, \gamma; \delta; p), \quad \dots \quad (22)$$

$$R(\gamma) > 0, R(\gamma+2m) > 0.$$

$$x^{\lambda-1} {}_4F_3(\alpha, \beta, \gamma, \lambda+m-k+\frac{1}{2}; \delta, \lambda, \lambda+2m; -x) \\ \underset{k, m}{=} \frac{\Gamma(\delta) \Gamma(\lambda) \Gamma(\lambda+2m)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma+m-k+\frac{1}{2})} p^{1-\lambda} E(\alpha, \beta, \gamma; \delta; p), \quad \dots \quad (23)$$

$$R(\lambda) > 0, R(\lambda+2m) > 0.$$

$$x^{\gamma-1} E\left(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; \frac{1}{x}\right) \\ \underset{k, m}{=} p^{1-\gamma} E(\alpha_1, \dots, \alpha_r, \gamma, \gamma+2m; \beta_1, \dots, \beta_s, \gamma+m-k+\frac{1}{2}; p), \quad \dots \quad (24)$$

$$R(\gamma) > 0, R(\gamma+2m) > 0.$$

$$\frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} x^{\lambda-m-\frac{1}{2}} {}_2F_1\left(a, b; c; -\frac{1}{x}\right) \\ \underset{k, m}{=} \sum_{m, -m} \left\{ \frac{\Gamma(-2m) \Gamma(a+k+m-\frac{1}{2}) \Gamma(b+k+m-\frac{1}{2})}{\Gamma(c+k+m-\frac{1}{2})} p^m \right. \\ \left. \times {}_2F_2(a+k+m-\frac{1}{2}, b+k+m-\frac{1}{2}; c+k+m-\frac{1}{2}, 1+2m; p) \right\}, \quad \dots \quad (25)$$

$$R(a+k \pm m - \frac{1}{2}) > 0, R(b+k \pm m - \frac{1}{2}) > 0.$$

3. THEOREM I. If

$$\phi(p) \stackrel{v}{=}_{k, m} h(x)$$

and

$$f(p) \stackrel{v}{=}_{\sigma, \mu} x^{2m-2\mu} h(x)$$

then

$$\begin{aligned} \phi(p) &= \frac{p^{m+k-\sigma-\mu+1}}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty t^{\sigma-k+m-\mu-1} (p+t)^{-1} f(p+t) \\ &\quad \times {}_2F_1\left(\frac{1}{2}-k+m, \sigma+\mu-k-m; \sigma-\mu-k+m; -\frac{t}{p}\right) dt, \end{aligned} \quad (26)$$

provided that $R(\sigma-k+m-\mu) > 0$ and the integral is convergent.*Proof.* We have

$$\phi(p) = p \int_0^\infty e^{-\frac{1}{2}px} (px)^{m-\frac{1}{2}} W_{k, m}(px) h(x) dx.$$

But Erdelyi (1939) has shown that

$$\begin{aligned} W_{k, m}(z) &= \frac{z^{k-\sigma}}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty e^{-\frac{1}{2}u} u^{\sigma-k+m-\mu-1} W_{\sigma, \mu}(u+z) \\ &\quad \times \left(1 + \frac{u}{z}\right)^{\mu-\frac{1}{2}} {}_2F_1\left(\frac{1}{2}-k+m, \sigma+\mu-k-m; \sigma-\mu-k+m; -\frac{u}{z}\right) du, \end{aligned}$$

 $R(\sigma-k+m-\mu) > 0$.If we put $z = px$ and $u = xt$, this takes the form

$$\begin{aligned} W_{k, m}(px) &= \frac{(px)^{k-\sigma}}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty e^{-\frac{1}{2}xt} (xt)^{\sigma-k+m-\mu-1} W_{\sigma, \mu}\{(p+t)x\} \\ &\quad \times \left(1 + \frac{t}{p}\right)^{\mu-\frac{1}{2}} {}_2F_1\left(\frac{1}{2}-k+m, \sigma+\mu-k-m; \sigma-\mu-k+m; -\frac{t}{p}\right) x dt. \end{aligned}$$

Hence

$$\begin{aligned} \phi(p) &= \frac{p}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty e^{-\frac{1}{2}px} (px)^{k-\sigma+m-\frac{1}{2}} h(x) dx \int_0^\infty e^{-\frac{1}{2}xt} (xt)^{\sigma-k+m-\mu-1} \\ &\quad \times W_{\sigma, \mu}\{(p+t)x\} \left(1 + \frac{t}{p}\right)^{\mu-\frac{1}{2}} {}_2F_1\left(\frac{1}{2}-k+m, \sigma+\mu-k-m; \sigma-\mu-k+m; -\frac{t}{p}\right) x dt \\ &= \frac{p^{k-\sigma+m-\mu+1}}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty t^{\sigma-k+m-\mu-1} {}_2F_1\left(\frac{1}{2}-k+m, \sigma+\mu-k-m; \sigma-\mu-k+m; -\frac{t}{p}\right) dt \\ &\quad \times \int_0^\infty e^{-\frac{1}{2}(p+t)x} \{(p+t)x\}^{\mu-\frac{1}{2}} W_{\sigma, \mu}\{(p+t)x\} x^{2m-2\mu} h(x) dx \\ &= \frac{p^{k-\sigma+m-\mu+1}}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty t^{\sigma-k+m-\mu-1} (p+t)^{-1} f(p+t) \\ &\quad \times {}_2F_1\left(\frac{1}{2}-k+m, \sigma+\mu-k-m; \sigma-\mu-k+m; -\frac{t}{p}\right) dt. \end{aligned}$$

The change in the order of integration is permissible by the application of de la Vallée Poussin's theorem (Bromwich, 1908, p. 457) when the generalised Laplace transforms of the functions involved exist and the resulting integral is absolutely convergent.

When $\sigma = -\mu + \frac{1}{2}$, the theorem may be put in the following form (Rathie, 1955).

COROLLARY. If

$$\phi(p) \stackrel{v}{=}_{k, m} h(x)$$

and

$$f(p) \doteq x^\lambda h(x)$$

then

$$\begin{aligned} \phi(p) &= \frac{p^{m+k+\frac{1}{2}}}{\Gamma(\lambda-m-k+\frac{1}{2})} \int_0^\infty t^{\lambda-m-k-\frac{1}{2}} (p+t)^{-1} f(p+t) \\ &\quad \times {}_2F_1\left(\frac{1}{2}-k+m, \frac{1}{2}-k-m; \lambda-k-m+\frac{1}{2}; -\frac{t}{p}\right) dt \quad \dots (27) \end{aligned}$$

provided that $R(\lambda-m-k+\frac{1}{2}) > 0$ and the integral is convergent.

4. We shall now use the theorem to evaluate two infinite integrals.

Example 1. Taking (6)

$$h(x) = e^{-ax} x^{\nu-1}$$

$$\begin{aligned} &\stackrel{v}{=}_{k, m} \frac{\Gamma(\nu) \Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})} p^{1-\nu} {}_2F_1\left(\nu, \nu+2m; \nu+m-k+\frac{1}{2}; -\frac{a}{p}\right) \\ &= \phi(p), \quad R(\nu) > 0, \quad R(\nu+2m) > 0, \end{aligned}$$

we have

$$\begin{aligned} x^{2m-2\mu} h(x) &= e^{-ax} x^{2m-2\mu+\nu-1} \\ &\stackrel{v}{=}_{\sigma, \mu} \frac{\Gamma(2m+\nu) \Gamma(2m+\nu-2\mu)}{\Gamma(2m+\nu-\mu-\sigma+\frac{1}{2})} p^{2\mu-2m-\nu+1} \\ &\quad \times {}_2F_1\left(2m+\nu, 2m+\nu-2\mu; 2m+\nu-\mu-\sigma+\frac{1}{2}; -\frac{a}{p}\right) \\ &= f(p), \quad R(\nu+2m) > 0, \quad R(\nu+2m-2\mu) > 0. \end{aligned}$$

Applying the theorem and replacing $\frac{1}{2}-k+m$ by α , $\sigma+\mu-k-m$ by β , $\sigma-\mu-k+m$ by γ , and $\nu+2m$ by δ , we get

$$\begin{aligned} {}_2F_1\left(\nu, \delta; \nu+\alpha; -\frac{a}{p}\right) &= \frac{\Gamma(\nu+\alpha) \Gamma(\nu+\gamma-\beta)}{\Gamma(\nu) \Gamma(\gamma) \Gamma(\alpha+\nu-\beta)} p^{\nu-\beta} \\ &\quad \times \int_0^\infty t^{\nu-1} (p+t)^{\beta-\gamma-\nu} {}_2F_1\left(\nu+\gamma-\beta, \delta; \nu+\alpha-\beta; -\frac{a}{p+t}\right) \\ &\quad \times {}_2F_1\left(\alpha, \beta; \gamma; -\frac{t}{p}\right) dt, \quad \dots (28) \end{aligned}$$

valid, by analytic continuation (A.C.), for $R(\gamma) > 0$, $R(\alpha+\nu-\beta) > 0$, $R(\nu) > 0$, $R(p) > R(a) > 0$.

Example 2. Take (25)

$$\begin{aligned} h(x) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} x^{k-m-\frac{1}{2}} {}_2F_1\left(a, b; c; -\frac{1}{x}\right) \\ &= \sum_{k, m, -m}^v \left\{ \frac{\Gamma(-2m)\Gamma(a+k+m-\frac{1}{2})\Gamma(b+k+m-\frac{1}{2})}{\Gamma(c+k+m-\frac{1}{2})} p^m \right. \\ &\quad \times {}_2F_2(a+k+m-\frac{1}{2}, b+k+m-\frac{1}{2}; c+k+m-\frac{1}{2}, 1+2m; p) \left. \right\} \\ &= \phi(p), R(a+k\pm m-\frac{1}{2}) > 0, R(b+k\pm m-\frac{1}{2}) > 0. \end{aligned}$$

Then (Rathie, 1952, p. 240)

$$\begin{aligned} x^\lambda h(x) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} x^{\lambda+k-m-\frac{1}{2}} {}_2F_1\left(a, b; c; -\frac{1}{x}\right) \\ &= \frac{\Gamma(a)\Gamma(b)\Gamma(\lambda+k-m-\frac{1}{2})}{\Gamma(c)} p^{\frac{1}{2}+\lambda-k-\lambda} \\ &\quad \times {}_2F_2(a, b; c, m-k-\lambda+\frac{1}{2}; p) \\ &\quad + \frac{\Gamma(a+\lambda+k-m-\frac{1}{2})\Gamma(b+\lambda+k-m-\frac{1}{2})\Gamma(\frac{1}{2}+m-k-\lambda)}{\Gamma(c+\lambda+k-m-\frac{1}{2})} p \\ &\quad \times {}_2F_2(a+\lambda+k-m-\frac{1}{2}, b+\lambda+k-m-\frac{1}{2}; c+\lambda+k-m-\frac{1}{2}, \lambda+k-m+\frac{1}{2}; p) \\ &= f(p), R(a+\lambda+k-m-\frac{1}{2}) > 0, R(b+\lambda+k-m-\frac{1}{2}) > 0. \end{aligned}$$

Applying the corollary and replacing $\frac{1}{2}-k+m$ by α , $\frac{1}{2}-k-m$ by β , $\lambda-m-k+\frac{1}{2}$ by γ , we get after a little simplification

$$\begin{aligned} \sum_{\alpha, \beta} \frac{\Gamma(a-\beta)\Gamma(b-\beta)\Gamma(\beta-\alpha)}{\Gamma(c-\beta)} p^\alpha {}_2F_2(a-\beta, b-\beta; c-\beta, 1+\alpha-\beta; p) \\ = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} {}_2F_1\left(\alpha, \beta; \gamma; -\frac{t}{p}\right) \times \\ \times \left[\frac{\Gamma(a)\Gamma(b)\Gamma(\gamma-\alpha-\beta)}{\Gamma(c)} (p+t)^{\alpha+\beta-\gamma} {}_2F_2(a, b; c, 1+\alpha+\beta-\gamma; p+t) \right. \\ \left. + \frac{\Gamma(a+\gamma-\alpha-\beta)\Gamma(b+\gamma-\alpha-\beta)\Gamma(\alpha+\beta-\gamma)}{\Gamma(c+\gamma-\alpha-\beta)} \times \right. \\ \left. \times {}_2F_2(a+\gamma-\alpha-\beta, b+\gamma-\alpha-\beta; c+\gamma-\alpha-\beta, 1+\gamma-\alpha-\beta; p+t) \right] dt, \quad \dots (29) \end{aligned}$$

valid, by A.C., for $R(a-\alpha) > 0$, $R(b-\alpha) > 0$, $R(a-\beta) > 0$, $R(b-\beta) > 0$, $R(\gamma) > 0$, $R(p) > 0$.

As a particular case of this result, if we take $b=c$ and use the relation

$$E(\alpha, \beta :: x) = \sum_{\alpha, \beta} \Gamma(\beta-\alpha)\Gamma(\alpha)x^\alpha {}_1F_1(\alpha; \alpha-\beta+1; x) \quad \dots \dots (30)$$

we get

$$\begin{aligned} E(a-\alpha, a-\beta :: p) &= \frac{p^{a-\alpha-\beta}}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} (p+t)^{\alpha+\beta-a-\gamma} E(a, a+\gamma-\alpha-\beta :: p+t) \times \\ &\quad \times {}_2F_1\left(\alpha, \beta; \gamma; -\frac{t}{p}\right) dt, \quad \dots (31) \end{aligned}$$

$R(\gamma) > 0$, $R(a-\alpha) > 0$, $R(a-\beta) > 0$, $R(p) > 0$.

Since

$$E(\tfrac{1}{2}-k-m, \tfrac{1}{2}-k+m; x) = \Gamma(\tfrac{1}{2}-k-m)\Gamma(\tfrac{1}{2}-k+m)x^{-k}e^{ix}W_{k,m}(x) \quad (32)$$

the above result is equivalent to the following integral representation for Whittaker function (cf. Meijer, 1941, p. 601)

$$\begin{aligned} W_{k,m}(p) &= \frac{\Gamma(\tfrac{1}{2}-\lambda-\mu)\Gamma(\tfrac{1}{2}-\lambda+\mu)}{\Gamma(2k-2\lambda)\Gamma(\tfrac{1}{2}-k-m)\Gamma(\tfrac{1}{2}-k+m)} p^{\lambda+\mu-k+\tfrac{1}{2}} \\ &\quad \times \int_0^\infty t^{2k-2\lambda-1} (p+t)^{-\mu-\tfrac{1}{2}} e^{it} W_{\lambda,\mu}(p+t) \\ &\quad \times {}_2F_1\left(k+m-\lambda-\mu, k-m-\lambda-\mu; 2k-2\lambda; -\frac{t}{p}\right) dt, \quad \dots \quad (33) \end{aligned}$$

$$R(2k-2\lambda) > 0, R(\tfrac{1}{2}-k \pm m) > 0, R(p) > 0.$$

THEOREM II. If

$$\phi(p) \doteq h(x)$$

$$p^{2-\lambda}h(p) \stackrel{v}{=}_{k,m} g(x)$$

and

$$p^{2-\mu}g(p) \doteq f(x)$$

then

$$\phi(p) = p^{1-\lambda} \int_0^\infty x^{-\mu} F_{\lambda,\mu}(px) f(x) dx,$$

provided that $R(\lambda) > 0$, $R(\lambda+2m) > 0$, $R(\mu) > 0$, $R(\mu+2m) > 0$, and the integral is convergent, and

$$\begin{aligned} F_{\lambda,\mu}(x) &= \sum_{\lambda,\mu} \frac{\Gamma(\lambda)\Gamma(\lambda+2m)\Gamma(\mu-\lambda)}{\Gamma(\lambda+m-k+\tfrac{1}{2})} x^\lambda \\ &\quad \times {}_2F_2(\lambda, \lambda+2m; \lambda+m-k+\tfrac{1}{2}, 1-\mu+\lambda; x). \end{aligned}$$

Proof. We know that if (Rathie, 1952, p. 243)

$$\phi(p) \doteq h(x)$$

and

$$p^{2-\lambda}h(p) \stackrel{v}{=}_{k,m} g(x)$$

then

$$\phi(p) = \frac{\Gamma(\lambda)\Gamma(\lambda+2m)}{\Gamma(\lambda+m-k+\tfrac{1}{2})} p \int_0^\infty t^{-\lambda} {}_2F_1\left(\lambda, \lambda+2m; \lambda+m-k+\tfrac{1}{2}; -\frac{p}{t}\right) g(t) dt.$$

But

$$t^{2-\mu}g(t) = t \int_0^\infty e^{-tx} f(x) dx.$$

Hence

$$\begin{aligned}\phi(p) &= \frac{\Gamma(\lambda)\Gamma(\lambda+2m)}{\Gamma(\lambda+m-k+\frac{1}{2})} p \int_0^\infty t^{\mu-\lambda-1} {}_2F_1\left(\lambda, \lambda+2m; \lambda+m-k+\frac{1}{2}; -\frac{p}{t}\right) \\ &\quad \times \left\{ \int_0^\infty e^{-tx} f(x) dx \right\} dt \\ &= \frac{\Gamma(\lambda)\Gamma(\lambda+2m)}{\Gamma(\lambda+m-k+\frac{1}{2})} p \int_0^\infty f(x) dx \int_0^\infty e^{-tx} t^{\mu-\lambda-1} {}_2F_1\left(\lambda, \lambda+2m; \lambda+m-k+\frac{1}{2}; -\frac{p}{t}\right) dt \\ &= p^{1-\lambda} \int_0^\infty x^{-\mu} F_{\lambda, \mu}(px) f(x) dx, \quad R(\mu) > 0, R(\mu+2m) > 0,\end{aligned}$$

by virtue of the integral (Rathie, 1952, p. 239)

$$\begin{aligned}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \int_0^\infty e^{-px} x^{a-1} {}_2F_1\left(a, b; c; -\frac{1}{x}\right) dx &= p^{-a} \frac{\Gamma(a)\Gamma(b)\Gamma(\alpha)}{\Gamma(c)} {}_2F_2(a, b; c, 1-\alpha; p) \\ &\quad + \frac{\Gamma(a+\alpha)\Gamma(b+\alpha)\Gamma(-\alpha)}{\Gamma(c+\alpha)} {}_2F_2(a+\alpha, b+\alpha; c+\alpha, 1+\alpha; p), \quad \dots \quad (34)\end{aligned}$$

$R(a+\alpha) > 0, R(b+\alpha) > 0, R(p) > 0$.

The reversion of the order of integration is justified when the Laplace and generalised transforms of the functions involved exist and the resulting integral is absolutely convergent.

It is easily seen that

$$F_{\lambda, \mu}(x) \sim A + Bx^{-2m} \text{ for large } x,$$

and

$$F_{\lambda, \mu}(x) = ax^\lambda + bx^\mu \text{ for small } x.$$

Example 1. Take (McLachlan and Humbert, 1950, p. 47)

$$\begin{aligned}h(x) &= 2x^{\lambda+m-1} K_{2m}(2\sqrt{x}) \\ &\doteq \Gamma(\lambda)\Gamma(\lambda+2m) e^{\frac{1}{2}p} p^{\frac{1}{2}-\lambda-m} W_{\frac{1}{2}-\lambda-m, m}\left(\frac{1}{p}\right) \\ &= pE\left(\lambda, \lambda+2m; \frac{1}{p}\right) \\ &= \phi(p), \quad R(\lambda) > 0, R(\lambda+2m) > 0;\end{aligned}$$

then (13)

$$\begin{aligned}p^{2-\lambda} h(p) &= 2p^{m+1} K_{2m}(2\sqrt{p}) \\ &\stackrel{v}{=} e^{-\frac{1}{x}} x^{k-m-\frac{1}{2}} = g(x), \\ &\quad k, m\end{aligned}$$

and (McLachlan and Humbert, 1950, p. 42)

$$\begin{aligned}p^{2-\mu} g(p) &= e^{-\frac{1}{p}} p^{k-m-\mu+\frac{1}{2}} \\ &\doteq x^{1(m+\mu-k-\frac{1}{2})} J_{m+\mu-k-\frac{1}{2}}(2\sqrt{x}) \\ &= f(x).\end{aligned}$$

Hence the theorem gives

$$E\left(\lambda, \lambda+2m :: \frac{1}{p}\right) = p^{-\lambda} \int_0^{\infty} x^{i(m-\mu-k-\frac{1}{2})} J_{m+\mu-k-\frac{1}{2}}(2\sqrt{x}) F_{\lambda, \mu}(px) dx, \quad \dots \quad (35)$$

valid, by A.C., for $R(\lambda+m-k+\frac{1}{2}) > 0$, $R(\mu+m-k+\frac{1}{2}) > 0$, $R(\mu+k-m) > 0$, $R(\mu+k+3m) > 0$, $R(p) > 0$.

When $k = -m + \frac{1}{2}$,

$$\begin{aligned} F_{\lambda, \mu}(x) &= \sum_{\lambda, \mu} \Gamma(\lambda) \Gamma(\mu-\lambda) x^{\lambda} {}_1F_1(\lambda; 1-\mu+\lambda; x) \\ &= E(\lambda, \mu :: x) \dots \dots \dots \dots \dots \dots \dots \quad (36) \end{aligned}$$

and we then have

$$E\left(\lambda, \lambda+2m :: \frac{1}{p}\right) = p^{-\lambda} \int_0^{\infty} x^{i(2m-\mu-1)} J_{2m+\mu-1}(2\sqrt{x}) E(\lambda, \mu :: px) dx, \quad (37)$$

$R(\lambda+2m) > 0$, $R(\mu+2m) > 0$, $R(\mu-2m+\frac{1}{2}) > 0$, $R(p) > 0$,

which is equivalent to a result given by Erdelyi.

Example 2. Next take (34)

$$\begin{aligned} h(x) &= \frac{\Gamma(\nu) \Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})} x^{\lambda-\nu-1} {}_2F_1\left(\nu, \nu+2m; \nu+m-k+\frac{1}{2}; -\frac{1}{x}\right) \\ &\doteq \frac{\Gamma(\nu) \Gamma(\nu+2m) \Gamma(\lambda-\nu)}{\Gamma(\nu+m-k+\frac{1}{2})} p^{1+\nu-\lambda} {}_2F_2(\nu, \nu+2m; \nu+m-k+\frac{1}{2}, 1+\nu-\lambda; p) \\ &\quad + \frac{\Gamma(\lambda) \Gamma(\lambda+2m) \Gamma(\nu-\lambda)}{\Gamma(\lambda+m-k+\frac{1}{2})} p {}_2F_2(\lambda, \lambda+2m; \lambda+m-k+\frac{1}{2}, 1+\lambda-\nu; p) \\ &= p^{1-\lambda} F_{\lambda, \nu}(p) = \phi(p), \quad R(\lambda) > 0, \quad R(\lambda+2m) > 0. \end{aligned}$$

Then

$$\begin{aligned} p^{2-\lambda} h(p) &= \frac{\Gamma(\nu) \Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})} p^{1-\nu} {}_2F_1\left(\nu, \nu+2m; \nu+m-k+\frac{1}{2}; \frac{1}{p}\right) \\ &\stackrel{\nu}{=} e^{-x} x^{\nu-1} = g(x), \quad R(\nu) > 0, \quad R(\nu+2m) > 0, \\ &\quad k, m \end{aligned}$$

and

$$\begin{aligned} p^{2-\mu} g(p) &= e^{-p} p^{\nu-\mu+1} \\ &\doteq \frac{(x-1)^{\mu-\nu-1}}{\Gamma(\mu-\nu)} = f(x), \quad R(\mu-\nu) > 0, \quad x > 1. \end{aligned}$$

Applying the theorem we get

$$F_{\lambda, \nu}(p) = \frac{1}{\Gamma(\mu-\nu)} \int_1^{\infty} x^{-\mu} (x-1)^{\mu-\nu-1} F_{\lambda, \mu}(px) dx, \quad \dots \quad (38)$$

$R(\mu-\nu) > 0$, $R(\nu) > 0$, $R(\nu+2m) > 0$, $R(p) > 0$,

which by the substitution $px = p+t$ gives

$$F_{\lambda, \nu}(p) = \frac{p^{\nu}}{\Gamma(\mu-\nu)} \int_0^{\infty} t^{\mu-\nu-1} (p+t)^{-\mu} F_{\lambda, \mu}(p+t) dt, \quad \dots \quad (39)$$

$R(\mu-\nu) > 0$, $R(\nu) > 0$, $R(\nu+2m) > 0$, $R(p) > 0$.

When $k = -m + \frac{1}{2}$, this yields by virtue of (36)

$$E(\lambda, \nu :: p) = \frac{p^\nu}{\Gamma(\mu - \nu)} \int_0^\infty t^{\mu - \nu - 1} (p+t)^{-\mu} E(\lambda, \mu :: p+t) dt, \quad \dots \quad (40)$$

$$R(\mu - \nu) > 0, R(\nu) > 0, R(p) > 0.$$

7. In this section we shall derive a few recurrence relations for the generalised transform and use them to obtain some recurrence formulae for the E -function.

THEOREM III. If

$$\phi_{k,m}(p) \stackrel{v}{=} h(x)$$

where $h(x)$ is independent of k and m , then

$$\phi_{k,m}(p) = \phi_{k-\frac{1}{2},m+\frac{1}{2}}(p) + (\frac{1}{2} - k - m) \phi_{k-1,m}(p) \quad \dots \quad (A)$$

and if

$$\phi_{k,m,\lambda}(p) \stackrel{v}{=} x^\lambda h(x)$$

$h(x)$ being independent of λ , k , and m , then

$$\phi_{k,m,\lambda}(p) = p \phi_{k-\frac{1}{2},m-\frac{1}{2},\lambda+1}(p) + (\frac{1}{2} - k + m) \phi_{k-1,m,\lambda}(p) \quad \dots \quad (B)$$

and

$$\frac{d^n}{dp^n} \left\{ \frac{1}{p} \phi_{k,m,\lambda}(p) \right\} = \frac{(-1)^n}{p} \phi_{k+\frac{1}{2},m-\frac{1}{2},\lambda+n}(p), \quad \dots \quad (C)$$

provided that the integrals involved are uniformly convergent and n is a positive integer.

Proof. (a) Since (Whittaker and Watson, 1935, p. 352)

$$W_{k,m}(z) = z^{\frac{1}{2}} W_{k-\frac{1}{2},m+\frac{1}{2}}(z) + (\frac{1}{2} - k - m) W_{k-1,m}(z) \quad \dots \quad (41)$$

we have

$$\begin{aligned} \phi_{k,m}(p) &= p \int_0^\infty e^{-ixx} (px)^{m-\frac{1}{2}} W_{k,m}(px) h(x) dx \\ &= p \int_0^\infty e^{-ixx} (px)^{m-\frac{1}{2}} \left\{ (px)^{\frac{1}{2}} W_{k-\frac{1}{2},m+\frac{1}{2}}(px) + \right. \\ &\quad \left. + (\frac{1}{2} - k - m) W_{k-1,m}(px) \right\} h(x) dx \\ &= \phi_{k-\frac{1}{2},m+\frac{1}{2}}(p) + (\frac{1}{2} - k - m) \phi_{k-1,m}(p). \end{aligned}$$

(b) The formula (B) may similarly be obtained by using

$$W_{k,m}(z) = z^{\frac{1}{2}} W_{k-\frac{1}{2},m-\frac{1}{2}}(z) + (\frac{1}{2} - k + m) W_{k-1,m}(z) \quad \dots \quad (42)$$

in

$$\phi_{k,m,\lambda}(p) = p \int_0^\infty e^{-ixx} (px)^{m-\frac{1}{2}} W_{k,m}(px) x^\lambda h(x) dx.$$

(c) The formula (C) follows from the above equation on differentiation n times with respect to p and using the result

$$\frac{d^n}{dx^n} \left\{ x^{m-\frac{1}{2}} e^{-ix} W_{k,m}(x) \right\} = (-1)^n x^{m-\frac{1}{2}(n+1)} e^{-ix} W_{k+\frac{1}{2},m-\frac{1}{2}n}(x) \quad \dots \quad (43)$$

provided that differentiation under the sign of integration is permissible.

To illustrate the application of the above formulae, we take

$$\begin{aligned} h(x) &= x^{\gamma-1} E\left(\alpha_1, \dots, \alpha_{r-2} : \beta_1, \dots, \beta_{s-1} : \frac{1}{x}\right) \\ &\stackrel{v}{=} p^{1-\gamma} E(\alpha_1, \dots, \alpha_{r-2}, \gamma, \gamma+2m : \beta_1, \dots, \beta_{s-1}, \gamma+m-k+\frac{1}{2} : p) \\ &= \phi_{k,m}(p). \end{aligned}$$

Applying the formula (A) and then replacing γ by α_r , $\gamma+2m$ by α_r , and $\gamma+m-k+\frac{1}{2}$ by β_s , we get

$$\begin{aligned} E(\alpha_1, \dots, \alpha_r : \beta_1, \dots, \beta_s : p) &= E(\alpha_1, \dots, \alpha_r+1 : \beta_1, \dots, \beta_s+1 : p) + \\ &\quad + (\beta_s - \alpha_r) E(\alpha_1, \dots, \alpha_r : \beta_1, \dots, \beta_s+1 : p) \quad (44) \end{aligned}$$

This result may also be derived from known recurrence relations for E -functions (MacRobert, 1941, p. 260).

A special case of this is

$$E(\alpha, \beta, \gamma : \delta : p) = E(\alpha, \beta, \gamma+1 : \delta+1 : p) + (\delta - \gamma) E(\alpha, \beta, \gamma : \delta+1 : p) \quad (45)$$

Now if we take $\gamma = \frac{1}{2} - k - m$, $\alpha = \delta = m - k - \frac{1}{2}$, and $\beta = m - k + \frac{1}{2}$ we get

$$\begin{aligned} E(\tfrac{1}{2} - k + m, \tfrac{1}{2} - k - m : : p) &= E(m - k - \tfrac{1}{2}, \tfrac{3}{2} - k - m : : p) + \\ &\quad + (2m - 1) E(m - k - \tfrac{1}{2}, \tfrac{1}{2} - k - m : : p) \quad \dots \quad (46) \end{aligned}$$

which by virtue of (32) yields

$$(m - k - \tfrac{1}{2}) W_{k,m}(p) = (\tfrac{1}{2} - k - m) W_{k,m-1}(p) + (2m - 1) p^{-\frac{1}{2}} W_{k+\frac{1}{2},m-\frac{1}{2}}(p) \quad (47)$$

This new recurrence formula for Whittaker function leads us to another recurrence formula for the generalised transform, viz.

$$\text{If} \quad \phi_{k,m,\lambda}(p) \stackrel{v}{=} x^\lambda h(x)$$

and $h(x)$ is independent of λ , k and m , then

$$(m - k - \tfrac{1}{2}) \phi_{k,m,\lambda}(p) = (\tfrac{1}{2} - k - m) p \phi_{k,m-1,\lambda+1}(p) + (2m - 1) \phi_{k+\frac{1}{2},m-\frac{1}{2},\lambda}(p) \quad (D)$$

To use this formula we take

$$\begin{aligned} x^\lambda h(x) &= x^{\lambda-1} E\left(\alpha_1, \dots, \alpha_{r-2} : \beta_1, \dots, \beta_{s-1} : \frac{1}{x}\right) \\ &\stackrel{v}{=} p^{1-\lambda} E(\alpha_1, \dots, \alpha_{r-2}, \lambda, \lambda+2m : \beta_1, \dots, \beta_{s-1}, \lambda+m-k+\tfrac{1}{2} : p) \\ &= \phi_{k,m,\lambda}(p). \end{aligned}$$

The formulae (D) and (C) then give us, on replacing λ by α_r , $\lambda+2m$ by α_r and $\lambda+m-k+\frac{1}{2}$ by β_s ,

$$\begin{aligned} &(\beta_s - \alpha_r - 1) E(\alpha_1, \dots, \alpha_r : \beta_1, \dots, \beta_s : p) \\ &= (\beta_s - \alpha_r) E(\alpha_1, \dots, \alpha_r - 1 + 1, \alpha_r - 1 : \beta_1, \dots, \beta_s : p) + \\ &\quad + (\alpha_r - \alpha_r - 1) E(\alpha_1, \dots, \alpha_r - 1 : \beta_1, \dots, \beta_s - 1 : p) \quad \dots \quad (48) \end{aligned}$$

and

$$\frac{d^n}{dp^n} \{ p^{-\alpha_r} E(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; p) \} \\ = (-1)^n p^{-\alpha_r-n} E(\alpha_1, \dots, \alpha_r+n; \beta_1, \dots, \beta_s; p) \quad (49)$$

and when $n=1$,

$$pE'(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; p) = \alpha_r E(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; p) - \\ - E(\alpha_1, \dots, \alpha_r+1; \beta_1, \dots, \beta_s; p) \quad \dots \quad (50)$$

An interesting special case of (48) is

$$(\alpha-\beta-1)E(\beta, \gamma; p) + (\gamma-\alpha)E(\beta+1, \gamma-1; p) \\ = (\gamma-\beta-1)E(\alpha, \beta, \gamma-1; \alpha-1; p) \quad \dots \quad (51)$$

From (49) we deduce that

$$\frac{d^n}{dp^n} \{ p^{-\beta} E(\alpha, \beta; p) \} = (-1)^n p^{-\beta-n} E(\alpha, \beta+n; p) \quad \dots \quad (52)$$

which is equivalent to

$$\frac{d^n}{dp^n} \{ p^{-m-\frac{1}{2}} e^{ip} W_{k,m}(p) \} = \frac{\Gamma(\frac{1}{2}-k+m+n)}{\Gamma(\frac{1}{2}-k+m)} (-1)^n p^{-i-m-in} e^{ip} W_{k-\frac{n}{2}, m+\frac{n}{2}}(p) \quad (53)$$

and when $n=1$,

$$pW'_{k,m}(p) = (\frac{1}{2}+m-\frac{1}{2}p)W_{k,m}(p) - (\frac{1}{2}-k+m)p^{\frac{1}{2}}W_{k-\frac{1}{2}, m+\frac{1}{2}}(p). \quad \dots \quad (54)$$

REFERENCES

- Bromwich, T. J. I'A. (1908). Infinite Series.
 Erdelyi, A. (1939). Integral representation for Whittaker function. *Proc. Benares Math. Soc.*, (2), 1, 39-53.
 Goldstein, S. (1932). Operational representation of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder functions. *Proc. Lond. Math. Soc.*, (2), 34, 103-25.
 MacRobert, T. M. (1941). Some formulae for the E -function. *Phil. Mag.*, (7), 31, 254-260.
 McLachlan, N. W., and Humbert, P. (1950). Supplement au formulaire pour le calcul symbolique, Fascicule 100.
 Meijer, C. S. (1936). Über Whittakersche Bez. Besselsche Funktionen und deren Produkte. *Nieuw Archief voor Wiskunde, Amsterdam*, (2), 18, 4tes Heft, 10-39.
 Meijer, C. S. (1941). Integraldarstellungen für Whittakersche Funktionen und ihre Produkte. *Proc. Neder. Akad. van Wetenschappen*, (5), 44, 601.
 Pasricha, B. R. (1943). Some integrals involving Whittaker functions. *Jour. Indian Math. Soc.*, 7, 46-50.
 Rathie, C. B. (1952). A study of a generalisation of the Laplace's integral. *Proc. National Acad. Sci. India*, 21, 231-49.
 Rathie, C. B. (1953). Some infinite integrals involving E -functions. *Jour. Indian Math. Soc.*, (4), 17, 167-75.
 Rathie, C. B. (1954). Some infinite integrals involving Bessel functions. *Proc. National Inst. Sci. India*, (1), 20, 62-69.
 Rathie, C. B. (1955). Some results involving hypergeometric and E -functions. *Proc. Glasgow Math. Assn.*, (3), 2.
 Varma, R. S. (1951). On a generalisation of Laplace integral. *Proc. National Acad. Sci. India*, 20.
 Whittaker, E. T., and Watson, G. N. (1935). Modern Analysis.

A NOTE ON THE OSCILLATIONS OF AN INFINITE CYLINDER SUBJECT TO RADIAL MAGNETIC FIELD

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I. INTRODUCTION

In a recent paper Ferraro and Memory (1952) have discussed the problem of oscillation of a star in its own magnetic field. Following them, we discuss here a case of torsional and radial oscillations of a cylindrical mass of highly conducting incompressible fluid in the presence of a permanent infinite axial magnetic line-pole of strength m /unit length. In numerical illustration of the problem we have considered the same values for radius, density and surface magnetic field as used by them. We have also compared the period of oscillations for the first two harmonics with those obtained by Ferraro and Memory.

II. FUNDAMENTAL EQUATIONS

The motion of incompressible fluid with the density ρ , the electrical conductivity σ , and the permeability μ ($= 1$), placed in a magnetic field \mathbf{H}_0 is described by Maxwell's equations in e.m. units

$$\text{curl } \mathbf{H} = 4\pi \mathbf{j} \quad \dots \dots \dots (1)$$

(neglecting displacement current)

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} \quad \dots \dots \dots (2)$$

$$\text{div } \mathbf{H} = 0 \quad \dots \dots \dots (3)$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{H}) \quad \dots \dots \dots (4)$$

and the hydrodynamic equations

$$\rho \frac{d\mathbf{V}}{dt} = \mathbf{j} \times \mathbf{H} - \text{grad } p \quad \dots \dots \dots (5)$$

$$\text{div } \mathbf{V} = 0. \quad \dots \dots \dots (6)$$

Here \mathbf{V} denotes the material velocity, \mathbf{H} and \mathbf{E} the magnetic and electric field strength respectively, \mathbf{j} the current density and p the pressure. For $\sigma = \infty$, it follows from (4) that

$$\mathbf{E} = -\mathbf{V} \times \mathbf{H}. \quad \dots \dots \dots (7)$$

Outside the cylinder, $\mathbf{j} = 0$ and the field equations are the same as (1) to (3) with $\mathbf{j} = 0$ in (1).

Using (1), (5) may be written

$$\rho \frac{d\mathbf{V}}{dt} = \frac{1}{4\pi} (\text{curl } \mathbf{H}) \times \mathbf{H} - \text{grad } p. \quad \dots \dots \dots (8)$$

Substituting (7) in (2), this becomes

$$\frac{\partial \mathbf{H}}{\partial t} = \text{curl} (\mathbf{V} \times \mathbf{H}) \quad \dots \quad (9)$$

which expresses the fact that the magnetic lines of force move with the fluid.

To form the equations of small oscillations, we write

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h} \quad \text{and} \quad p = P + p' \quad \dots \quad (10)$$

where \mathbf{H}_0 is the permanent field, neglecting squares and products of quantities of the first order \mathbf{V} , \mathbf{h} and p' (8) and (9) become approximately

$$\rho \frac{\partial \mathbf{V}}{\partial t} = \frac{1}{4\pi} (\text{curl } \mathbf{h}) \times \mathbf{H}_0 + \text{grad } p' \quad \dots \quad (11)$$

$$\frac{\partial \mathbf{h}}{\partial t} = \text{curl} (\mathbf{V} \times \mathbf{H}_0). \quad \dots \quad (12)$$

Operating (11) with a curl we get

$$4\pi\rho \frac{\partial}{\partial t} (\text{curl } \mathbf{V}) = \text{curl} \{ (\text{curl } \mathbf{h}) \times \mathbf{H}_0 \}. \quad \dots \quad (13)$$

Further since $\text{div } \mathbf{H}_0 = 0$ we find from (3)

$$\text{div } \mathbf{h} = 0. \quad \dots \quad (14)$$

Then (6), (12), (13) and (14) are the fundamental equations to determine the small oscillations.

III. SOLUTION OF PROBLEM

We solved these equations in cylindrical co-ordinate system and considered the radial and torsional oscillations only. Since $\text{div } \mathbf{V} = 0$ and $\text{div } \mathbf{h} = 0$, we may express \mathbf{V} and \mathbf{h} in terms of appropriate Stokes 'stream function', ψ and u respectively. At any time the family of curves $\psi = \text{constant}$ and $u = \text{constant}$ will then be the stream line of motion and the equations of the lines of force of the field of the currents induced by the motion of the fluid in permanent field respectively.

The resolutes of vectors \mathbf{V} and \mathbf{h} in the direction of r and θ increasing are given by

$$\left. \begin{aligned} v_r &= -\frac{1}{r} \frac{\partial \psi}{\partial \theta}, & v_\theta &= \frac{\partial \psi}{\partial r} \\ h_r &= -\frac{1}{r} \frac{\partial u}{\partial \theta}, & h_\theta &= \frac{\partial u}{\partial r}. \end{aligned} \right\} \quad \dots \quad (15)$$

The permanent magnetic field due to an infinite axial line pole of strength $m/\text{unit length}$ will be radial and its value will be given by

$$H_r = \frac{2m}{r}.$$

Let $\mu = \cos \theta$ and let Δ denote the operator

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1-\mu^2}{r^2} \frac{\partial^2}{\partial \mu^2} \quad \dots \quad (16)$$

hence this restriction over l will not interfere. To this solution we shall denote by $F_n^{(2)}(x)$.

The required solution of (26) is thus of the form

$$F_n(x) = AF_n^{(1)}(x) + BF_n^{(2)}(x), \quad \dots \quad \dots \quad \dots \quad (32)$$

where A and B are arbitrary constants.

IV. THE MAGNETIC FIELD ASSOCIATED WITH THE OSCILLATIONS

For points within the cylinder the stream function of the magnetic field is given by (19) and (25)

$$u = -i(4\pi\rho)^{\frac{1}{2}}x^{-1}\sum F_n^i(x)(1-\mu^2)^{\frac{1}{2}}P_n^1(\mu), \quad \dots \quad \dots \quad \dots \quad (33)$$

where F_n^i is the derivative of F_n with respect to x .

To find the magnetic field outside the cylinder we have to find solutions of the equations

$$\text{curl } \mathbf{H} = 0, \text{ div } \mathbf{H} = 0. \quad \dots \quad \dots \quad \dots \quad (34)$$

Since the field outside will only differ slightly from the field of the axial magnetic line pole, we write as before

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}'. \quad \dots \quad \dots \quad \dots \quad (35)$$

Using u' for the stream function of the induced field h' , we have from (34) that

$$\text{curl } \mathbf{h}' = 0 \text{ or } \Delta_1 u' = 0, \quad \dots \quad \dots \quad \dots \quad (36)$$

where Δ_1 is given by (24).

Boundary condition.—Let the value of x at the surface of the cylinder be denoted by x_1 , so that by (22)

$$a = \eta x_1. \quad \dots \quad \dots \quad \dots \quad \dots \quad (37)$$

To satisfy the continuity of the radial resolutes of the magnetic field at the surface, the solution of (36) must involve the Legendre function P_n^1 and must vanish at infinity. Such a solution is

$$u' = \sum_n B_n x^{-\sqrt{l}(1-\mu^2)^{\frac{1}{2}}} P_n', \quad \dots \quad \dots \quad \dots \quad (38)$$

where B_n 's are constants to be determined. Continuity of magnetic resolutes at the surface $x = x_1$ then gives two conditions:

$$x_1^{\sqrt{l}-1} F_n^i(x_1) = iB_n (4\pi\rho)^{-\frac{1}{2}} \quad \dots \quad \dots \quad \dots \quad (39)$$

$$\sqrt{l}B_n x_1^{1-\sqrt{l}} = i(4\pi\rho)^{\frac{1}{2}} [x_1 F_n^{ii}(x_1) - F_n^i(x_1)].$$

Elimination of B_n gives

$$(1-\sqrt{l})F_n^i(x_1) = x_1 F_n^{ii}(x_1) \quad \dots \quad \dots \quad \dots \quad (40)$$

and

$$u' = -i(4\pi\rho)^{\frac{1}{2}} \sum x_1^{\sqrt{l}-1} x^{-\sqrt{l}} F_n^i(x_1) (1-\mu^2)^{\frac{1}{2}} P_n'(\mu). \quad \dots \quad \dots \quad (41)$$

V. THE ELECTRIC FIELD ASSOCIATED WITH THE OSCILLATIONS

The electric field within the cylinder is given by (7) or since \mathbf{V} is small, approximately by

$$\mathbf{E} = -\mathbf{V} \times \mathbf{H}_0.$$

Since \mathbf{E} is wholly axial, its intensity E is given by

$$E = (4\pi\rho)^{\frac{1}{2}} \lambda \sum F_n^1(x) x^{-1} (1-\mu^2)^{\frac{1}{2}} P_n^1(\mu). \quad \dots \quad (42)$$

Outside the cylinder the electric field satisfies the condition

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} \quad \text{or} \quad \text{curl } \mathbf{E} = -\frac{\partial h'}{\partial t}$$

by (35). Using the stream function u' for the field h' given by (41), and the fact that E is continuous at the surface of the cylinder, we find

$$E = i\lambda u', \quad \dots \quad (43)$$

where u' is given by (41).

VI. THE VARIATION OF PRESSURE IN THE CYLINDER

This is most readily derived from the equation of motion (11), viz.

$$4\pi\rho \frac{\partial \mathbf{V}}{\partial t} = (\text{curl } \mathbf{h}) \times \mathbf{H}_0 - 4\pi \text{grad } p'.$$

Considering the azimuthal resolutes of this equation, we find that p' is of the form

$$p' = -(i\rho\lambda) \sum G_n(x) P_n(\mu), \quad \dots \quad (44)$$

where $G_n(x)$ is a function of x given by

$$G_n(x) = \left[x^{-1} \frac{\partial^3}{\partial x^3} - x^{-2} \frac{\partial^2}{\partial x^2} + x^{-3} \left\{ (1-l) + x^4 \right\} \frac{\partial}{\partial x} \right] F_n(x). \quad \dots \quad (45)$$

We shall denote by $G_n^{(1)}$ and $G_n^{(2)}$ the G functions corresponding to $F_n^{(1)}$ and $F_n^{(2)}$ respectively. Let

$$F_n^{(1)} = x^{2+\sqrt{l}} \sum_0^\infty A_k x^{4k}; \quad F_n^{(2)} = x^4 \sum_0^\infty A'_k x^{4k}, \quad \dots \quad (46)$$

where $A_0 = A'_0 = 1$ and A_k and A'_k are given by (30) and (31).

Then we find that

$$G_n^{(1)}(x) = \sum_0^\infty \frac{l}{4k+2+\sqrt{l}} A_k x^{4k+2+\sqrt{l}} \quad \dots \quad (47)$$

$$G_n^{(2)}(x) = 4(4-l) + \sum_0^\infty \frac{l}{4k+4} A'_k x^{4k+4}, \quad \dots \quad (48)$$

where $l = n(n+1)$. The series are absolutely and uniformly convergent for all values of x .

IX. THE MAGNETIC FIELD OF THE INDUCED CURRENT INSIDE THE CYLINDER

The corresponding stream function for the magnetic field of the induced current inside the cylinder is given by (62) and (63) for the case of the fundamental mode of the first two harmonics (omitting time factor)

$$u = -i(4\pi\rho)^{\frac{1}{2}} (3.414) c(x^{1.414} - k_3 x^2) \sin^2\theta \quad \dots \quad (62)$$

$$u = -i(4\pi\rho)^{\frac{1}{2}} (13.347) c(x^{2.449} - k_4 x^2) \sin^2\theta \cos\theta, \quad \dots \quad (63)$$

where

$$k_3 = 0.3627 \quad \dots \quad (64)$$

$$k_4 = 1.2102. \quad \dots \quad (65)$$

The magnetic lines of forces are given by $u = \text{const.}$ for the additional magnetic field in the fundamental mode of the first two harmonics are drawn in the first quadrant only (see Figs. 2(a) and 2(b)). The radial and azimuthal resolutes of the magnetic field of the induced current inside the cylinder are (omitting time factor)

$$h_r = -i(4\pi\rho)^{\frac{1}{2}} \eta(6.828) c(x^{0.414} - k_3 x) \sin\theta \cos\theta$$

$$h_\theta = -i(4\pi\rho)^{\frac{1}{2}} \eta(4.827) c(x^{0.414} - 1.414 k_3 x) \sin^2\theta \quad \dots \quad (66)$$

for the first harmonic and

$$h_r = -i(4\pi\rho)^{\frac{1}{2}} \eta(4.449) 3c(x^{2.449} - k_4 x^2) (3 \cos^2\theta - 1) \sin\theta$$

$$h_\theta = -i(4\pi\rho)^{\frac{1}{2}} \eta(4.449) 3c(2.449x^{1.449} - 2k_4 x) \sin^2\theta \cos\theta \quad \dots \quad (67)$$

for the second harmonic. The values of k_3 , k_4 and η are given by (64), (65) and (22) respectively.

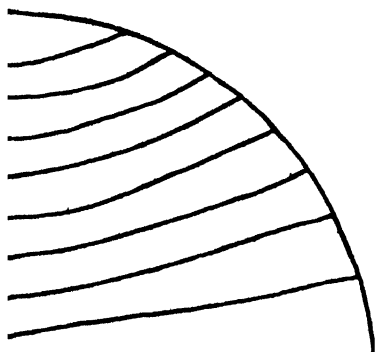


FIG. 2(a)

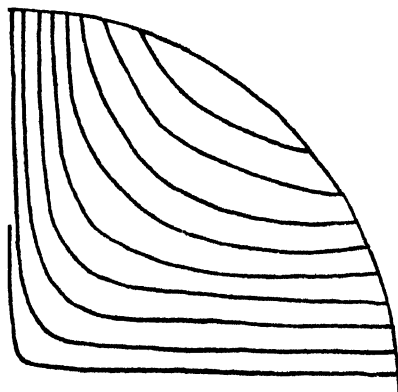


FIG. 2(b)

Fig. 2(a). Magnetic lines of force for the additional field in the fundamental mode of the first harmonic vibration.

Fig. 2(b). Magnetic lines of force for the additional field in the fundamental mode of the second harmonic vibration.

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SUMMARY

In the present paper, the discussion of the oscillation of a cylindrical fluid in the presence of permanent magnetic field has been considered. The magnetic field has been assumed due to infinite axial line magnetic pole.

REFERENCE

Ferraro, V. C. A., and Memory, D. F. (1952). Oscillations of a star in its own magnetic field. An illustrated problem, *M.N.*, 112, 361.

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THE SOLUTIONS OF CERTAIN HYPERGEOMETRIC EQUATIONS

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1. Certain hypergeometric functions of three variables have recently been defined by me in one of my papers (Shanti Saran, 1955). The elementary properties of these functions, including the partial differential equations, have been given in that paper. Later, the Pochhammer type of integral representations for the ten hypergeometric functions defined in Shanti Saran (1955) have been obtained (Shanti Saran, 1955). In this paper I have discussed the nature of the different solutions of the partial differential equations satisfied by two of these functions, viz. F_E and F_K , at the various regular singularities. The other systems of differential equations can be treated in a similar manner. In the first instance, I have obtained the series solution of the differential systems at the regular singularities at the origin. Later the general solution of these differential equations by the help of Pochhammer type of integrals has been obtained and the behaviour of these differential equations at various other singularities have been discussed.

The functions F_E and F_K are defined as

$$(1) \quad F_E(a_1, a_1, a_1, b_1, b_2, b_2; c_1, c_2, c_3; x, y, z) \\ = \sum_{m, n, p=0}^{\infty} \frac{(a_1, m+n+p)(b_1, m)(b_2, n+p)}{(1, m)(1, n)(1, p)(c_1, m)(c_2, n)(c_3, p)} x^m y^n z^p,$$

and

$$(2) \quad F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; x, y, z) \\ = \sum_{m, n, p=0}^{\infty} \frac{(a_1, m)(a_2, n+p)(b_1, m+p)(b_2, n)}{(1, m)(1, n)(1, p)(c_1, m)(c_2, n)(c_3, p)} x^m y^n z^p.$$

2. THE DIFFERENTIAL EQUATIONS SATISFIED BY F_E AND F_K

The differential equations satisfied by F_E and F_K are given by

$$(1) \quad F_E \begin{cases} [\theta(\theta+c_1-1)-x(\theta+\phi+\psi+a_1)(\theta+b_1)]W=0 \\ [\phi(\phi+c_2-1)-y(\theta+\phi+\psi+a_1)(\phi+\psi+b_2)]W=0 \\ [\psi(\psi+c_3-1)-z(\theta+\phi+\psi+a_1)(\phi+\psi+b_2)]W=0 \end{cases}$$

and

$$(2) \quad F_K \begin{cases} [\theta(\theta+c_1-1)-x(\theta+a_1)(\theta+\psi+b_1)]W=0 \\ [\phi(\phi+c_2-1)-y(\phi+\psi+a_2)(\phi+b_2)]W=0 \\ [\psi(\psi+c_3-1)-z(\phi+\psi+a_2)(\theta+\psi+b_1)]W=0 \end{cases}$$

respectively.

Using the classical method of solution by series, we assume a solution to be of the form

$$W = x^g y^h z^k \sum_{m,n,p=0}^{\infty} A_{m,n,p} x^m y^n z^p,$$

where g, h, k are suitable constants.

The indicial equations of the systems 2(1—2) are given by

$$g(g+c_1-1) = 0$$

$$h(h+c_2-1) = 0$$

$$k(k+c_3-1) = 0.$$

These give

$$g = 0 \text{ or } 1-c_1$$

$$h = 0 \text{ or } 1-c_2$$

$$k = 0 \text{ or } 1-c_3.$$

The above roots of the indicial equation lead to the following eight possible sets of values of the parameters g, h, k :—

$$\begin{array}{cccccccc} g = 0 & 0 & 0 & 1-c_1 & 1-c_1 & 0 & 1-c_1 & 1-c_1 \\ h = 0 & 1-c_2 & 0 & 0 & 1-c_2 & 1-c_2 & 0 & 1-c_2 \\ k = 0 & 0 & 1-c_3 & 0 & 0 & 1-c_3 & 1-c_3 & 1-c_3. \end{array}$$

These lead to the following general solutions of 2(1) and 2(2) valid in the neighbourhood of the origin :—

$$\begin{aligned} (3) \quad W = & AF_E(a_1, a_1, a_1, b_1, b_2, b_2; c_1, c_2, c_3; x, y, z) \\ & + B_1 x^{1-c_1} F_E(1-c_1+a_1, 1-c_1+a_1, 1-c_1+a_1, 1-c_1+b_1, b_2, b_2; \\ & \qquad \qquad \qquad 2-c_1, c_2, c_3; x, y, z) \\ & + B_2 y^{1-c_2} F_E(1-c_2+a_1, 1-c_2+a_1, 1-c_2+a_1, b_1, 1-c_2+b_2, 1-c_2+b_2; \\ & \qquad \qquad \qquad c_1, 2-c_2+c_3; x, y, z) \\ & + B_3 z^{1-c_3} F_E(1-c_3+a_1, 1-c_3+a_1, 1-c_3+a_1, b_1, 1-c_3+b_2, 1-c_3+b_2; c_1, \\ & \qquad \qquad \qquad c_2, 2-c_3; x, y, z) \\ & + D_1 x^{1-c_1} y^{1-c_2} F_E(2-c_1-c_2+a_1, 2-c_1-c_2+a_1, 2-c_1-c_2+a_1, 1-c_1+b_1, \\ & \qquad \qquad \qquad 1-c_2+b_2, 1-c_2+b_2; 2-c_1, 2-c_2, c_3; x, y, z) \\ & + D_2 y^{1-c_2} z^{1-c_3} F_E(2-c_2-c_3+a_1, 2-c_2-c_3+a_1, 2-c_2+c_3+a_1, b_1, \\ & \qquad \qquad \qquad 2-c_2-c_3+b_2, 2-c_2-c_3+b_1; c_1, 2-c_2, 2-c_3; x, y, z) \\ & + D_3 x^{1-c_1} z^{1-c_3} F_E(2-c_1-c_3+a_1, 2-c_1-c_3+a_1, 2-c_1-c_3+a_1, 1-c_1+b_1, \\ & \qquad \qquad \qquad 1-c_3+b_2, 1-c_3+b_2; 2-c_1, c_2, 2-c_3; x, y, z) \\ & + E x^{1-c_1} y^{1-c_2} z^{1-c_3} F_E(3-c_1-c_2-c_3+a_1, 3-c_1-c_2-c_3+a_1, 3-c_1-c_2-c_3 \\ & \qquad \qquad \qquad +a_1, 1-c_1+b_1, 2-c_2-c_3+b_2, 2-c_2-c_3+b_2; 2-c_1, 2-c_2, 2-c_3; x, y, z) \end{aligned}$$

and

$$\begin{aligned}
 W = & AF_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; x, y, z) \\
 & + B_1 x^{1-c_1} F_K(1-c_1+a_1, a_2, a_2, 1-c_1+b_1, b_2, 1-c_1+b_1; \\
 & \qquad \qquad \qquad 2-c_1, c_2, c_3; x, y, z) \\
 & + B_2 y^{1-c_2} F_K(a_1, 1-c_2+a_2, 1-c_2+a_2, b_1, 1-c_2+b_2, b_1; c_1, 2-c_2, c_3; \\
 & \qquad \qquad \qquad x, y, z) \\
 & + B_3 z^{1-c_3} F_K(a_1, 1-c_3+a_2, 1-c_3+a_2, 1-c_3+b_1, b_2, 1-c_3+b_1; \\
 & \qquad \qquad \qquad c_1, c_2, 2-c_3; x, y, z) \\
 & + D_1 x^{1-c_1} y^{1-c_2} F_K(1-c_1+a_1, 1-c_2+a_2, 1-c_2+a_2, 1-c_1+b_1, 1-c_2+b_2; \\
 & \qquad \qquad \qquad 1-c_1+b_1; 2-c_1, 2-c_2, c_3; x, y, z) \\
 & + D_2 y^{1-c_2} z^{1-c_3} F_K(a_1, 2-c_2-c_3+a_1, 2-c_2-c_3+a_1, 1-c_2+b_1, 1-c_2+b_2, \\
 & \qquad \qquad \qquad 1-c_3+b_1; c_1, 2-c_2, 2-c_3; x, y, z) \\
 & + D_3 x^{1-c_1} z^{1-c_3} F_K(1-c_1+a_1, 1-c_3+a_2, 1-c_3+a_2, 2-c_1-c_3+b_1, \\
 & \qquad \qquad \qquad b_2, 2-c_1-c_3+b_1; 2-c_1, c_2, 2-c_3; x, y, z) \\
 & + E x^{1-c_1} y^{1-c_2} z^{1-c_3} F_K(1-c_1+a_1, 2-c_2-c_3+a_2, 2-c_2-c_3+a_2, \\
 & \qquad \qquad \qquad 2-c_1-c_3+b_1, b_2, 2-c_1-c_3+b_1, 2-c_1, 2-c_2, 2-c_3; x, y, z).
 \end{aligned}$$

3. In this section Pochhammer-integral representations (Shanti Saran, 1955) are used to investigate the solutions of the differential systems 2(1) and 2(2).

Consider first the following integral representation (Shanti Saran, 1955) of F_E .

$$\begin{aligned}
 (1) \quad & F_E(a_1, a_1, a_1, b_1, b_2, b_2; c_1, c_2, c_3; x, y, z) \\
 & = \frac{\Gamma(k) \Gamma(k') \Gamma(2-k-k')}{(2\pi i)^2} \times \\
 & \times \int_C (-t)^{-k} (t-1)^{-k'} {}_2F_1(k, b_1; c_1; x/t) F_4(k', b_2; c_2, c_3; y/1-t, z/1-t) dt,
 \end{aligned}$$

where $a_1 = k+k'-1$. The contour C denotes a Pochhammer double-loop slung round the points 0 and 1 such that $|t| > |x|$, and $|\sqrt{y/1-t}| + |\sqrt{z/1-t}| < 1$ along the contour.

The above integral suggests that

$$(2) \quad W = \int_C t^{-k} (1-t)^{-k'} f_1(x/t) f_2(y/1-t, z/1-t) dt,$$

should be a solution of the system 2(1) satisfied by F_E where C is some closed contour and $f_1(u)$ and $f_2(v, w)$ are the solutions of the differential equations

$$(3) \quad [\theta_1(\theta_1+c_1-1)-u(\theta_1+k)(\theta_1+b_1)]f_1(u) = 0$$

$$(4) \quad \begin{cases} [\phi_1(\phi_1+c_2-1)-v(\phi_1+\psi_1+k')(\phi_1+\psi_1+b_2)]f_2(v, w) = 0 \\ [\psi_1(\psi_1+c_3-1)-w(\phi_1+\psi_1+k')(\phi_1+\psi_1+b_2)]f_2(v, w) = 0 \end{cases}$$

respectively, where $\theta_1, \phi_1, \psi_1 = u\partial/\partial u, v\partial/\partial v, w\partial/\partial w$ with $u = x/t, v = y/1-t, w = z/1-t$.

Now, denoting the differential system 2(1) by $L_1(W) = 0$, $L_2(W) = 0$ and $L_3(W) = 0$ respectively, we get from 3(2), with $a_1 = k + k' - 1$, that

$$L_1(W) = \int_C t^{-k}(1-t)^{-k'} [\{ f_2(\theta(\theta+c_1-1) - x(\theta+k)(\theta+b_1)) \} f_1 \\ - x(\phi+\psi+k'-1)f_2(\theta+b_1)f_1] dt.$$

Since $u = x/t$, $v = y/1-t$ and $w = z/1-t$, we have on using 3(3) that

$$L_1(W) = \int_C t^{-k}(1-t)^{-k'} \left[f_2 \left\{ \frac{x}{t} (\theta_1+k)(\theta_1+b_1) - x(\theta_1+k)(\theta_1+b_1) \right\} f_1 \right. \\ \left. - x(\phi_1+\psi_1+k'-1)f_2(\theta_1+b_1)f_1 \right] dt \\ = x \int_C t^{-k-1}(1-t)^{-k'} [f_2(1-t) (\theta_1+k)(\theta_1+b_1)f_1 \\ + (\phi_1+\psi_1+k'-1)f_2(\theta_1+b_1)f_1] dt \\ = x \int_C t^{-k-1}(1-t)^{-k'} [f_2(1-t) \{ t\partial/\partial t(t\partial/\partial t - b_1)f_1 \\ - kf_2(1-t)(t\partial/\partial t - b_1)f_1 \\ - (1-k')tf_2(t\partial/\partial t - b_1)f_1 \\ - t(1-t)\partial f_2/\partial t(t\partial/\partial t - b_1)f_1 \}] dt.$$

This gives on simplification

$$(5) \quad L_1(W) = x \int_C d [t^{-k}(1-t)^{1-k'} f_2 \{ t\partial f_1/\partial t - b_1 f_1 \}].$$

Similarly

$$(6) \quad L_2(W) = y \int_C d [t^{1-k}(1-t)^{-k'} \{ f_1(v\partial/\partial v + b_2)f_2 \}],$$

and

$$(7) \quad L_3(W) = z \int_C d [t^{1-k}(1-t)^{-k'} f_1(w\partial/\partial w + b_2)f_2].$$

Now 3(5-7) show that 3(2) will certainly be a solution of 2(1) whenever C is either a closed contour or else an open contour at the two ends of which

$$[t^{-k}(1-t)^{1-k'} f_2(t\partial/\partial t - b_1)f_1]$$

$$[t^{1-k}(1-t)^{-k'} f_1(v\partial/\partial v + b_2)f_2]$$

and

$$[t^{1-k}(1-t)^{-k'} f_1(w\partial/\partial w + b_2)f_2]$$

vanish where f_1 and f_2 are any of the solutions of the systems 3(4) and 3(5) respectively.

Next, from the Pochhammer type of integral representation of F_K (Shanti Saran, 1955), namely

$$(8) \quad F_K(a_1, a_2, a_3, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) \\ = \frac{\Gamma(k)\Gamma(k')\Gamma(2-k-k')}{(2\pi i)^2} \times \\ \times \int_C (-t)^{-k}(t-1)^{-k'} {}_2F_1(k, a_1; c_1; x/t) {}_2F_2(a_2, b_2, k'; c_2, c_3; y, z/1-t) dt$$

where $|t| > |x|$, $|y| + |z|/1-t| < 1$ along the contour and $b_1 = k+k'-1$, we see that the general solution of 2(2) will be of the type

$$(9) \quad W = \int_C t^{-k}(1-t)^{-k'} f_1(x/t) f_2(y, z/1-t) dt$$

provided the contour C is a closed contour or an open contour at the two ends of which

$$[t^{-k}(1-t)^{1-k'} f_2(t\partial/\partial t - a_1) f_1]$$

and

$$[t^{1-k}(1-t)^{-k'} f_1(v\partial/\partial v + a_2) f_2]$$

vanish.

Now, let us use in 3(2) the following branches of f_1 and f_2 valid in the vicinity of point $y = 1/x = 1/z = 0$, namely

$$u^{-b_1} {}_2F_1(b_1, 1+b_1-c_1, 1-b_1-k; 1/u)$$

and

$$w^{-b_2} {}_2F_4(1+b_2-c_3, b_2; c_2, 1+b_2-k'; v/w, 1/w)$$

respectively, we obtain a solution of 2(1) given by

$$W = Ax^{-b_1} z^{-b_2} F_R(1+b_2-c_3, b_1, 1+b_2-c_3, b_2, 1+b_1-c_1, b_2; c_2, 1+b_1+b_2-a_1, 1+b_1+b_2-a_1; y/z, 1/x, 1/z)$$

which is regular at $y = 1/x = 1/z = 0$.

Next, in 3(9) if we use a branch of f_1 in the vicinity of $1/x = 0$, we obtain a solution of 2(2) regular in the neighbourhood of $1/x = y = z = 0$. In fact, we get a power series in $1/x, y, z$, namely

$$W = A \sum_{m, n, p=0}^{\infty} \frac{(a_1, m)(a_2, n+p)(1+a_1-c_1, m)(b_2, n)}{(1, m)(1, n)(1, p)(1+a_1-b_1, m-p)(c_2, n)(c_3, p)} x^{-m} y^n (-z)^p,$$

where A is a constant. This function can be said to be a generalised Horn's function of three variables. For $y = 0$, this reduces to Horn's function H_2 .

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BIBLIOGRAPHY

- Appell, P. et Kampé de Fériet, J. (1926). Fonctions hypergéométriques et hypersphériques. Polynômes d'Hermite. Gauthier-Villars, Paris.
- Erdelyi, A. (1953). Higher transcendental functions. Vol. I.
- (1941). Integration of the differential equations of Appell's function F_4 . *Quart. Jour. of Math. (Oxford)*, 12, 68-77.
- Shanti Saran (1955). Hypergeometric functions of three variables. *Ganita*, 5, No. 2.
- (1955a). Integrals associated with hypergeometric functions of three variables. *Proc. Nat. Inst. Sc. of India*, 21A, No. 2, 1955.
- (1955b). A study of certain hypergeometric functions of three variables. *Thesis* approved for the degree of Ph.D., Lucknow University, pp. 76-100.

THE EQUATIONS OF MOTION OF PARTICLES IN THE UNIFIED FIELD THEORY OF EINSTEIN (1953)

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1. INTRODUCTION

As is well known the equations of motion of particles can be deduced from the field equations of general relativity by solving the latter so that the field is regular everywhere except at the particles. The usual method of approximation in general relativity utilizes the existence of the singularities of the field at the particles. The method followed by Einstein and co-workers (1938, 1949) is based on expansions in the powers of a parameter λ for the field potentials, the field equations having been split into terms according to the powers of λ . The split field equations demand integrability conditions. These integrability conditions are the equations of motion of the singularities of the field. Thus the equations of motion of the particles in general relativity turn out to be the integrability conditions of the field equations.

Einstein (1953) has postulated a set of field equations to represent the combined field of electromagnetism and gravitation. We find, on following the same technique for deriving the equations of motion, that these field equations also demand integrability conditions which are the required equations of motion. Our problem is to find out whether the equations of motion of the singularities contain terms representing Coulomb's law of force between charge particles and Newton's law of attraction between inertial particles.

Infeld (1951) and Callaway (1953) have applied the method of approximation to the sets of field equations postulated by Einstein in 1950 and 1953 respectively and shown that the equations of motion of charge particles, up to the first order of approximation, do not contain the term which accounts for Coulomb's law of force. We shall show here that a solution of the field equations (Einstein, 1953) can be found out so that the equations of motion of the singularities of the field contain a term accounting for Coulomb's law of force. The singularities considered by us here carry with them a charge cloud with density diminishing with distance and, in this respect, are different from those considered by Callaway.

2. PRELIMINARIES

The field equations postulated by Einstein (1953) are

$$\frac{\partial g_{\mu\nu}}{\partial x^\sigma} - g_{\rho\nu} \Gamma_{\mu\sigma}^\rho - g_{\mu\rho} \Gamma_{\sigma\nu}^\rho = 0, \quad \dots \dots \dots (2.1)$$

$$\partial(\sqrt{-g} g^{\mu\nu})/\partial x^\nu = 0, \quad \dots \dots \dots (2.2a)$$

$$\Gamma_{\mu\sigma}^\sigma = 0, \quad \dots \dots \dots (2.2b)$$

$$R_{\mu\nu} = 0, \quad \dots \dots \dots (2.3)$$

$$\frac{\partial R_{\mu\nu}}{\partial x^\sigma} + \frac{\partial R_{\sigma\mu}}{\partial x^\nu} + \frac{\partial R_{\nu\sigma}}{\partial x^\mu} = 0, \quad \dots \quad (2.4)$$

where

$$g = \det. |g_{\mu\nu}|,$$

$$g^{\mu\sigma}g_{\nu\sigma} = g^{\sigma\mu}g_{\sigma\nu} = \delta_\nu^\mu,$$

$$R_{\mu\nu} = R_{\mu\nu} + R_{\nu\mu} = \frac{\partial \Gamma_{\mu\nu}^\sigma}{\partial x^\sigma} - \frac{1}{2} \left[\frac{\partial \Gamma_{\mu\sigma}^\sigma}{\partial x^\nu} + \frac{\partial \Gamma_{\nu\sigma}^\sigma}{\partial x^\mu} \right] - \Gamma_{\mu\rho}^\sigma \Gamma_{\sigma\nu}^\rho + \Gamma_{\mu\nu}^\sigma \Gamma_{\sigma\rho}^\rho.$$

Here and in what follows the dummy suffix summation convention will be used. Greek indices refer to both space and time, running over the values 0, 1, 2, 3 and Latin indices refer to space co-ordinates only, running over the values 1, 2, 3. The suffix 0 refers to the time co-ordinate. A bar below two indices indicates symmetry and a hook in the same position indicates antisymmetry. $\Gamma_{\mu\nu}^\sigma$ are the affine connections to be determined by (2.1) in terms of the fundamental tensor $g_{\mu\nu}$ which consists of sixteen independent components. Equations (2.2a), (2.3) and (2.4) which are sixteen independent equations are to be solved for $g_{\mu\nu}$. The condition (2.2b) will be ensured by the equations (2.1) and (2.2a).

3. THE METHOD OF APPROXIMATION

The field equations will be solved for the case of quasi-stationary fields. For any field quantity ϕ , its derivatives with respect to the space co-ordinates x^s is of the order of ϕ while the derivative with respect to the time co-ordinate x^0 is of the order of $\lambda\phi$ where λ is of the order of smallness of v/c , v being a typical particle velocity and c the velocity of light. We define a new time $\tau (= \lambda x^0)$ so that the derivative of ϕ with respect to τ is of the order of ϕ . A derivative of ϕ with respect to (τ, x^s) is denoted by a comma:

$$\left. \begin{aligned} \frac{\partial \phi}{\partial \tau} &= \phi, {}_0; \\ \frac{\partial \phi}{\partial x^s} &= \phi, {}_s. \end{aligned} \right\} \dots \dots \dots (3.1)$$

The expansion of the field variables $g_{\mu\nu}$ in powers of λ may be taken as given in (3.2b) (Einstein, 1949):

$$\left. \begin{aligned} g_{\mu\nu} &= g_{\mu\nu} + g_{\mu\nu}; \\ g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}; \\ \gamma_{\mu\nu} &= h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} h_{\rho\sigma}; \\ \eta_{11} &= \eta_{22} = \eta_{33} = -\eta_{00} = -1; \\ \eta_{\mu\nu} &= 0, \mu \neq \nu; \\ \eta^{\mu\sigma} \eta_{\nu\sigma} &= \delta_\nu^\mu; \end{aligned} \right\} \dots \dots \dots (3.2a)$$

$$\left. \begin{aligned} \gamma_{00} &= \lambda^2 \gamma_{200} + \lambda^4 \gamma_{400} + \dots + \lambda^{2l} \gamma_{2l00} + \dots; \\ \gamma_{0m} &= \lambda^3 \gamma_{30m} + \lambda^5 \gamma_{50m} + \dots + \lambda^{2l+1} \gamma_{2l+10m} + \dots; \\ \gamma_{mn} &= \lambda^4 \gamma_{4mn} + \lambda^6 \gamma_{6mn} + \dots + \lambda^{2l+2} \gamma_{2l+2mn} + \dots; \\ g_{0m} &= \lambda^3 f_{30m} + \lambda^5 f_{50m} + \dots + \lambda^{2l+1} f_{2l+10m} + \dots; \\ g_{mn} &= \lambda^2 f_{2mn} + \lambda^4 f_{4mn} + \dots + \lambda^{2l} f_{2lmn} + \dots; \end{aligned} \right\} \dots \dots (3.2b)$$

In the above formulae an index below an entity indicates its order of smallness. Now the field equations (2.2a), (2.3) and (2.4) are expanded in powers of λ by substituting (3.2) for $g_{\mu\nu}$. The coefficients of λ^l ($l = 1, 2, 3 \dots$) in the field equations must vanish if the field equations are to be satisfied. The field equations will be solved here up to the order of λ^3 to find out the equations of motion up to the order of λ^4 . The field equations (2.2a), (2.3) and (2.4) up to the order of λ^3 give

$$f_{2mn, n} = 0, \quad \dots \dots \dots (3.3a)$$

$$f_{30m, m} = 0, \quad \dots \dots \dots (3.3b)$$

$$R_{200} \equiv \frac{1}{2} \gamma_{200, ss} = 0, \quad \dots \dots \dots (3.4a)$$

$$R_{2mn} \equiv \frac{1}{2} \delta_{mn} \gamma_{200, ss} = 0, \quad \dots \dots \dots (3.4b)$$

$$R_{30m} \equiv \frac{1}{2} (\gamma_{30m, ss} - \gamma_{30s, sm} + \gamma_{300, 0m}) = 0, \quad \dots \dots (3.4c)$$

$$\begin{aligned} & \frac{1}{6} \epsilon^{mnl0} (R_{2mn, l} + R_{2lm, n} + R_{2nl, m}) \\ &= \frac{1}{12} \epsilon^{mnl0} (f_{2nm, l} + f_{2ml, n} + f_{2ln, m}),_{ss} = 0, \quad \dots \dots (3.5a) \end{aligned}$$

$$\begin{aligned} & \frac{1}{2} \epsilon^{0mnl} (R_{30m, n} + R_{3n0, m} + R_{2mn, 0}) \\ &= \frac{1}{4} \epsilon^{0mnl} (f_{3m0, n} + f_{30n, m} + f_{2nm, 0}) = 0, \quad \dots \dots (3.5b) \end{aligned}$$

where δ_{mn} is Kronecker's delta symbol, $\epsilon^{\mu\nu\sigma\rho}$ is antisymmetric in all the four indices and ϵ^{1230} is unity.

The solution of the equations (3.3) and (3.4) is given by

$$f_{2mn} = \epsilon_{mns0} \phi_{20, s}, \quad \dots \dots \dots (3.6a)$$

$$f_{30m} = \epsilon_{ms0} \phi_{3s, i}, \quad \dots \dots \dots (3.6b)$$

$$\gamma_{200} = -4 \sum_{k=1}^N \frac{\hbar}{m} \psi, \quad \dots \dots \dots (3.7a)$$

$$\gamma_{0s} = 4 \sum_{k=1}^N m^{\dot{k}} \dot{\xi}^s \dot{\psi}, \quad \dots \dots \dots (3.7b)$$

$$\dot{\psi} = \left[(x^s - \dot{\xi}^s) (x^s - \dot{\xi}^s) \right]^{-\frac{1}{2}},$$

$$\epsilon_{\mu\nu\sigma\rho} = \epsilon^{\mu\nu\sigma\rho},$$

where ϕ_0 , ϕ_s are arbitrary functions of the co-ordinates x^μ , $m^{\dot{k}}$ are constants and $\dot{\xi}^s$ are the space co-ordinates of the k th particle at time τ . Here it may be noted that an index above an entity stands for the distinguishing number of the particle and a dot above a function stands for a differentiation with respect to the time τ . By the relation (3.6), equations (3.5a) and (3.5b) reduce to

$$-\frac{1}{2} \phi_{0,ssrr} = 0, \quad \dots \dots \dots (3.8a)$$

$$-\frac{1}{2} \left(\phi_{r,ss} - \phi_{s,sr} - \phi_{0,0r} \right)_{,tt} = 0. \quad \dots \dots \dots (3.8b)$$

A solution of (3.8a) may be taken as

$$\phi_0 = e^{\dot{k}} \dot{\psi} + \frac{1}{2} q^{\dot{k}} r, \quad \dots \dots \dots (3.9)$$

where $e^{\dot{k}}$ and $q^{\dot{k}}$ are functions of time only and $r^{\dot{k}}$, usually called as the distance from x^s to $\dot{\xi}^s$, is defined as

$$r^{\dot{k}} = \left[(x^s - \dot{\xi}^s) (x^s - \dot{\xi}^s) \right]^{\frac{1}{2}}.$$

It may be seen from (3.6) that the addition of the gradient of any function to ϕ_s does not alter f_{0m} . Such an addition is superfluous and may be considered as similar to the gauge transformation in the case of the classical electromagnetic theory. The integrability of (3.8b) demands

$$q^{\dot{k}} = 0. \quad \dots \dots \dots (3.10)$$

With the condition (3.10), a solution of (3.8b) may be given as

$$\phi_s = \left(e^{\dot{k}} \dot{\psi} + \frac{1}{2} q^{\dot{k}} r \right) \dot{\xi}^s. \quad \dots \dots \dots (3.11)$$

Now we are in a position to find out the equations of motion of the particles in the combined field of gravitation and electromagnetism, the gravitational field being given by (3.7) and the electromagnetic field being given by (3.6), (3.9), (3.10) and (3.11). Up to this approximation, there is no interaction between electromagnetic and gravitational fields.

4. THE EQUATIONS OF MOTION

The field equations that determine the motion of the particles are

$$\left. \begin{aligned} R_{mn} &= 0, \\ R_{00} &= 0. \end{aligned} \right\} \quad \dots \dots \dots (4.1)$$

We shall show now that the equations (4.1) demand integrability conditions. Equations (4.1) imply

$$R_{mn} + \frac{1}{2} \delta_{mn} (R_{00} - R_{ss}) = 0. \quad \dots \quad (4.2)$$

The equation (4.2) when written out explicitly gives

$$\begin{aligned} & \frac{1}{2} \left(\gamma_{mn, ss} - \gamma_{ms, sn} - \gamma_{ns, sm} + \delta_{mn} \gamma_{rs, rs} \right) \\ &= \frac{1}{2} \left(-\gamma_{0m, 0n} - \gamma_{0n, 0m} + 2 \delta_{mn} \gamma_{0s, 0s} - \delta_{mn} \gamma_{00, 00} \right) \\ & - \frac{1}{4} \gamma_{00, mn} - \frac{1}{8} \gamma_{00, m} \gamma_{00, n} + \frac{3}{8} \delta_{mn} \gamma_{00, s} \gamma_{00, s} \\ & - \frac{1}{2} \delta_{mn} \phi_{0, rs} \phi_{0, rs} + \frac{1}{2} f_{mr, s} f_{ns, r} \\ & - \frac{1}{4} \delta_{mn} \phi_{0, ss} \phi_{0, rr} + \phi_{0, ss} \phi_{0, mn} \\ & + \phi_{0, s} \phi_{0, smn} - \frac{1}{2} \phi_{0, m} \phi_{0, nrr} \\ & - \frac{1}{2} \phi_{0, n} \phi_{0, mrr} - \frac{1}{2} \delta_{mn} \phi_{0, s} \phi_{0, stt} \quad \dots \quad (4.3) \end{aligned}$$

The equation (4.3) may be written as

$$\Phi_{mn} = A_{mn}, \quad \dots \quad (4.4)$$

where Φ_{mn} stands for the expression on the left-hand side of the equation (4.3) and A_{mn} stands for the expression on the right side of (4.3). It may be seen (Einstein, 1949) that

$$\left. \begin{aligned} \Phi_{mn, n} &= 0, \\ \oint_{\bar{S}} \Phi_{mn} n_n dS &= 0, \end{aligned} \right\} \quad \dots \quad (4.5)$$

where \bar{S} stands for a closed two-dimensional surface enclosing the k th singularity and n_n is the unit outward normal to the surface \bar{S} . According to (4.4) and (4.5), A_{mn} should satisfy the equations

$$A_{mn, n} = 0, \quad \dots \quad (4.6a)$$

$$\oint_{\bar{S}} A_{mn} n_n dS = 0. \quad \dots \quad (4.6b)$$

A_{mn} contains only known functions of the co-ordinates. It may be seen that (4.6a) is satisfied everywhere in the regular region. The equation (4.6a) ensures that the left-hand side of (4.6b) is independent of x^s . The integral on the left side of (4.6b)

THE PROBLEM OF MOTION IN GENERAL RELATIVITY

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1. INTRODUCTION

In the early stages of the development of general relativity, it was assumed that a geodesic principle governs the motion of the particles. But later it was recognized that the field equations of general relativity are strong enough to determine the co-ordinates of the particles which are sources of the field. The problem of motion consists in solving the field equations so that the field is regular everywhere except at the particles. To do this, one has to take recourse to an approximation method. The problem of deriving the equations of motion from the field equations of empty space alone was tackled by Einstein, Infeld and Hoffmann in 1938, and Einstein and Infeld in 1940 and 1949. But from the earlier methods it does not follow that the field equations can be solved to any arbitrary high approximation. The method given in 1949 was considered to be satisfactory from the logical standpoint. We shall see in § 2 that this method also is not satisfactory from the mathematical point of view. A new version of the approximation method will be given in § 3 and § 4.

2. A MATHEMATICAL DISCREPANCY IN THE METHOD OF APPROXIMATION GIVEN IN 1949

The method of approximation starts with the expansion of the field variables $\gamma_{\mu\nu}$ which are defined by

$$\left. \begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, \\ \gamma_{\mu\nu} &= h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} h_{\rho\sigma}, \\ \eta_{11} &= \eta_{22} = \eta_{33} = -\eta_{00} = -1, \\ \eta_{\mu\nu} &= 0, \quad \mu \neq \nu, \end{aligned} \right\} \quad \dots \quad (2.1)$$

where $g_{\mu\nu}$ is the symmetric metric tensor of space-time. Here and in what follows the dummy suffix summation convention is generally adopted, exceptions being explicitly stated. Latin indices refer to space-co-ordinates running over the values 1, 2, 3 and the Greek indices refer to both space and time, running over the values 0, 1, 2, 3. The suffix 0 refer to the time co-ordinate. The power-series development for $\gamma_{\mu\nu}$ to represent the field of a non-radiating system of particles is known to be given by

$$\left. \begin{aligned} \gamma_{00} &= \lambda^2 \gamma_{00}^{(2)} + \lambda^4 \gamma_{00}^{(4)} + \dots + \lambda^{2l} \gamma_{00}^{(2l)} + \dots, \\ \gamma_{0m} &= \lambda^3 \gamma_{0m}^{(3)} + \lambda^5 \gamma_{0m}^{(5)} + \dots + \lambda^{2l+1} \gamma_{0m}^{(2l+1)} + \dots, \\ \gamma_{mn} &= \lambda^4 \gamma_{mn}^{(4)} + \lambda^6 \gamma_{mn}^{(6)} + \dots + \lambda^{2l+2} \gamma_{mn}^{(2l+2)} + \dots, \end{aligned} \right\} \quad \dots \quad (2.2)$$

where the parameter λ is of the order of smallness of v/c , v being a typical velocity of the particles and c the velocity of light. The indices at the bottom indicate the order of smallness. The derivative of a field quantity ϕ with respect to x^s is of the same order as ϕ while the derivative of ϕ with respect to x^0 is of the order of $\lambda\phi$. To treat the derivatives with respect to the time-co-ordinate on the same footing as the derivatives with respect to space-co-ordinates x^s , a new time τ ($= \lambda x^0$) is introduced so that $\partial\phi/\partial\tau$ is of the same order of ϕ . The derivatives of any entity with respect to (τ, x^s) are denoted by a comma. Thus

$$\left. \begin{aligned} \partial\phi/\partial\tau &= \frac{1}{\lambda} \partial\phi/\partial x^0 = \phi_{,0}; \\ \partial\phi/\partial x^s &= \phi_{,s}. \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (2.3)$$

Instead of the usual field equations

$$R_{\mu\nu} \equiv -\frac{\partial\Gamma_{\mu\nu}^{\sigma}}{\partial x^{\sigma}} + \frac{\partial\Gamma_{\mu\sigma}^{\sigma}}{\partial x^{\nu}} + \Gamma_{\mu\sigma}^{\rho} \Gamma_{\nu\rho}^{\sigma} - \Gamma_{\mu\nu}^{\sigma} \Gamma_{\sigma\rho}^{\rho} = 0, \quad \dots \quad \dots \quad (2.4)$$

a set of linear combinations of the field equations defined by

$$\Phi_{\mu\nu} + 2A_{\mu\nu} \equiv 2\left(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R_{\rho\sigma}\right) = 0, \quad \dots \quad \dots \quad (2.5)$$

where

$$\left. \begin{aligned} \Phi_{00} &= -\gamma_{00, ss}, \\ \Phi_{0m} &= -(\gamma_{0m, s} - \gamma_{0s, m}), s, \\ \Phi_{mn} &= -(\gamma_{mn, s} - \gamma_{ms, n}), s + (\delta_{mr}\gamma_{ns, s} - \delta_{mn}\gamma_{rs, s}), r, \end{aligned} \right\} \quad \dots \quad \dots \quad (2.6)$$

and

$$\left. \begin{aligned} 2A_{00} &= \gamma_{sr, sr} + 2A'_{00}, \\ 2A_{0m} &= \lambda(\gamma_{ms, s0} - \gamma_{00, m0}) + 2A'_{0m}, \\ 2A_{mn} &= -\lambda(\gamma_{0m, 0n} + \gamma_{0n, 0m} - 2\delta_{mn}\gamma_{0s, 0s}) + \lambda^2\gamma_{mn, 00} \\ &\quad - \lambda^2\delta_{mn}\gamma_{00, 00} + 2A'_{mn}. \end{aligned} \right\} \quad \dots \quad \dots \quad (2.7)$$

is to be solved by the method of approximation. Then the field equations (2.5) are split according to the powers of λ and are given as

$$\Phi_{00} + 2A_{00} = 0, \quad \dots \quad \dots \quad \dots \quad (2.8a)$$

$$\Phi_{0m} + 2A_{0m} = 0, \quad \dots \quad \dots \quad \dots \quad (2.8b)$$

$$\Phi_{mn} + 2A_{mn} = 0, \quad \dots \quad \dots \quad \dots \quad (2.8c)$$

$$l = 1, 2, 3 \dots$$

These equations are to be solved step by step without introducing singularities in empty space. A solution of the field equations (2.8a) for $l = 1$, viz.,

$$-\gamma_{00, ss} = 0 \quad \dots \quad \dots \quad \dots \quad (2.9)$$

is given by

$$\psi = 1/r = \left[(x^s - \xi^s)(x^s - \xi^s) \right]^{-1/2}, \quad \gamma_{00} = -4 \sum_{h=1}^N \frac{m_h}{2} \psi, \quad \dots \quad (2.10)$$

so that it may represent the field of N particles. ξ^s in the above formula are the space-co-ordinates of the k th particle at time τ . Starting with the solution (2.10) the field equations (2.8) for all l are to be solved consistently. In the process of solving the field equations (2.8b) and (2.8c) it becomes inevitable to have to add single poles $-4 \frac{k}{2l} m \psi$ and dipoles $-\frac{k}{2l} S_r \psi, r$ to γ_{00} . Here m and S_r are functions of time which will be determined by the integrability conditions of the field equations (2.8b) and (2.8c). The integrability conditions of the equations (2.8c) that determine S_r are

$$\oint_S \frac{k}{2l+2} 2A_{mr} n_r ds = 0, \quad \dots \quad (2.11)$$

where S is a two-dimensional surface enclosing the k th particle and n_r is the unit normal vector to the surface S . Equations (2.11) after the integration give

$$\frac{k}{2l+2} C_m - \frac{k}{2l} S_m + A_m = 0, \quad \dots \quad (2.12)$$

where

$$\frac{k}{2l+2} A_m = \sum_{p=1}^N \left(\frac{p}{2} \frac{k}{2l} m S_r - \frac{k}{2} \frac{p}{2l} S_r \right) \psi, r m + \left(\frac{k}{2} \frac{p}{2l} S_t - \frac{p}{2} \frac{k}{2l} S_t \right) \psi, t r m,$$

$$\psi = \left(\frac{p}{\psi} \right) (x^s = \xi^s), \psi, r \dots = \left(\frac{p}{\psi, r \dots} \right) (x^s = \xi^s),$$

$\sum_{p=1}^N$ stands for the summation of p from 1 to N excepting k and a dot above an entity stands for differentiation with respect to time thus:

$$\frac{k}{2l} S_m = d^2 \left(\frac{k}{2l} S_m \right) / d\tau^2.$$

$\frac{k}{2l} C_m$ in the equation (2.12) are certain known functions of the co-ordinates of the particles and their derivatives with respect to time τ .

By the device of adding single poles and dipoles to γ_{00} at each stage of the approximation the field equations can be solved to any desired degree of accuracy. Finally dipoles are removed by the use of the condition

$$\sum_{l=1} \lambda^{2l} \frac{k}{2l} S_r = 0, \quad \dots \quad (2.13)$$

which are the equations of motion of the particles.

The equation (2.13) implies that either

$$\frac{k}{2l} S_r = 0, \quad l = 1, 2, 3, \dots, \quad \dots \quad (2.14)$$

or that at least one of $\frac{k}{2l} S_r$ ($l = 1, 2, \dots$) is a function of λ . The equations (2.14) imply

$$\frac{k}{2l+2} C_m = 0, \quad l = 1, 2, 3, \dots, \quad \dots \quad (2.15)$$

which are to be satisfied by ξ^s . ξ^s are $3N$ functions of time. The infinite number of equations (2.15) will not in general be satisfied by any system of functions ξ^s .

Thus we are left with no alternative but to regard S_r as a function of λ . This implies that γ_{00} is a function of λ since we have added dipoles $-S_r \psi_r$ to γ_{00} . But the expansion (2.2) for $\gamma_{\mu\nu}$ implies that $\gamma_{\mu\nu}$ are independent of λ which will be vitiated if S_r is a function of λ . Actually the device of adding dipoles was to avoid the infinite number of equations (2.15) and get the equations

$$\sum_{l=1}^{\infty} \lambda^{2l+2} C_{2l+2}^k = 0, \quad \dots \quad (2.16)$$

This will be possible only by the violation of the assumption implied in the expansion (2.2), viz. that $\gamma_{\mu\nu}$ are independent of λ . Thus the method of adding and annihilating the dipole field also do not satisfactorily solve the problem of motion.

The fact that γ_{00} is required to be independent of λ was overlooked from the beginning. In the equations of motion, one expects Newtonian terms in the first approximation; and in the second approximation, those terms that account for the perihelion motion of Mercury. The expected equations of motion are of the type

$$\lambda^4 \left(m \ddot{\xi}^s - \sum_{p=1}^N m \frac{p}{2} \psi_{p,s} \right) + \lambda^6 C_s^k + \dots = 0. \quad \dots \quad (2.17)$$

The equation (2.17) implies that ξ^s are functions of the parameter λ . But a solution of the equation (2.9) was given by (2.10) which is a function of λ since ξ^s is a function of λ . To avoid this inconsistency in the procedure one may assume at the outset that ξ^s , the co-ordinates of the k th particle and M , the mass of the k th particle, are functions of λ ; and seek solutions of the field equations (2.8) for which $\gamma_{\mu\nu}$ are independent of λ . In §3 and §4 we shall see how the field equations can be solved consistently to any arbitrary order of approximation.

3. THE DEVELOPMENT OF ξ^s AND M

With the power-series development (2.2) for $\gamma_{\mu\nu}$ the field equations (2.5) are expanded in powers of λ and split according to powers of λ . In the l th stage of the approximation the equations to be solved are

$$\Phi_{00} + 2A_{00} = 0, \quad \dots \quad (3.1a)$$

$$\Phi_{0m} + 2A_{0m} = 0, \quad \dots \quad (3.1b)$$

$$\Phi_{mn} + 2A_{mn} = 0, \quad \dots \quad (3.1c)$$

where

$$\Phi_{00}^{2l} \equiv -\gamma_{00, ss}^{2l}, \quad \dots \quad (3.2a)$$

$$\Phi_{0m}^{2l+1} \equiv -\left(\gamma_{0m, s}^{2l+1} - \gamma_{0s, m}^{2l+1}\right), s, \quad \dots \quad (3.2b)$$

$$\Phi_{mn}^{2l+2} \equiv -\left(\gamma_{mn, s}^{2l+2} - \gamma_{ms, n}^{2l+2}\right), s + \left(\delta_{ms} \gamma_{nr, r}^{2l+2} - \delta_{mn} \gamma_{sr, r}^{2l+2}\right), s, \quad \dots \quad (3.2c)$$

and

$$\left. \begin{aligned} 2A_{00}^{2l} &\equiv \gamma_{rs, rs}^{2l} + 2A_{00}'^{2l}, \\ 2A_{0m}^{2l+1} &\equiv \left(-\gamma_{00, 0m}^{2l} + \gamma_{ms, s0}^{2l}\right) + 2A_{0m}'^{2l+1}, \\ 2A_{mn}^{2l+2} &\equiv -\gamma_{0m, 0n}^{2l+1} - \gamma_{0n, 0m}^{2l+1} + 2\delta_{mn} \gamma_{0s, 0s}^{2l+1} \\ &\quad + \gamma_{mn, 00}^{2l} - \delta_{mn} \gamma_{00, 00}^{2l} + 2A_{mn}'^{2l+2}. \end{aligned} \right\} \quad \dots \quad (3.3)$$

In the above formulae $A_b^{\mu\nu}$ contains all the non-linear terms of order b in the field equations (2.5). The field equations are to be solved so that $\gamma_{\mu\nu}$ are regular everywhere except at the positions of the particles, tending to zero as $r \rightarrow \infty$.

If a solution of the field equations in a specially chosen co-ordinate system is given, one can find out the solution of the field equations in any co-ordinate system connected with the chosen one by a given co-ordinate transformation. Hence, without any loss of generality, the field equations will now be solved in a co-ordinate system distinguished from others by the co-ordinate conditions:

$$\left. \begin{aligned} \gamma_{0m, m} - \gamma_{00, 0} &= 0, \\ \gamma_{mn, n} &= 0. \end{aligned} \right\} \quad \dots \quad (3.4)$$

We consider that the presence of the function

$$-4M\psi^{kk} \quad \dots \quad (3.5)$$

in γ_{00} is crucial for the representation of the field of N particles which are at the positions ξ^s at time τ . The character of the entire solution will depend on the harmonic function (3.5).

Schwarzschild's solution in isotropic co-ordinates to represent the field of a particle of mass m is given by

$$\left. \begin{aligned} \gamma_{00} &= -\frac{4m}{r} - \frac{2m^2}{r^2} + \dots, \\ \gamma_{0m} &= 0, \\ \gamma_{mn} &= \delta_{mn} \frac{m^2}{r^2} + \dots \end{aligned} \right\}$$

or as

$$\left. \begin{aligned} \gamma_{00} &= -4m/r - 2m^2/r^2 + \dots, \\ \gamma_{0m} &= 0, \\ \gamma_{mn} &= \delta_{mn} m^2/r^2 + \dots \end{aligned} \right\} \quad \dots \quad (3.6)$$

In considering (3.4) and (3.5) we are seeking a solution for the field of N particles analogous to the solution (3.6).

Since $\overset{k}{M}$ is at the most of the order of λ^2 , the expansions for $\overset{k}{M}$ and $\overset{k}{\xi^s}$ may be taken as

$$\left. \begin{aligned} \overset{k}{M} &= \lambda^2 \overset{k}{M}_2 + \lambda^3 \overset{k}{M}_3 + \lambda^4 \overset{k}{M}_4 + \dots, \\ \overset{k}{\xi^s} &= \overset{k}{\xi^s}_0 + \lambda \overset{k}{\xi^s}_1 + \lambda^2 \overset{k}{\xi^s}_2 + \dots, \end{aligned} \right\}$$

When (3.7) is substituted in $-4\overset{k}{M}\psi$ one gets, on expansion,

$$-4 \left(\lambda^2 \overset{k}{M}_2 + \lambda^3 \overset{k}{M}_3 + \dots \right) \overset{k}{\psi} + 4 \left(\sum_{l=2}^{\infty} \lambda^l \overset{k}{M}_l \right) \left(\sum_{b=1}^{\infty} \lambda^b \overset{k}{\xi^s}_b \right) \overset{k}{\psi}_{,s} + \dots, \quad \dots \quad (3.8)$$

where

$$\overset{k}{\psi} = (\overset{k}{R})^{-1} = \left[(x^s - \overset{k}{\xi^s}) (x^s - \overset{k}{\xi^s}) \right]^{-\frac{1}{2}}$$

Since $-4\overset{k}{M}\psi$ is to be present in γ_{00} , according to the expansions (2.2) and (3.7), one finds that

$$\overset{k}{M}_{2l+1} = \overset{k}{\xi^s}_{2l+1} = 0, \quad \dots \quad \dots \quad \dots \quad (3.9)$$

for all positive integral values of l . Thus in accordance with the power-series development (2.2) for $\gamma_{\mu\nu}$, the power-series development for $\overset{k}{\xi^s}$ and $\overset{k}{M}$ is

$$\overset{k}{\xi^s} = \overset{k}{\xi^s}_0 + \lambda \overset{k}{\xi^s}_1 + \lambda^2 \overset{k}{\xi^s}_2 + \dots + \lambda^{2l} \overset{k}{\xi^s}_{2l} + \dots, \quad \dots \quad \dots \quad (3.10a)$$

$$\overset{k}{M} = \lambda^2 \overset{k}{M}_2 + \lambda^4 \overset{k}{M}_4 + \dots + \lambda^{2l} \overset{k}{M}_{2l} + \dots, \quad \dots \quad \dots \quad (3.10b)$$

4. THE GENERAL OUTLINE OF THE METHOD

By the use of (3.10) the function $-4\overset{k}{M}\psi$ may be expanded in powers of λ as

$$\begin{aligned} -4\overset{k}{M}\psi &= \lambda^2 \left(-4\overset{k}{M}_2 \overset{k}{\psi} \right) + \lambda^4 \left(-4\overset{k}{M}_4 \overset{k}{\psi} + 4\overset{k}{M}_2 \overset{k}{\xi^s}_2 \overset{k}{\psi}_{,s} \right) + \dots \\ &\quad + \lambda^{2l} \left[-4\overset{k}{M}_{2l} \overset{k}{\psi} + 4 \sum_{a=1}^l \overset{k}{M}_{2(l-a)} \overset{k}{\xi^s}_{2a} \overset{k}{\psi}_{,s} \right. \\ &\quad \left. - 2 \sum_{a=1}^l \sum_{b=1}^l \overset{k}{M}_{2(l-a-b)} \overset{k}{\xi^s}_{2a} \overset{k}{\xi^s}_{2b} \overset{k}{\psi}_{,sr} + \dots \right] + \dots, \quad \dots \quad (4.1a) \end{aligned}$$

where $\overset{k}{M}$ for $a = 0, 1, 2, \dots$, is considered to be zero. The expansion (4.1a) is valid in the region given by

$$\left(x^s - \overset{k}{\xi^s}_0 \right) \left(x^s - \overset{k}{\xi^s}_0 \right) > \left(\overset{k}{\xi^s}_0 - \overset{k}{\xi^s}_0 \right) \left(\overset{k}{\xi^s}_0 - \overset{k}{\xi^s}_0 \right). \quad \dots \quad \dots \quad (4.1b)$$

The potential γ_{00} may be divided into two parts as

$$\gamma_{00} = \bar{\gamma}_{00} - 4M \overset{k}{\psi}, \quad \dots \quad \dots \quad \dots \quad (4.2)$$

where $\bar{\gamma}_{00}$ stands for the function which does not contain any harmonic function. The expansion (2.2) for $\gamma_{\mu\nu}$ now takes the form

$$\left. \begin{aligned} \gamma_{00} &= \lambda^2 \left(\bar{\gamma}_{00} - 4M \overset{k}{\psi} \right) + \lambda^4 \left(\bar{\gamma}_{00} - 4M \overset{k}{\psi} + 4M \overset{k}{\xi}^s \overset{k}{\psi}_s \right) + \dots, \\ \gamma_{0m} &= \lambda^3 \gamma_{0m} + \lambda^5 \gamma_{0m} + \dots + \lambda^{2l+1} \gamma_{0m} + \dots, \\ \gamma_{mn} &= \lambda^4 \gamma_{mn} + \lambda^6 \gamma_{mn} + \dots + \lambda^{2l+2} \gamma_{mn} + \dots, \end{aligned} \right\} \quad \dots \quad (4.3)$$

where

$$\begin{aligned} \gamma_{00} &= \bar{\gamma}_{00} - 4M \overset{k}{\psi} + 4 \sum_{a=1}^l M \overset{k}{\xi}^s \overset{k}{\psi}_s \\ &\quad - 2 \sum_{a=1}^l \sum_{b=1}^l M \overset{k}{\xi}^s \overset{k}{\xi}^r \overset{k}{\psi}_{sr} + \dots \quad \dots \quad \dots \quad (4.4) \end{aligned}$$

The variables $\bar{\gamma}_{00}$, γ_{0m} , γ_{mn} , M and $\overset{k}{\xi}^s$ appear for the first time in the field equations at the l th stage of the approximation. Hence these variables are to be determined by the equations (3.1). For the integrability of (3.1b) and (3.1c) certain integrals over closed surfaces, each enclosing one and only one of the particles, should vanish. The equations provided by the vanishing of these integrals are expected to determine M and $\overset{k}{\xi}^s$, while the equations (3.1) determine the field variables $\bar{\gamma}_{00}$, γ_{0m} , γ_{mn} .

For the uniqueness of the solution, it was assumed that no new harmonic function is to be added to a field variable at any stage of the approximation. The equation (3.1a) for $l = 1$ is

$$\bar{\gamma}_{00, ss} = 0, \quad \dots \quad \dots \quad \dots \quad (4.5a)$$

which gives the unique solution

$$\bar{\gamma}_{00} = 0 \quad \dots \quad \dots \quad \dots \quad (4.5b)$$

according to the assumption about the addition of harmonic functions. It is to be shown that the field equations can be solved to any desired degree of accuracy. We shall prove this by the method of induction. Let us assume that the field equations are solved up to the $(l-1)$ th stage of the approximation. It is to be shown that the equations (3.1) can be solved. Since the field equations in the lower approximations are assumed to be solved, the variables

$$\bar{\gamma}_{00}, \gamma_{0m}, \gamma_{mn}, M, \overset{k}{\xi}^s, \quad \dots \quad \dots \quad \dots \quad (4.6)$$

$$a = 1, 2, \dots, (l-1),$$

may be considered as known functions of the co-ordinates (x^s, τ) .

By the use of (4.4), the equations (3.1) are written out thus:

$$-\bar{\gamma}_{00, ss} + 2\bar{A}_{00} = 0, \quad \dots \quad (4.7a)$$

$$\Phi_{0m} + 2\bar{A}_{0m} + C_{0m} = 0, \quad \dots \quad (4.7b)$$

$$\Phi_{mn} + 2\bar{A}_{mn} + C_{mn} = 0, \quad \dots \quad (4.7c)$$

where

$$C_{0m} = 4 \left(\bar{M} \frac{k}{2l} \bar{\psi} \right)_{, 0m} - 4 \left(\bar{M} \frac{k}{2} \xi^s \frac{k}{2l-2} \bar{\psi} \right)_{, 0ms} \quad \dots \quad (4.7d)$$

and

$$\begin{aligned} C_{mn} = & -4 \left[\bar{M} \left(\xi^m \frac{k}{2l-2} \bar{\psi}_{, n} + \xi^n \frac{k}{2l-2} \bar{\psi}_{, m} - \delta_{mn} \xi^s \frac{k}{2l-2} \bar{\psi}_{, s} \right) \right]_{, 00} \\ & - \gamma_{00, n} \left(\bar{M} \xi^s \frac{k}{2} \frac{k}{2l-2} \bar{\psi}_{, sm} \right) - \gamma_{00, m} \left(\bar{M} \xi^s \frac{k}{2} \frac{k}{2l-2} \bar{\psi}_{, sn} \right) \\ & + 3\delta_{mn} \gamma_{00, s} \left(\bar{M} \xi^r \frac{k}{2} \frac{k}{2l-2} \bar{\psi}_{, rs} \right) \\ & - 2\gamma_{00, mn} \left(\bar{M} \xi^s \frac{k}{2} \frac{k}{2l-2} \bar{\psi}_{, s} \right) - 2\gamma_{00} \left(\bar{M} \xi^s \frac{k}{2} \frac{k}{2l-2} \bar{\psi}_{, smn} \right). \quad \dots \quad (4.7e) \end{aligned}$$

The function A_{0m} is split into two parts, viz. \bar{A}_{0m} , containing all those terms that are not dependent on \bar{M} and ξ^s and C_{0m} , containing all those terms that are dependent on \bar{M} and ξ^s . But the function A_{mn} is split into two parts, viz. \bar{A}_{mn} , containing those and only those terms that are not dependent on ξ^s , and C_{mn} , containing all those terms that are dependent on ξ^s .

The A_{00} contains only known functions (4.6) of the co-ordinates. The equation (4.7a), which is Poisson's equation in $\bar{\gamma}_{00}$, determines $\bar{\gamma}_{00}$ uniquely. Now the equation (4.7b) is to be solved for γ_{0m} . Since

$$\begin{aligned} \Phi_{0m, m} &= 0, \\ \oint_S \Phi_{0m} n_m dS &= 0, \end{aligned}$$

the conditions

$$2\bar{A}_{0m, m} + C_{0m, m} = 0, \quad \dots \quad (4.8a)$$

$$\left(2\bar{A}_{0m} + C_{0m} \right) n_m dS = 0, \quad \dots \quad (4.8b)$$

are to be satisfied for the integrability of (4.7b). Since we have to take $\overset{k}{S}$ in the regular region for the γ 's and since our procedure implies the expansion (4.1a), the surface $\overset{k}{S}$ is to be taken in the region given by (4.1b). This will be understood throughout the work. The four identities between the field equations, together with the field equations in the lower approximations, ensure the condition (4.8a), which in turn ensures that the left-hand side of (4.8b) is independent of the space-co-ordinates x^s . Since \bar{A}_{0m}^{2l+1} contain only known functions of the co-ordinates, the left-hand side of (4.8b) can be evaluated and the contribution from \bar{A}_{0m}^{2l+1} and C_{0m}^{2l+1} is given by

$$\frac{1}{4\pi} \oint_{\overset{k}{S}} C_{0m}^{2l+1} n_m dS = -4\overset{k}{M}_{2l}, \quad \dots \dots \dots (4.9)$$

$$\frac{1}{4\pi} \oint_{\overset{k}{S}} 2\bar{A}_{0m}^{2l+1} n_m dS = \overset{k}{C}_0^{2l+1}, \quad \dots \dots \dots (4.10)$$

where $\overset{k}{C}_0^{2l+1}$ is a known function of $\overset{k}{\xi^s}$, $a = 0, 1, 2, \dots (l-2)$. Thus the integrability condition (4.8b) gives

$$4\overset{k}{M}_{2l} - \overset{k}{C}_0^{2l+1} = 0. \quad \dots \dots \dots (4.11)$$

The equation (4.11) can always be solved for $\overset{k}{M}_{2l}$. Hence the integrability condition (4.8b) can be satisfied by choosing $\overset{k}{M}_{2l}$ according to the equation (4.11). Now the equation (4.7b), together with the condition (3.4), can be solved uniquely for γ_{0m}^{2l+1} .

So far we have seen that $\bar{\gamma}_{00}^{2l}$, $\overset{k}{M}_{2l}$ and γ_{0m}^{2l+1} become known from the equations (4.7a), (4.11) and (4.7b). The equation (4.7c) remains to be solved, the unknowns being γ_{mn}^{2l+2} and $\overset{k}{\xi^s}_{2l-2}$.

Since Φ_{mn}^{2l+2} satisfies the equations

$$\Phi_{mn,n}^{2l+2} = 0,$$

$$\oint_{\overset{k}{S}} \Phi_{mn}^{2l+2} n_n dS = 0,$$

the conditions,

$$2\bar{A}_{mn,n}^{2l+2} + C_{mn,n}^{2l+2} = 0, \quad \dots \dots \dots (4.12a)$$

$$\frac{1}{4\pi} \oint_{\overset{k}{S}} (2\bar{A}_{mn}^{2l+2} + C_{mn}^{2l+2}) n_n dS = 0, \quad \dots \dots \dots (4.12b)$$

are to be fulfilled for the integrability of (4.7c). The four identities, together with the field equations in the lower approximations, ensure the equation (4.12a). The equation (4.12a) ensures that the left-hand side of (4.12b) is at most a function of time. Since \bar{A}_{mn} is a known function of the co-ordinates the contribution to the integral coming from \bar{A}_{mn} can be evaluated. Therefore, we write

$$\frac{1}{4\pi} \oint_S \frac{2\bar{A}_{mn}}{2l+2} n_n dS = \frac{k}{2l+2} C_m, \quad \dots \dots \dots (4.13a)$$

and assume that we have calculated C 's. From the equation (4.7e) it may be seen that

$$\frac{1}{4\pi} \oint_S \frac{C'_{mn}}{2l+2} n_n dS = 4 \left(\frac{k}{2} \frac{M}{2l-2} \xi^m \right)_{,00} + 4M \sum_{p=1}^N \frac{p}{2} \left(\frac{p}{\xi^r} - \frac{k}{\xi^r} \right) \psi_{,rm}, \quad (4.13b)$$

where

$$\frac{p,k}{\psi} = \left(\frac{p}{\psi} \right) (x^s = \xi^s), \quad \frac{p,k}{\psi, r} \dots = \left(\frac{p}{\psi, r} \dots \right) (x^s = \xi^s).$$

Hence the integrability condition (4.12b) gives

$$4 \left(\frac{k}{2} \frac{M}{2l-2} \xi^m \right)_{,00} + 4M \sum_{p=1}^N \frac{p}{2} \left(\frac{p}{\xi^r} - \frac{k}{\xi^r} \right) \psi_{,rm} + \frac{k}{2l+2} C_m = 0. \quad \dots (4.14)$$

$\frac{k}{\xi^s}$ can be chosen such that the condition (4.14) is satisfied and hence the equations (4.7c), together with the co-ordinate conditions (3.4), can be solved for γ_{mn} . Thus from the equations (4.7), (4.11) and (4.14) the variables $\bar{\gamma}_{00}$, γ_{0m} , γ_{mn} , $\frac{k}{M}$ and $\frac{k}{\xi^s}$ become known. Since the field equations are satisfied by the Minkowskian metric $\eta_{\mu\nu}$ the field equations can be solved to any order of approximation. In solving the field equations we get the equations of the type (4.14) which determine $\frac{k}{\xi^s}$. Thus the equations (4.14) for $l = 1, 2, 3 \dots$, give the equations of motion of the particles. The co-ordinates of the particles given by

$$\xi^s = \xi^s_0 + \lambda^2 \xi^s_2 + \dots + \lambda^{2l-2} \xi^s_{2l-2} + \dots, \quad \dots \dots (4.15)$$

become known functions of time up to the term $\frac{k}{\xi^s_{2l-2}}$ when the field equations are solved up to the order $(2l+1)$. The equations (4.15) are the integrated equations of motion.

$-4M \frac{k}{2} \frac{\xi^s}{2l-2}$ serves the same purpose as $\frac{k}{S_r}$ given in the method of approximation followed by Einstein and Infeld in 1949. $\frac{k}{\xi^s}$ is a part of ξ^s , the co-ordinates of the k th particle. Thus this method is based on a purely mathematical deduction and no physical concept was used to get the equations of motion.

5. EQUATIONS OF MOTION UP TO THE SIXTH ORDER

By direct calculations the integrability conditions (4.14) for $l = 1$ and $l = 2$ can be found out and are given by

$$4 \left(\overset{h}{M} \overset{h}{\xi}^m - \overset{h}{M} \sum_{p=1}^{N'} \overset{p}{M} \overset{p}{\psi}_{,m} \right) = 0, \quad \dots \quad (5.1)$$

and

$$\begin{aligned} & \overset{h}{M} \left[\overset{h}{\xi}^m - \sum_{p=1}^{N'} \overset{p}{M} \left(\overset{h}{\xi}^s - \overset{p}{\xi}^s \right) \overset{p}{\psi}_{,sm} \right] \\ & + \sum_{p=1}^{N'} \overset{h}{M} \overset{p}{M} \left[-\frac{1}{2} \overset{p}{R}_{,sr} \overset{h}{\xi}^s \overset{p}{\xi}^r + \frac{1}{2} \overset{p}{\xi}^s \overset{p}{R}_{,sm} - 4 \overset{p}{\psi} \sum_{q=1}^N \overset{q}{M} \overset{q}{\psi}_{,m} \right. \\ & + \overset{p}{\psi}_{,s} \left\{ 4 \overset{h}{\xi}^s \overset{h}{\xi}^m + 4 \overset{p}{\xi}^s \overset{p}{\xi}^m - 4 \overset{h}{\xi}^s \overset{p}{\xi}^m - 3 \overset{p}{\xi}^s \overset{h}{\xi}^m \right\} \\ & + \overset{p}{\psi}_{,m} \left\{ -\overset{h}{\xi}^s \overset{h}{\xi}^s - \frac{3}{2} \overset{p}{\xi}^s \overset{p}{\xi}^s + 4 \overset{h}{\xi}^s \overset{p}{\xi}^s + 4 \sum_{q=1}^N \overset{q}{M} \overset{q}{\psi} \right. \\ & \left. \left. + \sum_{q=1}^N \overset{q}{M} \overset{q}{\psi} \right\} \right] = 0. \quad \dots \quad (5.2) \end{aligned}$$

The equations (5.1) and (5.2) together with (4.15) give the equations of motion

$$\begin{aligned} \lambda^4 \overset{h}{M} \left(\overset{h}{\xi}^m - \sum_{p=1}^{N'} \overset{p}{M} \overset{p}{\psi}_{,m} \right) = & -\lambda^0 \sum_{p=1}^{N'} \overset{h}{M} \overset{p}{M} \left[-\frac{1}{2} \overset{p}{R}_{,sr} \overset{h}{\xi}^s \overset{p}{\xi}^r + \frac{1}{2} \overset{p}{\xi}^s \overset{p}{R}_{,sm} \right. \\ & - 4 \overset{p}{\psi} \sum_{q=1}^N \overset{q}{M} \overset{q}{\psi}_{,m} + \overset{p}{\psi}_{,s} \left\{ 4 \overset{h}{\xi}^s \overset{h}{\xi}^m + 4 \overset{p}{\xi}^s \overset{p}{\xi}^m \right. \\ & \left. - 4 \overset{p}{\xi}^s \overset{h}{\xi}^m - 3 \overset{h}{\xi}^s \overset{p}{\xi}^m \right\} + \overset{p}{\psi}_{,m} \left\{ -\overset{h}{\xi}^s \overset{h}{\xi}^s - \frac{3}{2} \overset{p}{\xi}^s \overset{p}{\xi}^s \right. \\ & \left. \left. + 4 \overset{h}{\xi}^s \overset{p}{\xi}^s + 4 \sum_{q=1}^N \overset{q}{M} \overset{q}{\psi} + \sum_{q=1}^N \overset{q}{M} \overset{q}{\psi} \right\} \right]. \quad \dots \quad (5.3) \end{aligned}$$

The equations (5.3) are correct up to the sixth order of approximation. This is the equation which was solved by H. P. Robertson (1938) to find the perihelion motion of the planet Mercury. The $\overset{h}{\xi}^s$ which are appearing on the right-hand side of the equation (5.3) are the solutions of the Newtonian equations (5.1).

Our calculations show that $\overset{h}{M}$ is a constant, whereas $\overset{h}{M}$, $\overset{h}{M}$, etc. are functions of time. This result taken along with (3.9) shows how the associated mass parameter $\overset{h}{M}$ varies with time when the gravitational interaction is taken account of. This effect for a slowly varying field is in agreement with Mach's conjecture regarding the inertia of a particle.

ABSTRACT

A mistake has been noticed in the current theory of approximations giving the equations of motion of N particles in a relativistic gravitational field. The genesis of the mistake has been shown and the method of approximation is modified so as to give correct results. The new equations have the same form as the old ones considered by Robertson but in substance there is a difference. In the procedure given here the splitting up of terms of the field equations is the same as of the 1949 theory and the co-ordinate conditions are used in a form free from singularities. The calculations also show in what manner Mach's conjecture regarding the inertia of a particle is verified.

REFERENCES

- Einstein, A., Infeld, L., and Hoffmann, B. (1938). The Gravitational Equations and the Problem of Motion. *Ann. Math.*, **39**, 65-100.
Einstein, A., and Infeld, L. (1940). The Gravitational Equations and the Problem of Motion: ii. *Ann. Math.*, **41**, 455-464.
Einstein, A., and Infeld, L. (1949). On the Motion of Particles in General Relativity Theory. *Canad. J. Math.*, **1**, 209-241.
Robertson, H. P. (1938). The Two Body Problem in General Relativity. *Ann. Math.*, **39**, 101-104.

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INTERNAL BALLISTICS FOR POWER LAW OF BURNING WITH MOST GENERAL FORM FUNCTION

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1. INTRODUCTION

All the present theories of internal ballistics of guns assume a linear law of burning of the propellants as it renders the mathematical treatment of the ballistic problem comparatively easy. Since the experiments showed that many propellants burn according to the power law of burning, Clemmow in 1928 gave numerical solution of the equations of internal ballistics for power law of burning, assuming simple quadratic form function.

Now in the present-day advancement of the ballistic theories the need of a better form function has arisen. The simple quadratic form function is not applicable when we are considering a more digressive burning surface or when the propellant is a chopped one, e.g. chopped cord. Even for the most simple case of a cord we have a cubic form function although we generally take a quadratic one. The most general form function can be of the form $Z = \phi(f) \equiv (1-f)[1+\chi(f)]$, where $\chi(0) = 0$ which satisfies the two conditions that when $Z = 0$, $f = 1$ and when $Z = 1$, $f = 0$.

In the present communication the author has extended the results of Clemmow to the case of general form function $Z = \phi(f)$ and has deduced the results for other form functions, such as general cubic form function, for spherical propellants and for quadratic form function.

2. FUNDAMENTAL EQUATIONS

The four fundamental equations of internal ballistics for power law of burning, and having general form function, are:

Energy equation is

$$FCZ = p \left\{ A(x+l) - CZ \left(b - \frac{1}{\delta} \right) \right\} + \frac{1}{2}(\gamma-1)w_1v^2 \quad \dots \quad (1)$$

where $w_1 = 1.05w + \frac{1}{3}C$.

Dynamical equation is

$$w_1 \frac{dv}{dt} = Ap. \quad \dots \quad (2)$$

Form function is

$$\begin{aligned} Z &= \phi(f), \text{ where } \phi(f) \text{ is such that when } Z = 0, f = 1 \text{ and when } Z = 1, f = 0 \\ &= (1-f)[1+\chi(f)], \text{ where } \chi(f) \text{ is zero for } f = 0. \quad \dots \quad (3) \end{aligned}$$

Also there is a condition on equation (3) that it should give one positive root less than unity.

Rate of burning equation is

$$D \frac{df}{dt} = -\beta p^\alpha. \quad \dots \quad \dots \quad \dots \quad (4)$$

Making the following dimensionless transformation

$$\xi = 1 + \frac{x}{l}$$

$$\eta = \frac{vAD}{FC\beta} \left(\frac{FC}{Al} \right)^{1-\alpha}$$

$$\zeta = \frac{pAl}{FC}$$

and

$$M = \frac{A^2 D^2}{FC\beta^2 w_1} \left(\frac{FC}{Al} \right)^{2-2\alpha}$$

the above equations reduce to

$$Z = \zeta(\xi - BZ) + \frac{1}{2}(\gamma - 1)\eta^2/M \quad \dots \quad \dots \quad \dots \quad (5)$$

$$\eta \frac{d\eta}{d\xi} = M\zeta \quad \dots \quad \dots \quad \dots \quad \dots \quad (6)$$

$$Z = \phi(f) \quad \dots \quad \dots \quad \dots \quad \dots \quad (7)$$

and

$$\eta \frac{df}{d\xi} = -\zeta^\alpha. \quad \dots \quad \dots \quad \dots \quad \dots \quad (8)$$

3. SOLUTION OF THE EQUATIONS FOR $Z = \phi(f)$ WITH $B = 0$

Now the variable η can be easily eliminated, by differentiating (5) and using (6) we have

$$\begin{aligned} dZ &= \xi d\zeta + \zeta d\xi \\ &= \xi^{1-\gamma} d(\zeta^\gamma). \end{aligned} \quad \dots \quad \dots \quad \dots \quad (9)$$

From (6) and (8) we get

$$\frac{d\eta}{df} = -M\zeta^{1-\alpha} \quad \dots \quad \dots \quad \dots \quad (10)$$

and equation (8) can be written as

$$\eta = -\zeta^\alpha \frac{d\xi}{df}. \quad \dots \quad \dots \quad \dots \quad (11)$$

Substituting the value of η from (11) in (10) we get

$$\frac{d}{df} \left(\zeta^\alpha \frac{d\xi}{df} \right) = M\zeta^{1-\alpha}. \quad \dots \quad \dots \quad \dots \quad (12)$$

Now make the following substitutions:

$$Y = \zeta^\gamma \quad \dots \quad \dots \quad \dots \quad (13)$$

so that equation (9) reduces to

$$dZ = \xi^{1-\gamma} dY \quad \dots \quad \dots \quad \dots \quad (14)$$

Mathematically, the function $\phi(f)$ being known, we can express f as a function of Z , $f = \phi_1(Z)$ (say).

Therefore differentiating this function we get

$$df = \phi'_1(Z) dZ. \quad \dots \dots \dots (15)$$

From (9), (12) and (15) we get

$$\frac{d}{dZ} \cdot \frac{dZ}{df} \left[\zeta^\alpha \frac{d\zeta}{dZ} \cdot \frac{dZ}{df} \right] = M \zeta^{1-\alpha}$$

or

$$\frac{1}{\phi'_1(Z)} \frac{d}{dZ} \left[\zeta^\alpha \frac{1}{\phi'_1(Z)} \frac{d\zeta}{dZ} \right] = M \zeta^{1-\alpha}. \quad \dots \dots \dots (16)$$

Performing this differentiation we get

$$\frac{1}{\phi'_1(Z)} \left[\xi'' \zeta^\alpha \frac{1}{\phi'_1(Z)} + \alpha \zeta^{\alpha-1} \zeta' \xi' \frac{1}{\phi'_1(Z)} + \zeta^\alpha \xi' \frac{d}{dZ} \left\{ \frac{1}{\phi'_1(Z)} \right\} \right] = M \zeta^{1-\alpha}. \quad \dots (17)$$

This can be written as

$$\left\{ \frac{1}{\phi'_1(Z)} \right\}^2 \left[\xi'' + \alpha \xi' \frac{\zeta'}{\zeta} \right] + \xi' \frac{1}{\phi'_1(Z)} \frac{d}{dZ} \left\{ \frac{1}{\phi'_1(Z)} \right\} = M \zeta^{1-2\alpha}. \quad \dots (18)$$

Differentiating (14) we get

$$(\gamma-1) \frac{\xi'}{\xi} = \frac{Y''}{Y'}. \quad \dots \dots \dots (19)$$

Differentiating this equation (19) again we get

$$\frac{\xi''}{\xi'} = \frac{Y'''}{Y''} + (2-\gamma) \frac{Y''}{(\gamma-1)Y'} \quad \dots \dots \dots (20)$$

and from (13)

$$\frac{\zeta'}{\xi} = \frac{Y'}{Y} - \gamma \frac{\xi'}{\xi} = \frac{Y'}{Y} - \frac{\gamma Y''}{(\gamma-1)Y'}. \quad \dots \dots \dots (21)$$

Substituting these values in (18) we get

$$\begin{aligned} & \left\{ \frac{1}{\phi'_1(Z)} \right\}^2 \left[\frac{Y'''}{Y''} + (n-2) \frac{Y''}{Y'} + \alpha \frac{Y'}{Y} \right] + \frac{1}{\phi'_1(Z)} \frac{d}{dZ} \left\{ \frac{1}{\phi'_1(Z)} \right\} \\ & = \frac{Q(Y')^{2-2n}}{Y'' Y^{2\alpha-1}} \quad \dots \dots \dots (22) \end{aligned}$$

where

$$n = \frac{\gamma(1-\alpha)}{\gamma-1} \text{ and } Q = (\gamma-1)M.$$

For the particular function $\phi(f)$ we can find out $\phi_1(Z)$ and hence the equation (22) can always be integrated numerically to give a series of values of Y , Y' and Y'' in terms of Z , for a given propellant, i.e. given n and α and a given value of Q representing the loading conditions. For finite shot-start pressure the initial conditions are

$$Y_0 = \zeta_0, Z = Z_0 = \zeta_0$$

and also from (14) we have $Y' = 1$ and $Y'' = 0$.

For zero shot-start pressure the initial conditions are

$$Y_0 = 0, Z_0 = 0, Y' = 1 \text{ and } Y'' = 0.$$

We can develop a series solution of the equation (22) if the particular form of $\phi(f)$ is assumed and the initial calculations can be checked up.

Here the shot-travel is given by equation (14) as

$$\xi^{\gamma-1} = Y'. \quad \dots \quad (23)$$

The pressure is given by equation (13) as

$$\zeta = \frac{Y}{\xi^\gamma} = \frac{Y}{(Y')^{\frac{\gamma}{\gamma-1}}}. \quad \dots \quad (24)$$

For maximum pressure $\frac{d\zeta}{dZ} = 0$ and therefore differentiating equation (23) and putting $\frac{d\zeta}{dZ} = 0$ we have

$$\frac{Y'}{Y} = \frac{\gamma}{\gamma-1} \frac{Y''}{Y'} \quad \dots \quad (25)$$

and this equation can be solved numerically from the tabulated values of Y , Y' and Y'' .

At all-burnt position, $Z = 1$.

Therefore the velocity can be obtained from the energy equation in the following form:

$$\begin{aligned} \eta^2 &= 2M(Z - \zeta\xi)/(\gamma-1) \\ &= 2M \left(Z - \frac{Y}{Y'} \right) / (\gamma-1). \end{aligned}$$

Hence

$$v^2 = 2FC \left(Z - \frac{Y}{Y'} \right) / (\gamma-1) w_1. \quad \dots \quad (26)$$

After all-burnt there is simple adiabatic expansion and

$$\zeta \xi^\gamma = Y_2 \quad \dots \quad (27)$$

and therefore muzzle velocity is given by

$$v_s^2 = \frac{2FC}{(\gamma-1)w_1} \left(1 - \frac{Y_2}{\xi_s^{\gamma-1}} \right). \quad \dots \quad (28)$$

Now in further analysis we will assume different forms of the function $\phi(f)$ and show how these equations are modified.

4. SOLUTION FOR CUBIC FORM FUNCTION

$$Z = (1-f)(1+\theta f+\theta' f^2).$$

This form function can be written as

$$Z = 1 + (\theta-1)f + (\theta'-\theta)f^2 - \theta'f^3$$

or
$$f^3 + \frac{(\theta-\theta')}{\theta'} f^2 + \frac{(1-\theta)}{\theta'} f + \frac{(Z-1)}{\theta'} = 0. \quad \dots \quad (29)$$

This further reduces to the form

$$f'^3 + p_1 f' + q_1 = 0 \quad \dots \quad (30)$$

where
$$p_1 = \frac{1-\theta}{\theta'} - \frac{1}{3} \left(\frac{\theta-\theta'}{\theta'} \right)^2$$

and
$$q_1 = \frac{(Z-1)}{\theta'} - \frac{(1-\theta)(\theta-\theta')}{3\theta'^2} + \frac{2}{27} \frac{(\theta-\theta')^3}{\theta'^3}.$$

The solutions of the equation (30) are

$$f'_1 = (A)^{\frac{1}{3}} + (B)^{\frac{1}{3}}$$

$$f'_2 = \omega(A)^{\frac{1}{3}} + \omega^2(B)^{\frac{1}{3}}$$

$$f'_3 = \omega^2(A)^{\frac{1}{3}} + \omega(B)^{\frac{1}{3}}$$

where
$$A = -\frac{q_1}{2} + \sqrt{\frac{q_1^2}{4} + \frac{p_1^3}{27}}.$$

$$B = -\frac{q_1}{2} - \sqrt{\frac{q_1^2}{4} + \frac{p_1^3}{27}}.$$

Hence
$$f_1 = [(A)^{\frac{1}{3}} + (B)^{\frac{1}{3}}] - \frac{(\theta-\theta')}{3\theta'}$$

$$f_2 = [\omega(A)^{\frac{1}{3}} + \omega^2(B)^{\frac{1}{3}}] - \frac{(\theta-\theta')}{3\theta'}$$

$$f_3 = [\omega^2(A)^{\frac{1}{3}} + \omega(B)^{\frac{1}{3}}] - \frac{(\theta-\theta')}{3\theta'}.$$

Now the values of the roots depend on the values of θ and θ' , so that let us consider the most general root given by

$$f = \frac{K_1}{\theta_1} \left[-\{Z+k\} + \sqrt{(Z+k)^2 + k_1} \right]^{\frac{1}{3}} + \frac{K_2}{\theta_1} \left[-\{Z+k\} - \sqrt{(Z+k)^2 + k_1} \right]^{\frac{1}{3}} - K_3$$

where
$$k = \frac{2}{27} \frac{(\theta-\theta')^3}{\theta'^2} - \frac{(1-\theta)(\theta-\theta')}{3\theta'} - 1, \quad \theta_1 = (2\theta')^{\frac{1}{3}}$$

$$k_1 = \frac{4\theta'^2 p_1}{27} \quad \text{and} \quad k_3 = \frac{\theta-\theta'}{3\theta'}.$$

Also K_1 and K_2 can have values 1, 1; ω , ω^2 and ω^2 , ω respectively. Let us make the substitution $(Z+k) = Z_1$ (say)

$$\therefore f = \frac{K_1}{\theta_1} \left[-Z_1 + \sqrt{Z_1^2 + k_1} \right]^{\frac{1}{3}} + \frac{K_2}{\theta_1} \left[-Z_1 - \sqrt{Z_1^2 + k_1} \right]^{\frac{1}{3}} - K_3.$$

Here
$$\phi_1(Z_1) = \frac{K_1}{\theta_1} \left[-Z_1 + \sqrt{Z_1^2 + k_1} \right]^{\frac{1}{3}} + \frac{K_2}{\theta_1} \left[-Z_1 - \sqrt{Z_1^2 + k_1} \right]^{\frac{1}{3}} - K_3 \quad (32)$$

$$\therefore \frac{1}{\phi_1'(Z_1)} = \frac{3\theta_1\sqrt{Z_1^2+k_1}}{[-K_1\{-Z_1+\sqrt{Z_1^2+k_1}\}^{\frac{1}{2}}+K_2\{-Z_1-\sqrt{Z_1^2+k_1}\}^{\frac{1}{2}}]}$$

or $\left\{\frac{1}{\phi_1'(Z_1)}\right\}^2 = \frac{9\theta_1^2(Z_1^2+k_1)}{[-K_1\{-Z_1+\sqrt{Z_1^2+k_1}\}^{\frac{1}{2}}+K_2\{-Z_1-\sqrt{Z_1^2+k_1}\}^{\frac{1}{2}}]^2} \dots (33)$

Also

$$\frac{1}{\phi_1'(Z_1)} \frac{d}{dZ_1} \left\{ \frac{1}{\phi_1'(Z_1)} \right\} = \frac{9\theta_1^2 Z_1 \{-K_1(-Z_1+\sqrt{Z_1^2+k_1})^{\frac{1}{2}}+K_2(-Z_1-\sqrt{Z_1^2+k_1})^{\frac{1}{2}}\} - 3\theta_1^2 \sqrt{Z_1^2+k_1} \{K_1(-Z_1+\sqrt{Z_1^2+k_1})^{\frac{1}{2}}+K_2(-Z_1-\sqrt{Z_1^2+k_1})^{\frac{1}{2}}\}}{[-K_1\{-Z_1+\sqrt{Z_1^2+k_1}\}^{\frac{1}{2}}+K_2\{-Z_1-\sqrt{Z_1^2+k_1}\}^{\frac{1}{2}}]^3} \dots (34)$$

Hence our equation (22) reduces to:

$$\frac{(Z_1^2+k_1)}{[-K_1\{-Z_1+\sqrt{Z_1^2+k_1}\}^{\frac{1}{2}}+K_2\{-Z_1-\sqrt{Z_1^2+k_1}\}^{\frac{1}{2}}]^2} \left[\frac{Y'''}{Y''} + (n-2) \frac{Y''}{Y'} + \alpha \frac{Y'}{Y} \right] + \frac{9\theta_1^2 Z_1 \{-K_1(-Z_1+\sqrt{Z_1^2+k_1})^{\frac{1}{2}}+K_2(-Z_1-\sqrt{Z_1^2+k_1})^{\frac{1}{2}}\} - 3\theta_1^2 \sqrt{Z_1^2+k_1} \{K_1(-Z_1+\sqrt{Z_1^2+k_1})^{\frac{1}{2}}+K_2(-Z_1-\sqrt{Z_1^2+k_1})^{\frac{1}{2}}\}}{9\theta_1^2 [-K_1\{-Z_1+\sqrt{Z_1^2+k_1}\}^{\frac{1}{2}}+K_2\{-Z_1-\sqrt{Z_1^2+k_1}\}^{\frac{1}{2}}]^3} = \frac{Q_1(Y')^{2-2n}}{Y'' Y^{2\alpha-1}}$$

where $n = \frac{\gamma(1-\alpha)}{(\gamma-1)}$ and $Q_1 = \frac{(\gamma-1)M}{9\theta_1^2}$.

In this case this is the equation which gives Y , Y' and Y'' in terms of Z_1 .

Further analysis is the same as has already been discussed in section 3.

As an example let us consider the case of a cord. In this case Z is a cubic in f , given by

$$Z = (1-f) \left(1+f+\frac{f^2}{\lambda} \right)$$

λ is generally of the order of 200.

$$\therefore Z = (1-f)(1+f+0.005f^2).$$

This can be written as

$$0.005f^3 + 0.995f^2 + (Z-1) = 0$$

$$\text{or } f^3 + 199f^2 + 200(Z-1) = 0.$$

Here there is one change of sign, because $(Z-1)$ is negative; hence there is only one positive root. Also the discriminant of the transformed cubic is positive, therefore $K_1 = K_2 = 1$. Now we can set up our equation (35) in this case and can integrate this numerically.

5. SOLUTION FOR THE CUBIC FORM FUNCTION

$$Z = (1-f)(1+f+f^2).$$

This form function can be written as

$$Z = (1-f^3). \quad \dots \quad (37)$$

From this equation we get

$$f = (1-Z)^{\frac{1}{3}}. \quad \dots \quad (38)$$

Therefore our

$$\phi_1(Z) = (1-Z)^{\frac{1}{3}}.$$

Hence our equation (22) reduces to

$$9(1-Z)^{\frac{1}{3}} \left[\frac{Y'''}{Y''} + (n-2) \frac{Y''}{Y'} + \alpha \frac{Y'}{Y} \right] - 6(1-Z)^{\frac{1}{3}} = \frac{Q(Y')^{2-2n}}{Y'' Y^{2\alpha-1}}. \quad \dots \quad (39)$$

This can be further written as

$$(1-Z)^{\frac{1}{3}} \left[\frac{Y'''}{Y''} + (n-2) \frac{Y''}{Y'} + \alpha \frac{Y'}{Y} \right] - \frac{2}{3}(1-Z)^{\frac{1}{3}} = \frac{Q_1(Y')^{2-2n}}{Y'' Y^{2\alpha-1}} \quad \dots \quad (40)$$

where $n = \frac{\gamma(1-\alpha)}{(\gamma-1)}$ and $Q_1 = \frac{(\gamma-1)M}{9}$.

In this case, this is the equation which gives Y , Y' and Y'' in terms of Z . Further analysis we have to do exactly in the same way as done in section 3.

6. SOLUTION FOR THE GENERAL QUADRATIC FORM FUNCTION

$$Z = (1-f)(1+\theta f).$$

This form function can be written as

$$Z = 1 + (\theta-1)f - \theta f^2. \quad \dots \quad (41)$$

From this equation we get

$$f = \left[\frac{-(\theta-1)}{2\theta} + \frac{\sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} \right]$$

(Since we have to consider positive root, we take the positive sign.)

Therefore

$$\phi_1(Z) = \left[\frac{-(\theta-1) + \sqrt{(\theta+1)^2 - 4\theta Z}}{2\theta} \right] \quad \dots \quad (42)$$

Hence our equation (22) reduces to

$$\left[(\theta+1)^2 - 4\theta Z \right] \left[\frac{Y'''}{Y''} + (n-2) \frac{Y''}{Y'} + \alpha \frac{Y'}{Y} \right] - 2\theta = \frac{Q(Y')^{2-2n}}{Y'' Y^{2\alpha-1}}. \quad \dots \quad (43)$$

This can be further modified as has been done in Clemmow's paper. Let us have

$$qY = q\zeta^\gamma = Y_1 \text{ (say)}$$

and

$$Z_1 = qZ \quad \text{where} \quad q = \frac{4\theta}{(1+\theta)^2}.$$

Therefore the equation (43) modifies to

$$\left(1 - Z_1\right) \left[\frac{Y_1'''}{Y_1''} + (n-2) \frac{Y_1''}{Y_1'} + \alpha \frac{Y_1'}{Y_1} \right] - \frac{1}{2} = \frac{Q(Y_1)^{2-2n}}{Y_1'' Y_1^{2\alpha-1}} \quad \dots \quad (44)$$

This is the equation which gives on numerical integration the values of Y , Y' and Y'' in terms of Z_1 .

The rest of the treatment is the same as done by Clemmow.

7. SOLUTION FOR THE FORM FUNCTION $Z = (1-f)$.

Here our $\phi_1(Z) = (1-Z)$.

Hence our equation (22) reduces to

$$\frac{Y'''}{Y''} + (n-2) \frac{Y''}{Y'} + \alpha \frac{Y'}{Y} = \frac{M(\gamma-1)(Y')^{2-2n}}{Y'' Y^{2\alpha-1}} \quad \dots \quad (45)$$

and we have to proceed exactly in the same way as in section 3. The series solution for this equation has been given by Clemmow.

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SUMMARY

In this communication the author deals with the problem of internal ballistics of guns for power law of burning and using propellants which have the most general form function, $Z = \phi(f)$; where function $\phi(f)$ is such that when $Z = 0, f = 1$ and when $Z = 1, f = 0$. Further from this solution the author has deduced the results for a general cubic form function given by $Z = (1-f)(1+\theta f+\theta' f^2)$, and from this for a cubic $Z = (1-f^3)$ which represents the form function for spherical propellants. Also the results for a general quadratic form function given by $Z = (1-f)(1+\theta f)$ have been deduced.

REFERENCES

- Clemmow, C. A. (1928). A theory of internal ballistics based on a pressure-index law of burning for propellants. *Phil. Trans. Roy. Soc.*, A 227, 345.
H. M. Stationery Office (1951). *Internal Ballistics*.

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